Waves in shallow water, I

Lecture 4

600 m pier at Duck, NC - Hurricane Grace, 1991

Waves in shallow water, I

This lecture: Korteweg-de Vries equation

A. Derive the Korteweg-de Vries (KdV) equation as a model of waves of moderate amplitude in shallow water

B. Properties of the KdV equation

C. Accuracy of KdV as a model of water waves

Waves in shallow water
(coming attractions)

Lecture 7:

A. Tsunami of 2004, Hurricane Katrina, 2006

B. Kadomtsev-Petviashvili (KP) equation:

a 2-D generalization of KdV

theory and experiment

Lecture 8:

The shallow water equations

(a different model for waves in shallow water)

A. Derive the KdV equation

General method to derive approximate models of water waves (KdV, KP, NLS, 3-wave,…)

Start with “exact” water wave equations

1) Identify a specific limit of interest

(ex: small amplitude waves in shallow water)

2) Scale equations to show that limit explicitly

3) Solve eq’ns approximately, order-by-order

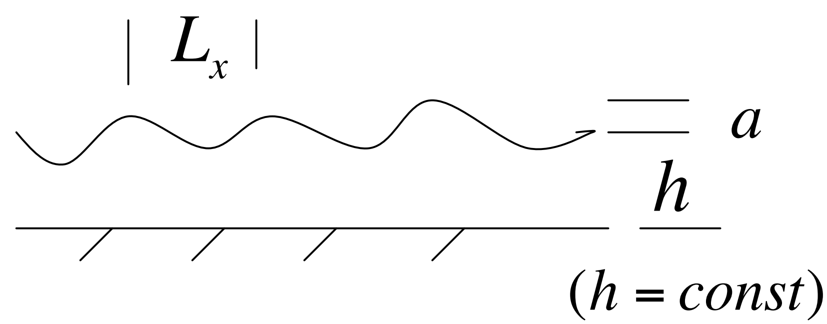
4) Introduce “multiple scales” as needed

1) Limit of interest for KdV (or KP)

Assume:

- Small amplitude

$$a << h$$



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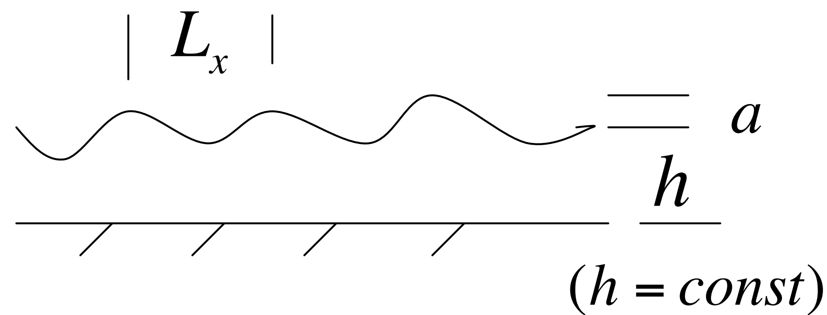

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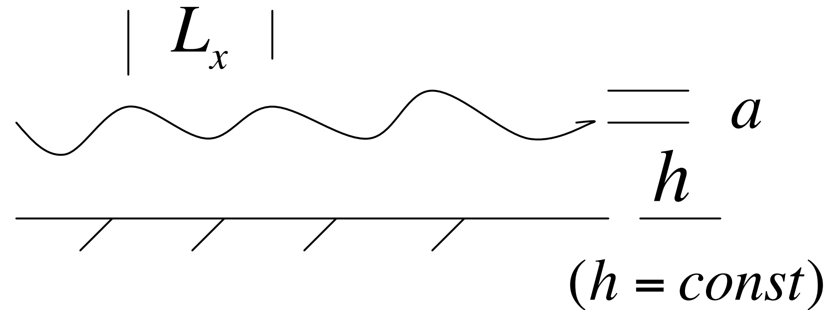

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If exactly true ➔ KdV

If approximately true ➔ KP

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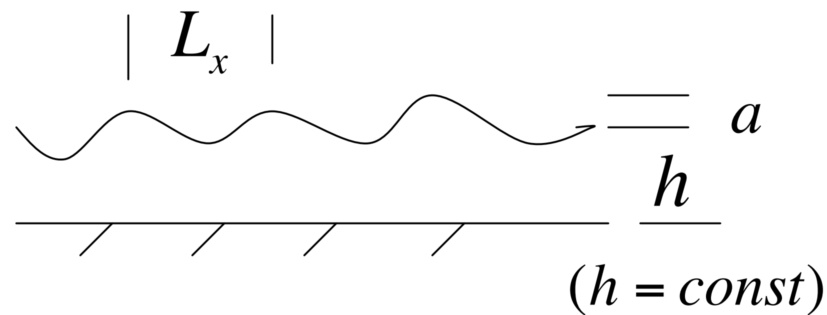

- Small amplitude

  $a << h$

- Shallow water (long waves) $h << L_x$
- Motion primarily in one direction

  If exactly true ➔ KdV

  If approximately true ➔ KP

- All small effects balance

KdV: $\frac{a}{h} = \varepsilon << 1, \quad (\frac{h}{L_x})^2 = O(\varepsilon), \qquad \partial_y \equiv 0.$

2) Scale variables to impose ε–limit

Characteristic length: h

➔ $z^* = \dfrac{z}{h}, \qquad x^* = \dfrac{x}{L_x} = \sqrt{\varepsilon}\,\dfrac{x}{h}, \qquad \eta = \varepsilon h[\eta^*(x^*, t^*, \varepsilon)]$

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Characteristic speed: $\sqrt{gh}$

➔ $u = \varepsilon\sqrt{gh}[u^*]$ ➔ $\phi = h\sqrt{\varepsilon g h}[\phi^*(x^*, z^*, t^*, \varepsilon]$

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Characteristic length: *h*

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Characteristic time: $T = \frac{L_x}{\sqrt{gh}} = \frac{h}{\sqrt{\varepsilon g h}}$

➔ $t^* = \frac{\sqrt{\varepsilon g h}}{h}t$

We will also need a "slow time": $\tau = \varepsilon t^* = \varepsilon\frac{\sqrt{\varepsilon g h}}{h}t$

3) Solve equations, order by order

a) Laplace' equation, and b.c. at z = -h:

Write ϕ as a convergent Taylor series in ($z+h$).

After some algebra,

$$\phi = h\sqrt{\varepsilon g h}\,[\phi_0(x^*, t^*, \varepsilon) + \sum_{m=1}^{\infty} \frac{(-\varepsilon)^m}{(2m)!} \cdot (1+z^*)^{2m} \cdot \frac{\partial^{2m}\phi_0}{\partial (x^*)^{2m}}]$$

Where series converges, this is exact.

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Where series converges, this is exact.

$$u \sim \varepsilon\sqrt{gh}\cdot[\hat{u}^*(x^*,t^*,\varepsilon) - \frac{\varepsilon}{2}(1+z^*)^2\frac{\partial^2\hat{u}^*}{\partial(x^*)^2} + O(\varepsilon^2)],$$

➔

$$w \sim -\varepsilon\sqrt{\varepsilon g h}\cdot[(1+z^*)\frac{\partial\hat{u}^*}{\partial x^*} - \frac{\varepsilon}{6}(1+z^*)^3\frac{\partial^3\hat{u}^*}{\partial(x^*)^3} + O(\varepsilon^2)].$$

3) Solve equations, order by order

b) At free surface:

- Two evolution equations, to be solved for

$$\eta^{*}(x^{*}, t^{*}, \varepsilon), \quad \hat{u}^{*}(x^{*}, t^{*}, \varepsilon).$$

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b) At free surface:

- Two evolution equations, to be solved for

$$\eta^*(x^*, t^*, \varepsilon), \quad \hat{u}^*(x^*, t^*, \varepsilon).$$

- Solve these approximately, by expanding each in formal asymptotic series, and solving order by order:

$$\eta^* \sim \eta_1(x^*, t^*, \varepsilon) + \varepsilon\eta_2(x^*, t^*, \varepsilon) + O(\varepsilon^2),$$

$$\hat{u}^* \sim u_1(x^*, t^*, \varepsilon) + \varepsilon u_2(x^*, t^*, \varepsilon) + O(\varepsilon^2).$$

3) Solve equations, order by order

c) At O(ε),

$$\frac{\partial \eta_1}{\partial t^*} + \frac{\partial u_1}{\partial x^*} = 0, \qquad \frac{\partial u_1}{\partial t^*} + \frac{\partial \eta_1}{\partial x^*} = 0.$$

Solution:

$$\eta_1(x^*, t^*, \varepsilon) = f(x^* - t^*, \varepsilon) + F(x^* + t^*, \varepsilon),$$
$$u_1(x^*, t^*, \varepsilon) = f(x^* - t^*, \varepsilon) - F(x^* + t^*, \varepsilon).$$

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$$u_1(x^*, t^*, \varepsilon) = f(x^* - t^*, \varepsilon) - F(x^* + t^*, \varepsilon).$$

d) At O(ε^2): $$\frac{\partial \eta_2}{\partial t^*} + \frac{\partial u_2}{\partial x^*} = RHS_a, \qquad \frac{\partial u_2}{\partial t^*} + \frac{\partial \eta_2}{\partial x^*} = RHS_b.$$

Find (η_2, u_2) grow linearly in t^*. (Bad!)

4) Introduce slow time-scale

- Find $\eta(x,t,\varepsilon) \sim \varepsilon h[\eta_1 + \varepsilon\eta_2 + O(\varepsilon^2)]$

$$\sim \varepsilon h[(bdd) + (\underset{\Uparrow}{\varepsilon t^*})(bdd) + O(\varepsilon^2)]$$

so formal expansion disordered when εt^*= O(1).

4) Introduce slow time-scale

- Find $\eta(x,t,\varepsilon) \sim \varepsilon h[\eta_1 + \varepsilon\eta_2 + O(\varepsilon^2)]$

$$\sim \varepsilon h[(bdd) + (\varepsilon t^*)(bdd) + O(\varepsilon^2)]$$

so formal expansion disordered when εt^*= O(1).

- Introduce a slow time-scale, $\tau = \varepsilon t^*$, to eliminate this problem. Then

$$\eta_j(x^*, t^*, \varepsilon) \to \hat{\eta}_j(x^*, t^*, \tau), \qquad \frac{\partial}{\partial t^*} \to \frac{\partial}{\partial t^*} + \varepsilon\frac{\partial}{\partial \tau}.$$

- <u>Choose</u> $(\frac{\partial \eta_1}{\partial \tau}, \frac{\partial u_1}{\partial \tau})$ to eliminate unphysical growth at εt^*= O(1).

5) Result

At O(ε), $$\eta_1(x^*, t^*, \tau) = f(x^* - t^*, \tau) + F(x^* + t^*, \tau),$$
$$u_1(x^*, t^*, \tau) = f(x^* - t^*, \tau) - F(x^* + t^*, \tau),$$

essentially as before. Define $r = x^* - t^*, \quad s = x^* + t^*$.

5) Result - 2 KdV equations

At O(ε), $$\eta_1(x^*,t^*,\tau) = f(x^* - t^*,\tau) + F(x^* + t^*,\tau),$$
$$u_1(x^*,t^*,\tau) = f(x^* - t^*,\tau) - F(x^* + t^*,\tau),$$

essentially as before. Define $r = x^* - t^*, \quad s = x^* + t^*.$
Then at O(ε^2),

KdV:
$$2\frac{\partial f}{\partial \tau} + 3f\frac{\partial f}{\partial r} + \left(\frac{1}{3} - \frac{\sigma}{\rho g h^2}\right)\frac{\partial^3 f}{\partial r^3} = 0,$$
$$-2\frac{\partial F}{\partial \tau} + 3F\frac{\partial F}{\partial s} + \left(\frac{1}{3} - \frac{\sigma}{\rho g h^2}\right)\frac{\partial^3 F}{\partial s^3} = 0.$$

Also find η_2, u_2.

B. Properties of the KdV equation

What did Korteweg & de Vries know?

Rescale equation:

$$\partial_\tau v + 6v\partial_\xi v + \partial_\xi^3 v = 0$$

Solitary wave:

$$v(\xi,\tau) = 2p^2 \operatorname{sec} h^2\{p(\xi - 4p^2\tau + \xi_0)\}$$

Properties of the KdV equation

Solitary waves in the ocean

 

Both photos taken in Hawaii by Robert Odom

App. Phys. Lab., U of Washington

see www.amath.washington.edu/~bernard/kp/waterwaves.html

Properties of the KdV equation

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$$\partial_\tau v + 6v\partial_\xi v + \partial_\xi^3 v = 0$$

Solitary wave:

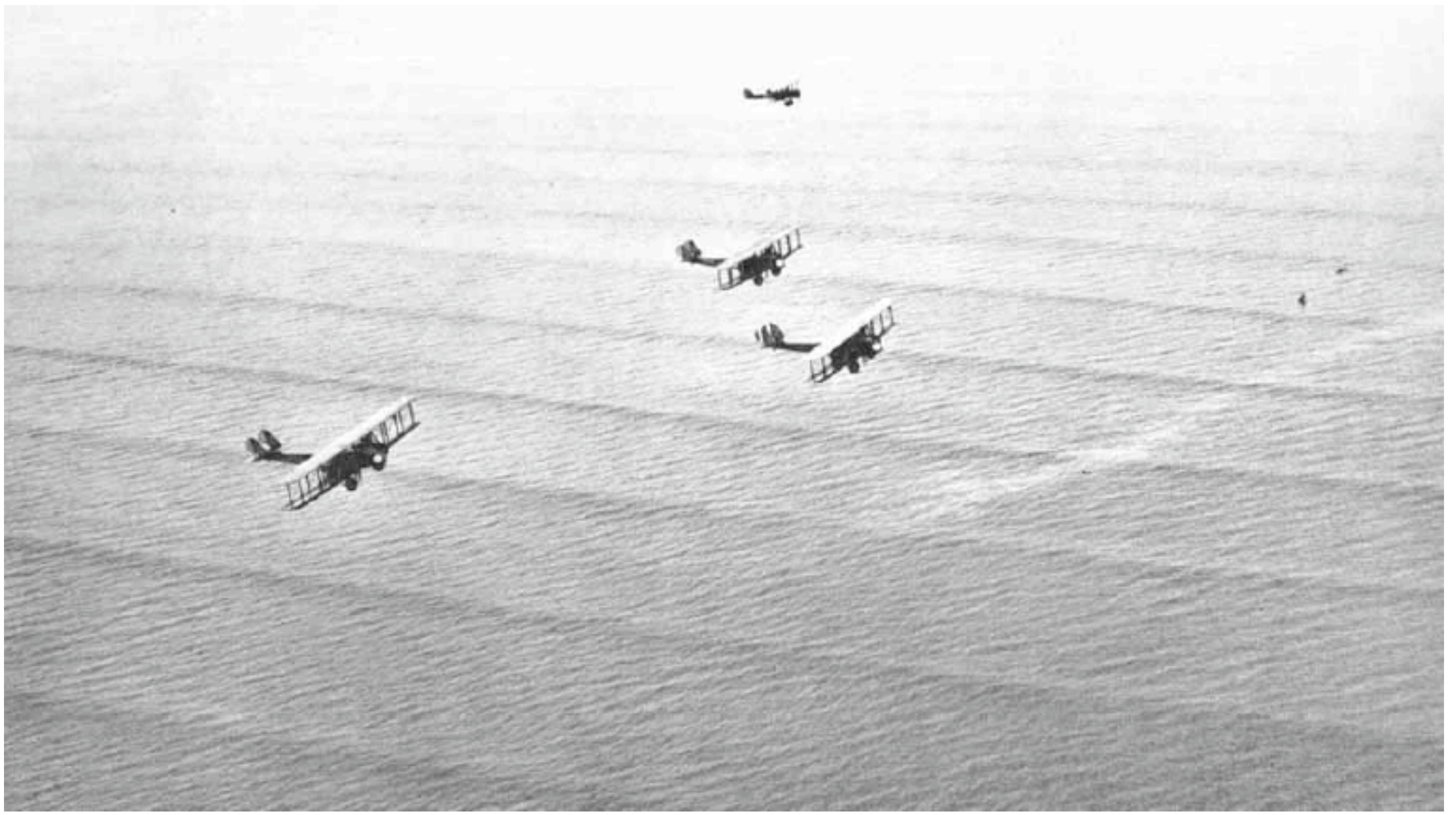
$$v(\xi,\tau) = 2p^2 \sec h^2\{p(\xi - 4p^2\tau + \xi_0)\}$$

Periodic “cnoidal” wave:

$$v = 2p^2\kappa^2 cn^2\{p(\xi - c\tau + \xi_0;\kappa)\} + v_0$$

Properties of the KdV equation

A cnoidal wavetrain near Panama



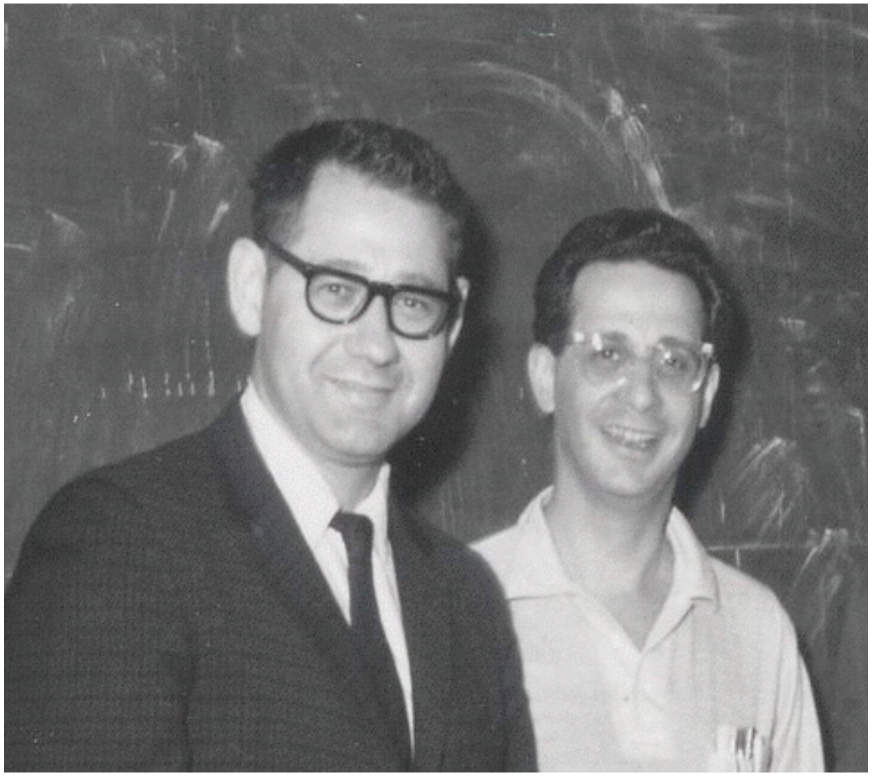
National Geographic, 1933

Properties of the KdV equation

Cnoidal waves near Lima, Peru

photo by Anna Segur, 2004

Properties of the KdV equation



Miracles!

discovered mostly by
Zabusky & Kruskal (1965),
Gardner, Greene, Miura…

Consider

$$\partial_\tau v + 6v\partial_\xi v + \partial_\xi^3 v = 0, \qquad -\infty < \xi < \infty,$$

with $v \to 0$ rapidly as $|\xi| \mapsto \infty,$ and infinitely differentiable

Miracles of the KdV equation

1) A <u>conservation law</u> is a relation of the form

$$\partial_\tau\{density\} + \partial_\xi\{flux\} = 0.$$

KdV has ***infinitely many*** conservation laws:

$$\partial_\tau\{v\} + \partial_\xi\{3v^2 + \partial_\xi^2 v\} = 0,$$
$$\partial_\tau\{v^2\} + \partial_\xi\{4v^3 + 2v\partial_\xi^2 v - \tfrac{1}{2}(\partial_\xi v)^2\} = 0,$$
$$\partial_\tau\{v^3 - \tfrac{1}{2}(\partial_\xi v)^2\} + \partial_\xi\{....\} = 0,$$
$$\partial_\tau\{v^4 - 2v(\partial_\xi v)^2 + \tfrac{1}{5}(\partial_\xi^2 v)^2\} + \partial_\xi\{....\} = 0,....$$

➔ Any solution of KdV is *very* constrained.

Miracles of the KdV equation

2) The inverse scattering transform

Let $v(\xi,\tau)$ be smooth, real and $v \to 0$ rapidly as $|\xi|^2 \to \infty$

Consider the Schrödinger equation,

$$\partial_\xi^2 \psi + [\lambda + v(\xi,\tau)]\psi = 0,$$

Miracles of the KdV equation

2) The inverse scattering transform

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Consider the Schrödinger equation,

$$\partial_\xi^2 \psi + [\lambda + v(\xi,\tau)]\psi = 0,$$

and

$$\partial_\tau \psi = [\partial_\xi v + \alpha]\psi + [4\lambda - 2v]\partial_\xi \psi.$$

Miracles of the KdV equation

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$$\partial_\tau \psi = [\partial_\xi v + \alpha]\psi + [4\lambda - 2v]\partial_\xi \psi.$$

$$\partial_\xi^2(\partial_\tau \psi) = \partial_\tau(\partial_\xi^2 \psi) \qquad \Leftrightarrow \qquad \partial_\tau v + 6v\partial_\xi v + \partial_\xi^3 v = 0.$$

Miracles of the KdV equation

2) Inverse scattering transform as a nonlinear Fourier transform:

Start with

$$v(\xi, 0) \xrightarrow{\text{(KdV)}} v(\xi, \tau)$$

Miracles of the KdV equation

2) Inverse scattering transform as a nonlinear Fourier transform:

Start with

$$\begin{tikzcd} v(\xi, 0) \arrow[rr, "\text{direct scattering}"] \arrow[d, "\text{(KdV)}"'] & & \text{scattering data}|_{\tau=0} \arrow[d, "\text{(linear ODEs)}"] \\ v(\xi, \tau) & & \text{scattering data}|_{\tau> 0} \arrow[ll, "\text{inverse scattering}"'] \end{tikzcd}$$

Miracles of the KdV equation

2) Inverse scattering transform - to solve KdV as an initial value problem

- At $\tau = 0$, solve $\partial_{\xi}^{2}\psi + [\lambda + v(\xi,0)]\psi = 0.$

Miracles of the KdV equation

2) Inverse scattering transform - to solve KdV as an initial-value problem

- At $\tau = 0$, solve $\partial_\xi^2 \psi + [\lambda + v(\xi, 0)]\psi = 0.$
- Every discrete eigenvalue represents one solitary wave (or "soliton").
- Continuous spectrum leads to an oscillatory wave train, which disperses (as in a linear problem).
- Arbitrary initial data evolves into N solitons, plus dispersing oscillatory waves.
- Everything is predicted explicitly.

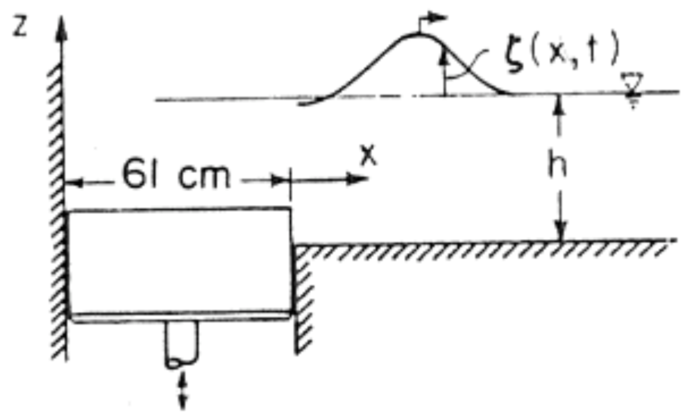
C. Accuracy of the KdV model

1) How accurately does a KdV solution predict the behaviour of actual water waves, in the appropriate limit?

2) How accurately does a KdV solution approximate the corresponding solution of the water wave equations, in the appropriate limit?

C(1). Laboratory tests of KdV

Experimental equipment (J.L. Hammack)

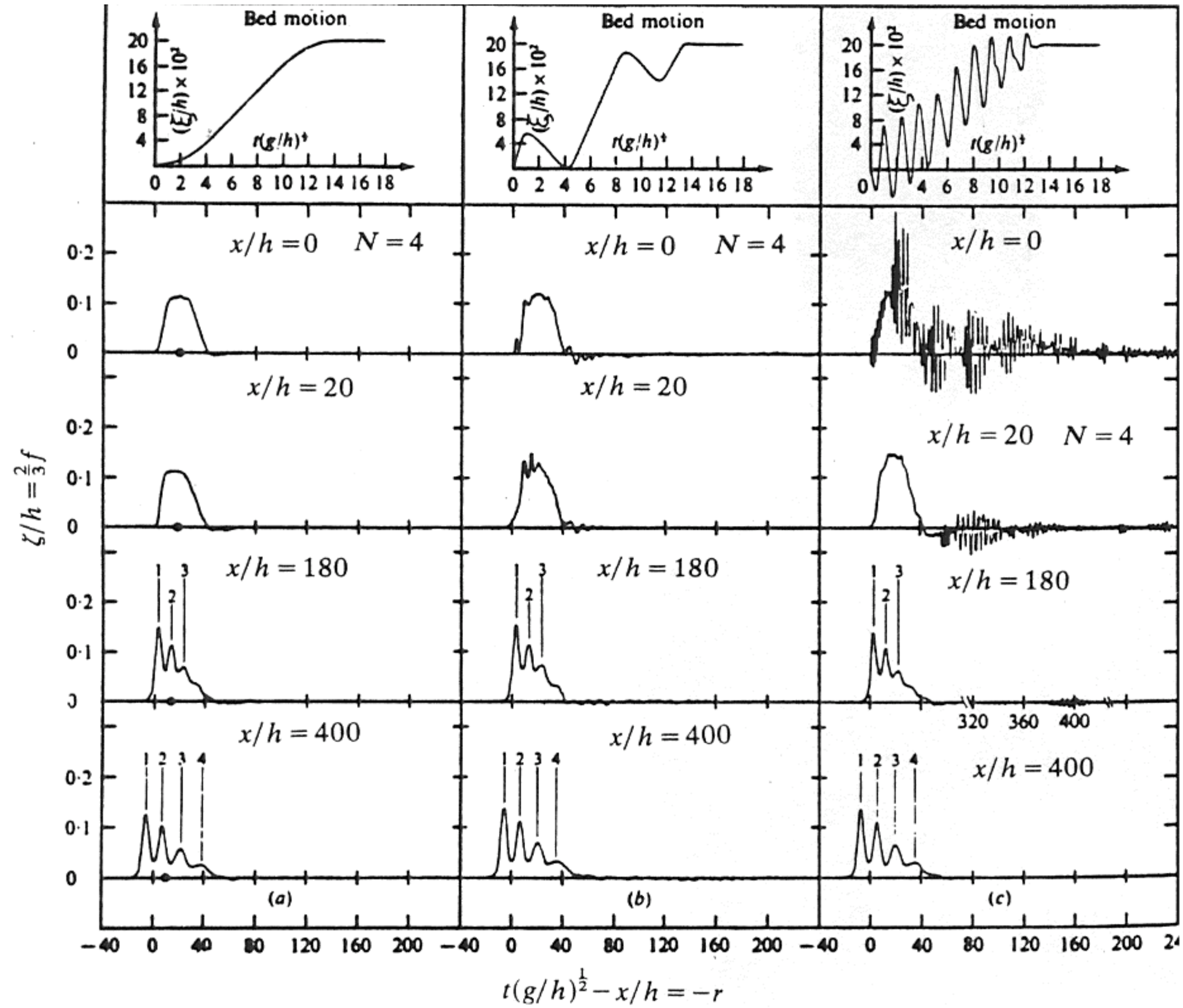


References: Hammack, 1973,

Hammack & Segur, 1974, 1978

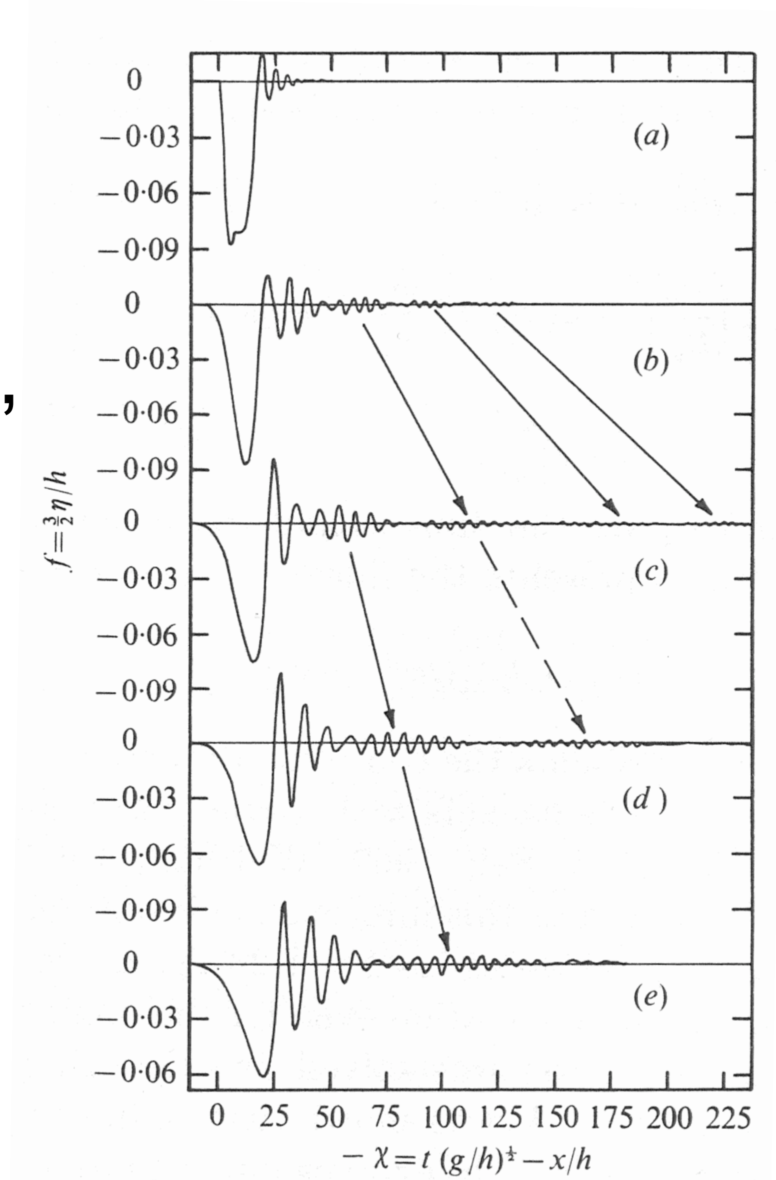
Laboratory tests of KdV

Positive
Initial data
(solitons!)



Laboratory tests of KdV

Negative
initial data:
(dispersive waves,
no solitons)



C(2). Mathematical accuracy

1) Craig (1985)
2) G. Schneider & C. E. Wayne (2000, 2002)
3) Bona, Colin & Lannes (2005),
   Bona, Chen & Saut (2002, 2004)
4) J.D. Wright (2006)
5) Shen & Sun (1991), Beale (1991)
   Vanden-Broeck (1991)

Waves in shallow water

Thursday: tsunami of 2004, KP theory,
shallow water equations, …

Forty years later

Martin Kruskal (d. 2006)

Peter Lax

Clifford Gardner

Robert Miura

John and Alice Greene

(d. 2007)

Waves in shallow water, I

Lecture 5

600 m pier at Duck, NC - Hurricane Grace, 1991