Nonlinear Waves: Woods Hole GFD Program 2009

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Lecture 3: Nonlinear waves in a variable medium

The motivation here is to describe internal solitary waves in the coastal ocean, where the background is typically **not a uniform state**, as we have assumed so far in obtaining asymptotic models such as the Korteweg-de Vries (KdV) and extended Korteweg-de Vries (eKdV) equations.

Hence the aim is to extend these models to incorporate a variable background state. On the assumption that the background state is **slowly varying** in some sense relative to the waves, the outcome will be KdV-type equations, but with **variable coefficients**, namely the variable-coefficient Korteweg-de Vries (vKdV) equation, or the variable-coefficient extended Korteweg-de Vries (veKdV) equation. These equations are not integrable in general, and so we must then seek a combination of asymptotic and numerical solutions.

Strait of Messina

based on the article *Internal Waves in the Strait of Messina Studied by a Numerical Model and Synthetic Aperture Radar Images from the ERS 1/2 Satellites* by P. Brandt, A. Rubino, W. Alpers, and J. O. Backhaus [1997] with permission of the authors

Overview

The Strait of Messina separates the Italian Peninsula from the Italian island of Sicily and connects the Tyrrhenian Sea in the north with the Ionian Sea in the south (Figure 1). The strait is a narrow channel, whose smallest cross-sectional area is 0.3 km ^{2} in the sill region. There, the mean water depth is 80 m. While in the southern part of the strait the water depth increases rapidly (a depth of 800 m is encountered approximately 15 km south of the sill), in the northern part it increases more gently (a depth of 400 m is encountered approximately 15 km north of the sill). Throughout the year, two different water masses are present in the Strait of Messina: the Tyrrhenian surface water (TSW) and the colder and saltier Levantine Intermediate Water (LIW).

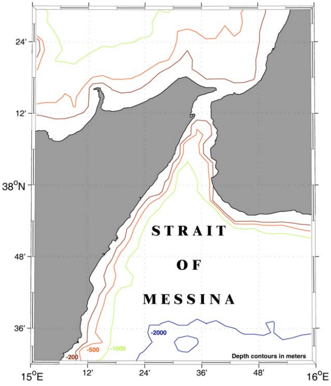
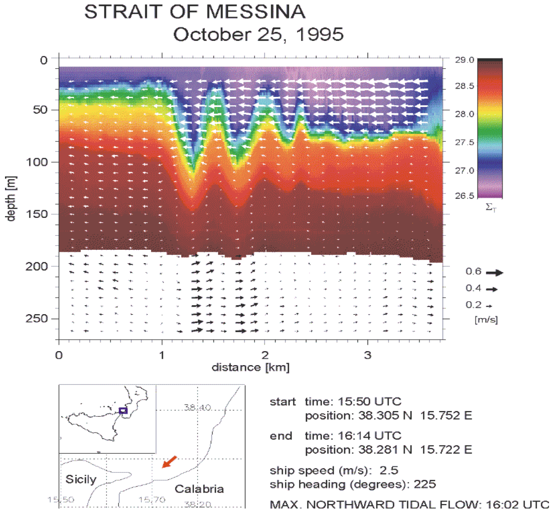
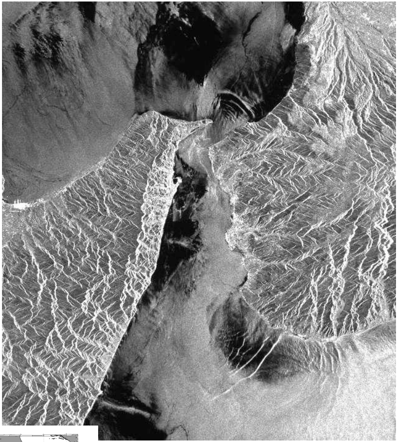


Figure 1. Bathymetry of the Strait of Messina and surrounding region.

3.2: Strait of Messina



3.3: Strait of Messina



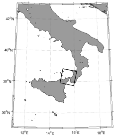


Figure 5. ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. ©ESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR http://www.ifm.uni-hamburg.de/ers-sar/]

3.4: Northwest Shelf

Australian Northwest Shelf

Overview

The Australian Northwest shelf extends roughly 2000 km along the coast of Western Australia (Figure 1). The region is influenced by part of the South Equatorial Current that runs southwest out of the Arafura Sea.

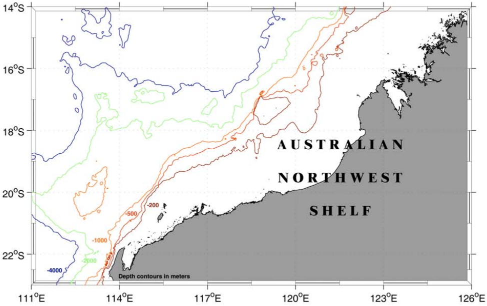


Figure 1. Bathymetry of Australian Northwest Shelf [Smith and Sandwell, 1997]

Observations

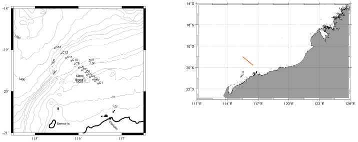
There has been considerable scientific investigation into the internal tide and associated internal waves on the Australian Northwest Shelf [Holloway, 1987, 1994; Smyth and Holloway, 1988; Holloway et al. 1997, 1998, 1999a, 199b; Holloway and Pelinovsky, 2001; Pelinovsky et al, 1995]

Observations on the Northwest Australian Shelf by Holloway show long internal waves of semi-diurnal tidal origin (internal tides) with wavelengths of approximately 20 km evolving into a variety of nonlinear waveforms. These forms include bores on both the leading and trailing faces of the long internal tide, as well as short period (approximately 10 minutes, close to the buoyancy period) internal solitary waves. The nonlinear features develop as the waves propagate shoreward into decreasing water depth with a propagation speeds of approximately 0.4 m/s.

3.5: Northwest Shelf

An Atlas of Oceanic Internal Solitary Waves (February 2004)
by Global Ocean Associates
Prepared for Office of Naval Research – Code 322 PO

Australian Northwest Shelf



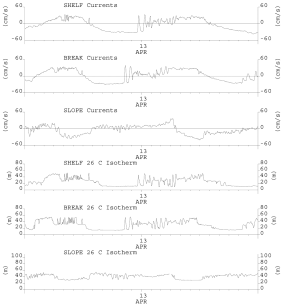


Figure 2. Time series of isotherm displacements and onshore currents are shown from 3 moorings, (*Slope, Break* and *Shelf*), located in 78 to 109 m water depths, and a few kilometers apart at the outer edge of the Australian Northwest continental shelf. The plots show a variety of nonlinear wave forms including bores on both the leading and trailing faces of the long internal tide, as well as short period (approximately 10 minutes, close to the buoyancy period) internal solitary waves. [After Holloway and Pelinovsky, 2001]

3.6: Northwest Shelf

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Australian Northwest Shelf

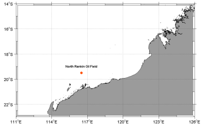
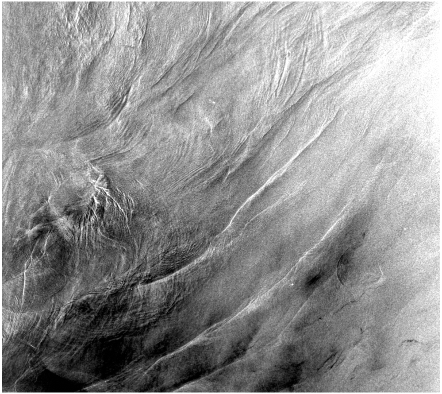


Figure 3. RADARSAT-1 (C-band HH) Standard Mode SAR image acquired over the North Rankin oil and gas field of Western Australia on 12 February 1997 (orbit 11881). The image shows a complex pattern of internal wave signatures. Imaged area is approximately 100 km x 100 km. ©CSA 1997. [Image courtesy of George Cresswell, CSIRO Marine Research, Hobart, Tasmania, Australia. RADARSAT image acquired as part of ADRO Project #72 Cresswell and Tildesley]



3.7: Linear Waves

The simplest model for linear waves in an inhomogeneous medium is the one-dimensional wave equation with a variable speed $c(x)$ (for example, water waves on variable depth $h(x)$ where $c = (gh)^{1/2}$)

$$u_{tt} - (c^2(x)u_x)_x = 0\,. \tag{1}$$

This has the WKB asymptotic solution

$$u \sim a(x,t)f(t-\tau(x))\,, \quad \text{where} \quad \tau(x) = \int^x \frac{dx}{c(x)}\,, \tag{2}$$

$$\text{and} \quad (a^2)_t + (ca^2)_x = 0\,. \tag{3}$$

Here it assumed that the phase function $f(\xi)$ ($\xi = t - \tau(x)$) is rapidly varying relative to the amplitude function $a(x,t)$ and the speed $c(x)$. For example, $f(\xi) = \exp(-i\omega\xi)$ where $\omega >> 1$ and $a = a(x) \propto c^{-1/2}(x)$ is the well-known high-frequency approximation. Equation (3) expresses conservation of **wave action**, where the action density here is a^2, and in this context usually coincides with the wave energy density.

3.8: Variable coefficient KdV equation

We now extend the KdV equation to a variable background,

$$u_t + cu_x + \frac{cQ_x}{2Q} u + \mu u u_x + \beta u_{xxx} = 0 \,. \qquad (4)$$

Here the inhomogeneous background implies that the linear phase speed $c = c(x)$, and that likewise the coefficients are $\mu = \mu(x), \beta = \beta(x)$. The linear magnification factor $Q = Q(x)$ is such that in the linear long wave theory Qu^2 is the wave action flux (for water waves $Q = c$). Equation (4) is derived using the usual KdV balance, where if $\partial/\partial x \sim \epsilon << 1$, then the linear dispersion $\epsilon^2 \sim \partial^2/\partial x^2$ is balanced by nonlinearity $u \sim \epsilon^2$. To this balance we add **weak inhomogeneity** so that Q_x/Q scales as ϵ^3. Within this basic balance of terms, (4) transforms into

$$u_\tau + \frac{Q_\tau}{2Q} u + \frac{\mu}{c} u u_X + \frac{\beta}{c^3} u_{XXX} = 0 \,, \qquad (5)$$

$$\text{where} \quad \tau = \int_0^x \frac{dx'}{c(x')}, \qquad X = \tau - t \,. \qquad (6)$$

The coefficients depend explicitly on τ, which describes evolution along the wave path. **Equations (4) and (5) are asymptotically equivalent**, and differ only in terms $O(\epsilon^7)$, which is the error term term in both.

3.9: Variable coefficient KdV equation

$$u_\tau + \frac{Q_\tau}{2Q} u + \frac{\mu}{c} u u_X + \frac{\beta}{c^3} u_{XXX} = 0 \,,$$

This governing equation can be cast into several equivalent forms. That most commonly used is the variable-coefficient KdV equation, obtained here by putting

$$A = Q^{1/2} u \,, \quad \text{so that} \quad A_\tau + \alpha A A_X + \lambda A_{XXX} = 0 \,, \tag{7}$$

$$\text{where} \quad \alpha = \frac{\mu}{c Q^{1/2}} \,, \quad \lambda = \frac{\beta}{c^3} \,. \tag{8}$$

The coefficients $\alpha = \alpha(\tau), \lambda = \lambda(\tau)$, and there are two conservation laws

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} A \, dX = 0 \,, \tag{9}$$

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} A^2 \, dX = 0 \,. \tag{10}$$

Although often referred to as conservation of "mass" and "momentum", the latter is strictly that for **wave action flux**, and the former is only asymptotically that for the physical mass.

3.10: Slowly-varying periodic waves

We now suppose that the coefficients α, λ are slowly varying, and write

$$\alpha = \alpha(T)\,, \quad \lambda = \lambda(T)\,, \quad T = \sigma\tau\,, \quad \sigma << 1\,. \tag{11}$$

Then seek a standard multi-scale expansion for a **modulated periodic wave**, namely

$$A = A_0(\theta, T) + \sigma A_1(\theta, T) + \cdots\,, \tag{12}$$

$$\theta = k\{X - \frac{1}{\sigma}\int^T V(T)\,dT\}\,. \tag{13}$$

Here A **is periodic in the phase** θ with a fixed period of 2π, while k is a fixed constant. Substitution into (7) yields at the leading orders

$$-VA_{0\theta} + \alpha A_0 A_{0\theta} + \lambda k^2 A_{0\theta\theta\theta} = 0\,, \tag{14}$$

$$-VA_{1\theta} + \alpha(A_0 A_1)_\theta + \lambda k^2 A_{1\theta\theta\theta} = -\frac{1}{k}A_{0T}\,. \tag{15}$$

Each of these is an ordinary differential equation with θ as the independent variable, and T as a parameter.

3.11: Cnoidal waves

The solution of (14) is the well-known cnoidal wave

$$A_0 = a\{b(m) + \mathrm{cn}^2(\gamma\theta; m)\} + d\,, \tag{16}$$

$$\text{where} \quad b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \quad \alpha a = 12m\lambda\gamma^2k^2\,, \tag{17}$$

$$\text{and} \quad V = \alpha d + \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}\,. \tag{18}$$

Here $cn(x;m)$ is the Jacobian elliptic function of modulus $m, 0 < m < 1$, $K(m), E(m)$ are the elliptic integrals of the first and second kind. The amplitude is a, the mean value of A over one period is d, $\gamma = K(m)/\pi$ and the spatial period is $2\pi/k$. As the modulus $m \to 1$, this becomes a **solitary wave**, since then $b \to 0$ and $\mathrm{cn}^2(x) \to \mathrm{sech}^2(x)$; in this limit $\gamma \to \infty, k \to 0$ with $\gamma k = K$ held fixed. As $m \to 0, \gamma \to 1/2$, and it reduces to sinusoidal waves of small amplitude $a \sim m$ and wavenumber k. The cnoidal wave (16) contains three free parameters, which here depend on the slow variable T; we take these to be the amplitude a, the mean level d and the modulus m, so that equations (17, 18) then determine k, V respectively.

The task now is to determine how a, d, m, depend on T. There are two principal methods used to achieve this. One is the so-called **Whitham averaging method**, where one seeks three appropriate conservation laws for the vKdV equation (7), inserts the periodic cnoidal wave into these laws, and then averages over the phase θ. It is important that, in addition, one should also use the law for **conservation of waves**, namely

$$k_T + \omega_X = 0\,, \tag{19}$$

where here $X = \epsilon x$ and $\omega = kV$. But in the present case, there is no X-dependence, and so this readily yields the result that k **is a constant**. For the vKdV equation, it is convenient to take the remaining two conservations laws as those for "mass" and "momentum" (9, 10)

$$\frac{\partial}{\partial T}\int_0^{2\pi} A d\theta = 0\,, \tag{20}$$

$$\frac{\partial}{\partial T}\int_0^{2\pi} A^2 d\theta = 0\,. \tag{21}$$

3.13: Modulation equations

Substitution of the cnoidal wave (16) into the "mass" equation (20) readily shows that **the mean level d is a constant** . Hence the remaining variable, namely the wave amplitude a, can now be found by substituting (16) into the "momentum" equation (21). The result is

$$a^2\{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{cn}^4(\gamma\theta; m)\, d\theta \;-\; b(m)^2\} \;=\; \text{constant}\,,$$

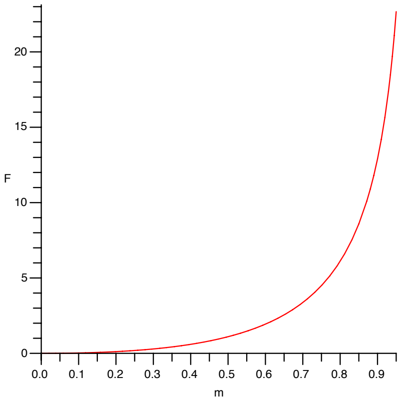
or $$\frac{a^2}{m^2}\{(2-3m)(1-m)+\frac{(4m-2)E(m)}{K(m)}-3m^2b(m)^2\}=\text{constant}\,.$$

Finally substitution of the expressions (17) yields a relation for the variation of the modulus m,

$$F(m) \equiv K^2\{(4-2m)EK - 3E^2 - (1-m)K^2\} = \text{constant}\,\frac{\alpha^2}{\lambda^2}\,. \qquad (22)$$

Here $K = K(m), E = E(m)$. Numerical evaluation shows that $F(m)$ is a monotonically increasing function of m, and so as α/λ increases so does m. Thus, if the nonlinear coefficient $\alpha \to 0$ then $m \to 0$ since then $F(m) \sim m^2$. In this limit $m \sim \alpha$, but the amplitude a remains finite. But, if the dispersive coefficient $\lambda \to 0$, (e.g. internal waves in shallow water) then $m \to 1$ and the waves become more like solitary waves.

3.14: Modulated water waves



For water waves, $c = (gh)^{1/2}, Q = c, \mu = 3c/2h, \beta = ch^2/6$ and so $\alpha/\lambda \propto h^{-9/4}, F(m) \propto h^{-9/2}$. Thus as the waves propagate towards the shore $h \to 0$, $m \to 1$ and the amplitude $a \propto h^{-3/4}$, so that the **surface elevation** $\eta \propto h^{-1}$. For linear waves $\eta \propto h^{-1/4}$.

3.15: Asymptotic expansion

An alternative to the Whitham averaging procedure, is to continue the asymptotic expansion to the next order, and invoke the condition that A_1 is a periodic function of θ. Indeed, it is implicit in the Whitham averaging procedure that the higher-order terms in the expansion have this property. Although it can be shown that the presence of a suitable underlying Lagrangian usually ensures that this is so, we shall nevertheless verify it directly here for the first-order term. This is given by (15) in which the right-hand side is now a known periodic function of θ, given by (16). A necessary and sufficient condition for A_1 to be periodic in θ is that the right-hand side of (15) should be orthogonal to the periodic solutions of the adjoint to the homogeneous operator on the left-hand side. This adjoint is

$$-VA_{1\theta} + \alpha A_0 A_{1\theta} + \lambda k^2 A_{1\theta\theta\theta} = 0\,. \tag{23}$$

It is readily seen that two solutions of (23) are $1, A_0$, both of which are periodic. A third solution can be found by the variation-of-parameters method, but it is not periodic. Hence there are two orthogonality conditons, the first showing that d is a constant, while the second condition recovers the momentum conservation law (21). These are then supplemented as before by the equation for conservation of waves.

3.16: Slowly varying solitary waves

These results for a slowly-varying periodic wave cannot be extrapolated to the case of a slowly-varying solitary wave, as the limits $m \to 1$ and $\sigma \to 0$ do not commute. In physical terms, the basis for the validity of the slowly-varying periodic wave is that the local period (i.e. $1/kV$) should be much less than the scale of the variable medium (i.e. $1/\sigma$). The limit $m \to 1$ in (16, 17, 18) requires that $\gamma \to \infty, k \to 0$ with $\gamma k = K$ held constant, and so the period technically becomes infinite. A new concept of slowly-varying is needed, which in physical terms is that the half-width should be much less than $1/\sigma$. Technically we proceed as above and invoke a multi-scale asymptotic expansion of the same form (11, 12), but now replace the phase (13) with

$$\phi = X - \frac{1}{\sigma} \int^{T} V(T)\, dT \,. \tag{24}$$

A is not now required to be periodic in ϕ, and is defined in the domain $-\infty < \phi < \infty$, and we require that A remains bounded as $\phi \to \pm\infty$. We can suppose that $\lambda > 0$ (otherwise transpose A, x with $-A, -x$). Then, small-amplitude waves will propagate in the negative x-direction, and we can assume that $A \to 0$ **as** $\phi \to \infty$. However, it will transpire that we cannot impose this boundary condition as $\phi \to -\infty$.

The counterpart of (14, 15) is

$$-VA_{0\phi} + \alpha A_0 A_{0\phi} + \lambda A_{0\phi\phi\phi} = 0\,, \tag{25}$$
$$-VA_{1\phi} + \alpha (A_0 A_1)_\phi + \lambda A_{1\phi\phi\phi} = -A_{0T}\,. \tag{26}$$

But now the solution for A_0 is the solitary wave

$$A = a\,\mathrm{sech}^2(K\phi)\,, \quad \text{where} \quad V = \frac{\alpha a}{3} = 4\lambda K^2\,. \tag{27}$$

A background d can be added as in (16), but is a constant, and can be removed by a Galilean transformation. At the next order, we seek a solution of (26) for A_1 which is bounded as $\phi \to \pm\infty$, and further $A_1 \to 0$ as $\phi \to \infty$. As before, the adjoint equation to (26) is

$$-VA_{1\phi} + \alpha A_0 A_{1\phi} + \lambda A_{1\phi\phi\phi} = 0\,. \tag{28}$$

The two bounded solutions are $1, A_0$, but only the second solution satisfies the condition that $A_1 \to 0$ as $\phi \to \infty$. Hence **only one othogonality condition can be imposed**, namely that the right-hand side of (26) is orthogonal to A_0, which leads to

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} A_0^2\, d\phi = 0\,. \tag{29}$$

3.18: Slowly varying solitary waves

As the solitary wave (27) has just one free parameter (for instance, the amplitude a), this equation suffices to determine its variation. Substituting (27) into (29) leads to the law

$$a^3 = \text{constant}\frac{\alpha}{\lambda} . \tag{30}$$

This agrees with the $m \to 1$ limit of the periodic wave case (22). But the vKdV equation (7) possesses two conservation laws (9, 10)

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} A dx = 0 , \quad \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} A^2 dx = 0 ,$$

for "mass" and "momentum" respectively. The second condition (10) is just (29) and leads to (30). But since this completely defines the slowly-varying solitary wave, we now see that this **cannot simultaneously conserve total mass**. This is apparent when one examines the solution of (26) for A_1, from which it is readily shown that although $A_1 \to 0$ as $\phi \to \infty$, $A_1 \to H_1$ as $\phi \to -\infty$ where

$$VH_1 = -\frac{\partial M_0}{\partial T} , \quad M_0 = \int_{-\infty}^{\infty} A_0 \, dx , \quad \text{or} \quad H_1 = \frac{6}{\alpha K}\frac{a_T}{a} . \tag{31}$$

3.19: Slowly varying solitary waves

This non-uniformity in the slowly-varying solitary wave has been recognized for some time. The remedy is the construction of a **trailing shelf** A_s of small amplitude $O(\sigma)$ but long length-scale $O(1/\sigma)$, which thus has $O(1)$ mass, but $O(\sigma)$ momentum. It resides **behind the solitary wave**, nd to leading order has a value independent of T, so that

$$A_s = \epsilon A_s(X) \,, \quad X = \sigma x \,, \quad \text{for} \quad X < \Phi(T) = \int^T V(T)\, dT \,. \qquad (32)$$

It is determined by its value at the location $X = \Phi(T)$ of the solitary wave, namely $A_s(\Phi(T)) = H_1(T)$ (31). Its polarity depends on the sign of λa_T, that is, it has the same polarity when the wave amplitude is growing, but the opposite polarity when the wave is decaying. It may readily be verified that the slowly-varying solitary wave and the trailing shelf together satisfy conservation of mass. At higher orders in σ the shelf itself will evolve and may generate secondary solitary waves.

3.20: Critical point $\alpha = 0$

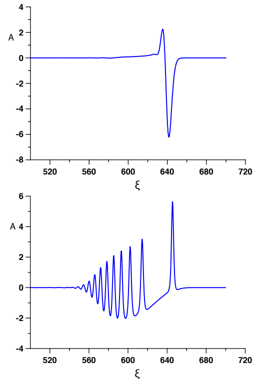
The adiabatic expression (30)

$$a^3 = \text{constant}\frac{\alpha}{\lambda} .$$

shows that the **critical point** where $\alpha = 0$ is a site where we may expect a dramatic change in the wave structure. As α passes through zero, assume that there is a critical point $\tau = 0$ where $\alpha = 0$. Then as $\alpha \to 0$, it follows from (30) that the wave amplitude also decreases to zero as $|\alpha|^{1/3}$, while the the mass M_0 of the solitary wave grows as $|\alpha|^{-1/3}$. Thus the amplitude A_1 of the trailing shelf grows as $|\alpha|^{-8/3}$, with the opposite polarity to the wave. Essentially the trailing shelf passes through the critical point as a disturbance of the opposite polarity to that of the original solitary wave, which then being in an environment with the opposite sign of α, can generate **a train of solitary waves of the opposite polarity, riding on a pedestal of the same polarity as the original wave**.

3.21: Critical point $\alpha = 0$

$\lambda = 1$, α varies from -1 to 1. The upper panel is when $\alpha = 0$ and the lower panel is when $\alpha = 1$. This is **conversion of a depression wave into a train of elevation waves riding on a negative pedestal**.



Lecture 3: References

Internal solitary waves in a variable medium **Grimshaw, R.** 2007 *Gesellschaft fur Angewandte Mathematik* **30**, 96-109.

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