Nonlinear Waves: Woods Hole GFD Program 2009

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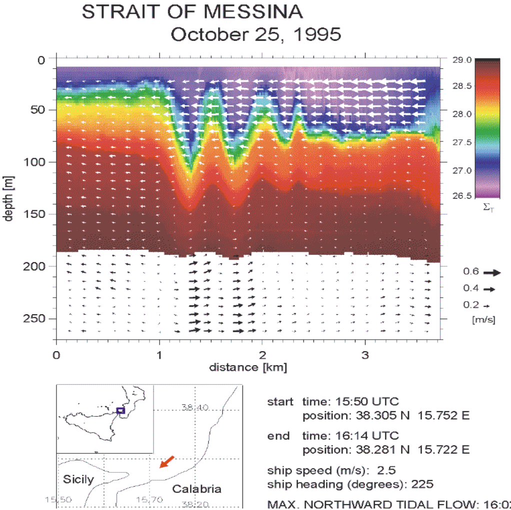
Lecture 2: Derivation of the Korteweg-de Vries Equation

Here we shall give an outline of the derivation of the KdV equation for **surface and internal waves**.

Thus we consider an inviscid, incompressible fluid which is bounded above by a free surface and below by a flat rigid boundary. We shall suppose that the flow is two-dimensional and can be described by the spatial coordinates (x, z) where x is horizontal and z is vertical. This configuration is appropriate for the modelling of internal solitary waves in coastal seas, and also in straits, fjords or lakes provided that the effect of lateral boundaries can be ignored. The case of water waves emerges as a special case. With some modifications this model can also be used to describe atmospheric solitary waves.

Morning Glory Waves of the Gulf of Carpentaria

Internal solitary waves in the ocean



start time: 15:50 UTC
position: 38.305 N 15.752 E

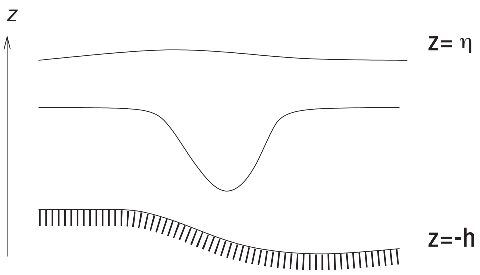
end time: 16:14 UTC
position: 38.281 N 15.722 E

ship speed (m/s): 2.5
ship heading (degrees): 225

MAX. NORTHWARD TIDAL FLOW: 16:02 UTC

2.1: Derivation for surface and internal waves

In the basic state the fluid has **density** $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that $p_{0z} = -g\rho_0$ describes the basic hydrostatic equilibrium, and a horizontal shear flow $u_0(z)$ in the $x-$direction. We shall consider only the case when the bottom is flat, that is h is a constant. Extensions to variable depth are possible and would lead to a variable-coefficient KdV equation.



2.2: Euler equations

In standard notation, the equations of motion relative to this basic state are

$$\rho_0(u_t + u_0 u_x + w u_{0z}) + p_x = -\rho_0(u u_x + w u_z)\,,$$
$$-\rho(u_t + u_0 u_x + w u_{0z} + u u_x + w u_z)\,, \qquad (1)$$
$$p_z + g\rho = -(\rho_0 + \rho)(w_t + u_0 w_x + u w_x + w w_z)\,, \qquad (2)$$
$$g(\rho_t + u_0 \rho_x) - \rho_0 N^2 w = -g(u\rho_x + w\rho_z)\,, \qquad (3)$$
$$u_x + w_z = 0\,. \qquad (4)$$

Here $(u_0 + u, w)$ are the velocity components in the (x, z) directions, $\rho_0 + \rho$ is the density, $p_0 + p$ is the pressure and t is time. $N(z)$ is the **buoyancy frequency**, defined by

$$\rho_0 N^2 = -g\rho_{0z}\,. \qquad (5)$$

The boundary conditions are

$$w = 0 \quad \text{at} \quad z = -h\,, \qquad (6)$$
$$p_0 + p = 0\,, \quad \text{at} \quad z = \eta\,, \qquad (7)$$
$$w = \eta_t + u_0\eta_x + u\eta_x\,, \quad \text{at} \quad z = \eta\,. \qquad (8)$$

2.3: Linear long waves

Then to describe internal solitary waves we seek solutions whose horizontal length scales are much greater than h, and whose time scales are much greater than N^{-1}. We shall also assume that the waves have small amplitude. Then the dominant balance is obtained by equating to zero the terms on the left-hand side of (1 - 4), together with the linearization of the free surface boundary conditions. We then obtain the set of equations describing linear long wave theory. To proceed it is useful to use the **vertical particle displacement** ζ as the primary dependent variable. It is defined by

$$\zeta_t + u_0\zeta_x + u\zeta_x + w\zeta_z = w\,. \tag{9}$$

Note that it then follows that the perturbation density field is given by $\rho = \rho_0(z-\zeta) - \rho_0(z) \approx \rho_0 N^2\zeta$ as $\zeta \to 0$, where we have assumed that as $x \to -\infty$, the density field relaxes to its basic state. The isopycnal surfaces (i.e. $\rho_0 + \rho =$ constant) are then given by $z = z_0 + \zeta$ where z_0 is the level as $x \to -\infty$. In terms of ζ, the kinematic boundary condition (8) becomes simply $\zeta = \eta$ at $z = \eta$.

2.4: Linear long waves

Linear long wave theory is now obtained by omitting the right-hand side of equations (1 - 4), and simultaneously linearizing boundary conditions (7,8). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z)\,, \tag{10}$$

while the remaining dependent variables are then given by

$$u = (c - u_0)A\phi_z\,, \quad p = \rho_0(c - u_0)^2 A\phi_z\,, \quad \rho = \rho_0 N^2 A\phi\,. \tag{11}$$

Here c is the linear long wave speed, and the modal functions $\phi(z)$ are defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0\,, \quad \text{for} \quad h < z < 0\,, \tag{12}$$

$$\phi = 0 \quad \text{at} \quad z = -h\,, \quad (c - u_0)^2\phi_z = g\phi\,, \quad \text{at} \quad z = 0\,. \tag{13}$$

Equation (12) is the well-known **Taylor-Goldstein** equation, here in the long-wave limit.

2.5: Linear long-wave modes

Typically, the boundary-value problem (12, 13) defines an infinite sequence of **linear long-wave modes**, $\phi_n^{\pm}(z)$, $n = 0, 1, 2, \ldots$, with corresponding speeds $c_n^{\pm}$. Here, the superscript "$\pm$" indicates waves with $c_n^+ > u_M = \max u_0(z)$ and $c_n^- < u_M = \min u_0(z)$ respectively. We shall confine our attention to these regular modes, and consider only stable shear flows. Nevertheless, we note that there may also exist singular modes with $u_m < c < u_M$ for which an analogous theory can be developed. Note that it is useful to let $n = 0$ denote the **surface gravity waves** for which c scales with $\sqrt{gh}$, and then $n = 1, 2, 3, \ldots$ denotes the **internal gravity waves** for which c scales with Nh. In general, the boundary-value problem (12, 13) is readily solved numerically. Typically, the surface mode ϕ_0 has no extrema in the interior of the fluid and takes its maximum value at the surface $z = 0$, while the internal modes $\phi_n^{\pm}(z)$, $n = 1, 2, 3, \ldots$, have $n - 1$ extremal points in the interior of the fluid, and vanish near $z = 0$ (and, of course, also at $z = -h$).

2.6: Linear long-wave modes

It can now be shown that, within the context of linear long wave theory, any localised initial disturbance will evolve into a set of outwardly propagating modes, each given by an expression of the form (10). Indeed, it can be shown that the solution of the linearised long wave equations is given asymptotically by

$$\zeta \sim \sum_{n=0}^{\infty} A_n^{\pm}(x - c_n^{\pm} t)\phi_n^{\pm}(z)\,, \quad \text{as} \quad t \to \infty\,. \tag{14}$$

Here the amplitudes $A_n^{\pm}(x)$ are determined from the initial conditions, Assuming thats the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, it is sufficient for **large times to consider just a single mode**. Henceforth, we shall omit the indices and assume that the mode has speed c, amplitude A and modal function $\phi(z)$. Then, as time increases, the hitherto neglected **nonlinear terms** begin to have an effect, and cause wave steepening. However, this is opposed by the terms representing **linear wave dispersion**, also neglected in the linear long wave theory. A balance between these two effects emerges as time increases and the outcome is the KdV equation for the wave amplitude.

2.7: Asymptotic expansion

The formal derivation of the evolution equation requires the introduction of **two small parameters**, α and ϵ, respectively characterising the wave amplitude and dispersion. The **KdV balance** requires $\alpha = \epsilon^2$, with a corresponding **timescale** of ϵ^{-3}. The asymptotic analysis required is well understood, so we shall give only a brief outline here. We introduce the scaled variables

$$T = \epsilon \alpha t\,, \quad X = \epsilon(x - ct)\,, \tag{15}$$

and then put

$$\zeta = \alpha A(X, T)\phi(z) + \alpha^2 \zeta_2 + \dots\,, \tag{16}$$

with similar expressions analgous to (11) for the other dependent variables. At leading order, we get the linear long wave theory for the modal function $\phi(z)$ and the speed c, defined by the modal equations (12, 13). Since the modal equations are homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ζ (to $0(\alpha)$) at the depth z_m.

Then, at the next order, we obtain the equation for ζ_2,

$$\{\rho_0(c-u_0)^2\zeta_{2Xz}\}_z + \rho_0 N^2\zeta_{2X} = M_2\,, \quad \text{for} \quad -h < z < 0\,; \tag{17}$$

$$\zeta_{2X} = 0 \text{ at } z = -h\,, \quad \rho_0(c-u_0)^2\zeta_{2Xz} - \rho_0 g\zeta_{2X} = N_2 \text{ at } z = 0\,. \tag{18}$$

Here the **inhomogeneous terms** M_2, N_2 are known in terms of $A(X,T)$ and $\phi(z)$, and are given by

$$M_2 = 2\{\rho_0(c-u_0)\phi_z\}_z A_T + 3\{\rho_0(c-u_0)^2\phi_z^2\}_z AA_X - \rho_0(c-u_0)^2\phi A_{XXX}\,, \tag{19}$$

$$N_2 = 2\{\rho_0(c-u_0)\phi_z\}A_T + 3\{\rho_0(c-u_0)^2\phi_z^2\}AA_X\,. \tag{20}$$

The left-hand side of equations (17, 18)) are identical to the equations defining the modal function (i.e. (12, 13)), and hence can be solved only if a certain compatibility condition is satisfied. In general the compatibility condition is that the inhomogenous terms in (17, 18) should be orthogonal to the solutions of the adjoint of the modal equations (12, 13). Here, rather than following this general procedure, we obtain the compatibility condition by a direct construction of ζ_2.

2.9: Compatibility condition

Thus a formal solution of (17) which satisfies the first boundary condition in (18) is

$$\zeta_{2X} = A_{2X}\phi + \phi \int_{-h}^{z} \frac{M_2\psi}{W}\,dz - \psi \int_{-h}^{z} \frac{M_2\phi}{W}\,dz\,, \tag{21}$$

$$\text{where} \qquad W = \rho_0(c-u_0)^2\{\phi_z\psi - \psi_z\phi\}\,. \tag{22}$$

Here $\psi(z)$ is a solution of the modal equation (12) which is linearly independent of $\phi(z)$, and so, in particular $\psi(-h) \neq 0$. Here W is the Wronskian and is a constant independent of z. Indeed, the expression (22) can be used to obtain ψ explicitly in terms of ϕ. Next, we insist that the expression (21) for ζ_{2X} should satisfy the second boundary condition in (18). The result is the compatibility condition

$$\int_{-h}^{0} M_2\phi\,dz = [N_2\phi]_{z=0}\,. \tag{23}$$

Note that the amplitude A_2 is left undetermined at this stage.

2.10 Korteweg-deVries equation

Substituting the expressions (19, 20) into (23) we obtain the required evolution equation for A, namely the **KdV equation**

$$A_T + \mu A A_X + \lambda A_{XXX} = 0 \,. \tag{24}$$

where the coefficients μ and λ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_z^3 \, dz \,, \tag{25}$$

$$I\lambda = \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi^2 \, dz \,, \tag{26}$$

$$\text{where} \qquad I = 2\int_{-h}^{0} \rho_0 (c - u_0) \phi_z^2 \, dz \,. \tag{27}$$

The KdV equation (24) is to be solved with the initial condition $A(X, T = 0) = A_0(X)$ where $A_0(X)$ is determined from the linear long wave theory, and is in essence the projection of the original initial conditions onto the relevant linear long wave mode. As mentioned before, localized initial conditions lead to the generation of a finite number of solitary waves, or **internal solitons**.

$$I\mu = 3\int_{-h}^{0} \rho_0 (c-u_0)^2 \phi_z^3 \, dz\,, \tag{28}$$

$$I\lambda = \int_{-h}^{0} \rho_0 (c-u_0)^2 \phi^2 \, dz\,, \tag{29}$$

$$\text{where} \qquad I = 2\int_{-h}^{0} \rho_0 (c-u_0) \phi_z^2 \, dz\,. \tag{30}$$

Confining attention to waves propagating to the right, so that $c > u_M = \max u_0(z)$, we see that I and λ are always positive. For the surface mode, $\phi_z > 0$ and $\phi(0) = 1$ so we see that $\mu > 0$. Further, recalling that for the internal modes the modal functions are normalised so that $\phi(z_m) = 1$ where z_m is an extremal point, then it is readily shown that for the usual situation of a near-surface pycnocline, μ is negative for the first internal mode. However, in general μ can take either sign, and in some special situations may even be zero. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically.

2.12 Water waves

To illustrate the procedure, consider first the case of **water waves**. We put the density $\rho =$ constant so that then $N^2 = 0$ (5). Then

$$\phi = \frac{z+h}{h} \quad \text{for} \quad -h < z < 0\,, \quad c = (gh)^{1/2}\,. \tag{31}$$

$$\text{and so} \quad \mu = \frac{3c}{2h}\,, \quad \lambda = \frac{ch^2}{6}\,. \tag{32}$$

Thus the KdV equation for water waves is, in the original variables,

$$\zeta_t + c\zeta_x + \frac{3c}{2h}\zeta\zeta_x + \frac{ch^2}{6}\zeta_{xxx} = 0\,. \tag{33}$$

Note that here $z_m = 0$ so that $A = \zeta$, the free surface displacement, to leading order. For zero surface tension, this is the equation derived by Korteweg and de Vries in 1895 (and first by Boussinesq in 1870's).

2.13 Interfacial waves

Similarly, for **interfacial waves**, let the density be a constant ρ_1 in an upper layer of height h_1 and $\rho_2 > \rho_1$ in the lower layer of height $h_2 = h - h_1$. That is

$$\rho_0(z) = \rho_1 H(z + h_1) + \rho_2 H(-z - h_1)\,, \quad \rho_0 N^2 = g(\rho_2 - \rho_1)\delta(z + h_1)\,.$$

Here $H(z)$ is the Heaviside function and $\delta(z)$ is the Dirac δ-function. For simplicity, we shall assume that $\rho_1 \approx \rho_2$, the usual situation in the ocean, and also then the upper boundary condition for $\phi(z)$ becomes just $\phi(0) \approx 0$. Then we find that

$$\phi = \frac{z + h}{h_2} \text{ for } -h < z < h_1\,, \quad \phi = -\frac{z}{h_1} \text{ for } -h_1 < z < 0\,, \tag{34}$$

$$c^2 = \frac{g(\rho_2 - \rho_1)}{\rho_2}\frac{h_1 h_2}{h_2 + h_2\,,} \quad \mu = \frac{3c(h_1 - h_2)}{h_1 h_2}\,, \quad \lambda = \frac{c h_1 h_2(h_2^2 + h_1^2)}{h_1 + h_2}\,. \tag{35}$$

Note that the nonlinear coefficient μ for these interfacial waves is negative (positive) when $h_1 < (>) h_2$ (that is, the when interface is closer (further) to the free surface than to the bottom), The case when $h_1 \approx h_2$ leads to the necessity to use an extended KdV equation which contains a cubic nonlinear term.

2.14 Higher-order KdV equations

Proceeding to the next highest order will yield an equation set analogous to (17, 18) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. We shall not give details here, but note that using the transformation $A + \alpha A_2 \to A$, and then combining the KdV equation (24) with the evolution equation for A_2 will lead to a **higher-order KdV** equation

$$A_T + \mu A A_X + \lambda A_{XXX}$$
$$+\alpha\{\lambda_1 A_{XXXXX} + \sigma A^2 A_X + \mu_1 A A_{XXX} + \mu_2 A_X A_{XX}\} = 0\,. \tag{36}$$

Explicit expressions for the coefficients are known. Note that to be **Hamiltonian**, $\mu_2 = 2\mu_1$. However, this equation is not unique, as the near-identity transformation $A \to A + \alpha(aA^2 + bA_{XX})$ asymptotically reproduces the same equation but with altered coefficients,

$$(\lambda_1, \sigma, \mu_1, \mu_2) \to (\lambda_1, \sigma - a\mu, \mu_1, \mu_2 - 6a\lambda + 2b\mu)\,.$$

Further when $\mu \neq 0, \lambda \neq 0$, the enhanced transformation,

$$A \to A + \alpha(aA^2 + bA_{XX} + a' A_X \int^X A\, dX + b' X A_T)\,,$$

can asymptotically reduce (36) to the KdV equation.

2.15 Extended KdV equation

A particularly impotant special case arises when the coefficient μ (24) is close to zero. Then the cubic nonlinear term in the higher-order KdV equation is the most important, and the KdV equation (24) is replaced by the **extended KdV** (or Gardner) equation,

$$A_T + \mu A A_X + \alpha\sigma A^2 A_X + \lambda A_{XXX} = 0 \,. \tag{37}$$

For $\mu \approx 0$, a rescaling is needed and the optimal choice is to assume that μ is $0(\epsilon)$, and then replace A with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2. This is an integrable equation, in canonical form

$$A_t + 6AA_x + 6\beta A^2 A_x + A_{xxx} = 0 \,. \tag{38}$$

Like the KdV equation, the Gardner equation is integrable by the inverse scattering transform. Here the coefficient β can be either positive or negative, and the structure of the solutions depends crucially on which sign is appropriate.

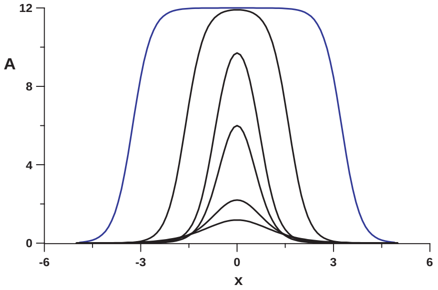
The solitary wave solutions are given by

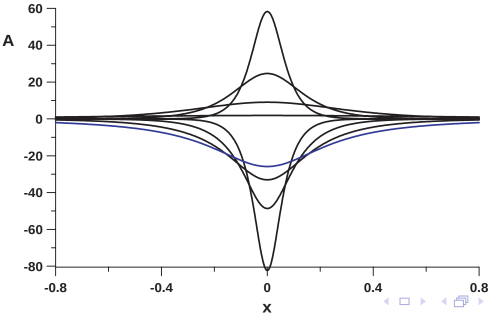
$$A = \frac{a}{b + (1-b)\cosh^2 \gamma(x - Vt)}\,, \tag{39}$$

$$\text{where} \quad V = a(2+\beta a) = 4\gamma^2\,, \quad b = \frac{-\beta a}{(2+\beta a)}\,. \tag{40}$$

There are two cases to consider. If $\beta < 0$, then there is a single family of solutions such that $0 < b < 1$ and $a > 0$. As b increases from 0 to 1, the amplitude a increases from 0 to a maximum of $-1/\beta$ while the speed V also increases from 0 to a maximum of $-1/\beta$. In the limiting case when $b \to 1$ the solution (39) describes the so-called "thick" solitary wave, which has a flat crest of amplitude $a_m = -1/\beta$. The figure shows solitary wave solutions (39) of the extended KdV equation; upper panel for $\beta < 0$; lower panel for $\beta > 0$.

2.17 Solitary waves of the eKdV equation





2.18 Other long-wave models

When the density stratification is confined to a near-surface layer, lying above a **deep ocean with constant density**, a different scaling is needed. In the upper layer, the long-wave scaling used above still holds, but this needs to be matched to a different scaling in the deep lower layer, where optimally it can is assumed that the vertical scale matches the horizontal scale. In particular $h = H/\epsilon$. In this scenario, the KdV equation (24) is replaced by the **intermediate long-wave (ILW) equation**

$$A_\tau + \mu A A_\theta + \delta \mathcal{L}(A_\theta) = 0\,, \tag{41}$$

$$\text{where} \quad \mathcal{L}(A) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k \coth kH \exp(ik\theta)\mathcal{F}(A)dk\,, \tag{42}$$

$$\text{and} \quad \mathcal{F}(A) = \int_{-\infty}^{\infty} A \exp(-ik\theta))d\theta\,. \tag{43}$$

Here the nonlinear coefficient μ is again given by (25) with $-h$ now replaced by $-\infty$, while the dispersive coefficient δ is defined by $I\delta = (\rho_0 c^2 \phi^2)_{z\to-\infty}$. In the limit $H \to \infty$, $k \coth kH \to |k|$ and (41) becomes the **Benjamin-Ono (BO) equation**. In the opposite limit $H \to 0$, (41) reduces to a KdV equation. Both are also integrable equations.

Lecture 2: References

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