Nonlinear Waves: Woods Hole GFD Program 2009

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Examples of wave motion:

Water waves, atmospheric and oceanic internal waves (gravity waves), sound waves (music), electromagnetic waves (light, radio), elastic waves (earthquakes), etc.

An atmospheric gravity wave train in Northern Australia: the "Morning Glory".

1.1: Linear Waves

For simplicity, assume only one space dimension, and that a typical field variable is $u(x,t)$. Linear waves can then be represented by the Fourier component,

$$u = Real[A \exp(ikx - i\omega t)], \tag{1}$$

where k is the *wavenumber*, ω is the wave *frequency* and A is the wave amplitude, which may also be a function of k. The full solution is obtained by a superposition of such components.

The wave dynamics are determined by the dispersion relation

$$\omega = \omega(k), \tag{2}$$

whose precise form is determined by the physical system under consideration. For instance, for water waves,

$$\omega^2 = gk \tanh(kh), \tag{3}$$

where h is the still water depth, and g is gravity. Here there are two branches of the dispersion relation.

1.2: Linear Waves

An important feature of linear waves is that the dispersion relation captures the full system in Fourier space. That is, if the physical system takes the schematic form

$$D(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x})\, u = 0\,, \quad \text{then} \quad D(\omega, k) = 0\,, \tag{4}$$

whose solutions are the branches $\omega = \omega(k)$. For stable waves, ω is real-valued for all real-valued k. There are two important velocities,

$$\text{Phase Velocity}: \quad c = \frac{\omega}{k}\,, \quad \text{and} \quad \text{Group Velocity}: \quad c_g = \frac{d\omega}{dk}\,. \tag{5}$$

For a dispersive wave system, they are different. The phase of the wave (e.g. a wave crest) propagates with velocity c, but the wave energy propagates with the velocity c_g. The wave energy E for each Fourier component is typically given by an expression of the form $E = F(k)|A|^2$. For instance, for water waves $E = g|A|^2/2$ where A is the surface elevation above the still-water depth.

1.3: Nonlinear Waves

In general, as a linear dispersive wave system evolves, each Fourier component with wavenumber k propagates with its own group velocity, and so the system disperses. Then nonlinearity, that is the necessity to take account that the amplitude is finite and not infinitesimally small, typically arises in three scenarios.

(1) **Long waves**: Here $k \to 0$. Because the dispersion relation can be made to satisfy the antisymmetry condition $\omega(k) = -\omega(-k)$ (ensuring real-valued solutions), it follows that when also $\omega(0) = 0$, we have that $\omega = c_0 k + O(k^3)$, and so $c_g = c_0 + O(k^2)$, with **weak dispersion**.
(2) **Wave packets**: Here it is assumed that the wave energy is concentrated around a finite wavenumber k_0 say. Consequently, there is again only **weak dispersion**, and approximately the wave group propagates with a constant group velocity $c_{g0} = c_g(k = k_0)$.
(3) **Resonant wave interactions**: Due to nonlinearity, two linear waves with wavenumbers $k_{1,2}$ say, will interact to form another wave with wavenumber $k_0 = k_1 + k_2$. If the corresponding frequencies are resonant, that is $\omega_0 \approx \omega_1 + \omega_2$ $(\omega_i = \omega(k = k_i))$, then there can be a strong effect.

1.4: Korteweg-de Vries (KdV) equation

Here we consider the long-wave regime, where $k \to 0$, and assume that we can use the approximate dispersion relation

$$\omega = c_0 k - \beta k^3 \,, \tag{6}$$

with an error of $O(k^5)$. This translates to an evolution equation

$$u_t + c_0 u_x + \beta u_{xxx} = 0 \,, \tag{7}$$

where we recall that $-i\omega = \partial/\partial t, ik = \partial/\partial x$ for each Fourier component. The dominant term is $u_t + c_0 u_x \approx 0$, showing that the wave propagates with speed c_0 unchanged, except for the effect of the weak dispersion due to the term u_{xxx}. This small effect needs to be balanced by nonlinearity, and in many physical systems this has the form $\mu u u_x$, for some constant coefficient μ. Thus the model equation takes the form

$$u_t + c_0 u_x + \mu u u_x + \beta u_{xxx} = 0 \,. \tag{8}$$

This is the famous **Korteweg-de Vries (KdV)** equation, first derived in the water-wave context in 1895, and subsequently found to hold in many physical systems.

1.5: KdV equation, solitons

The KdV equation is, in the reference frame moving with speed c_0 (transform $x \to x - c_0 t$),

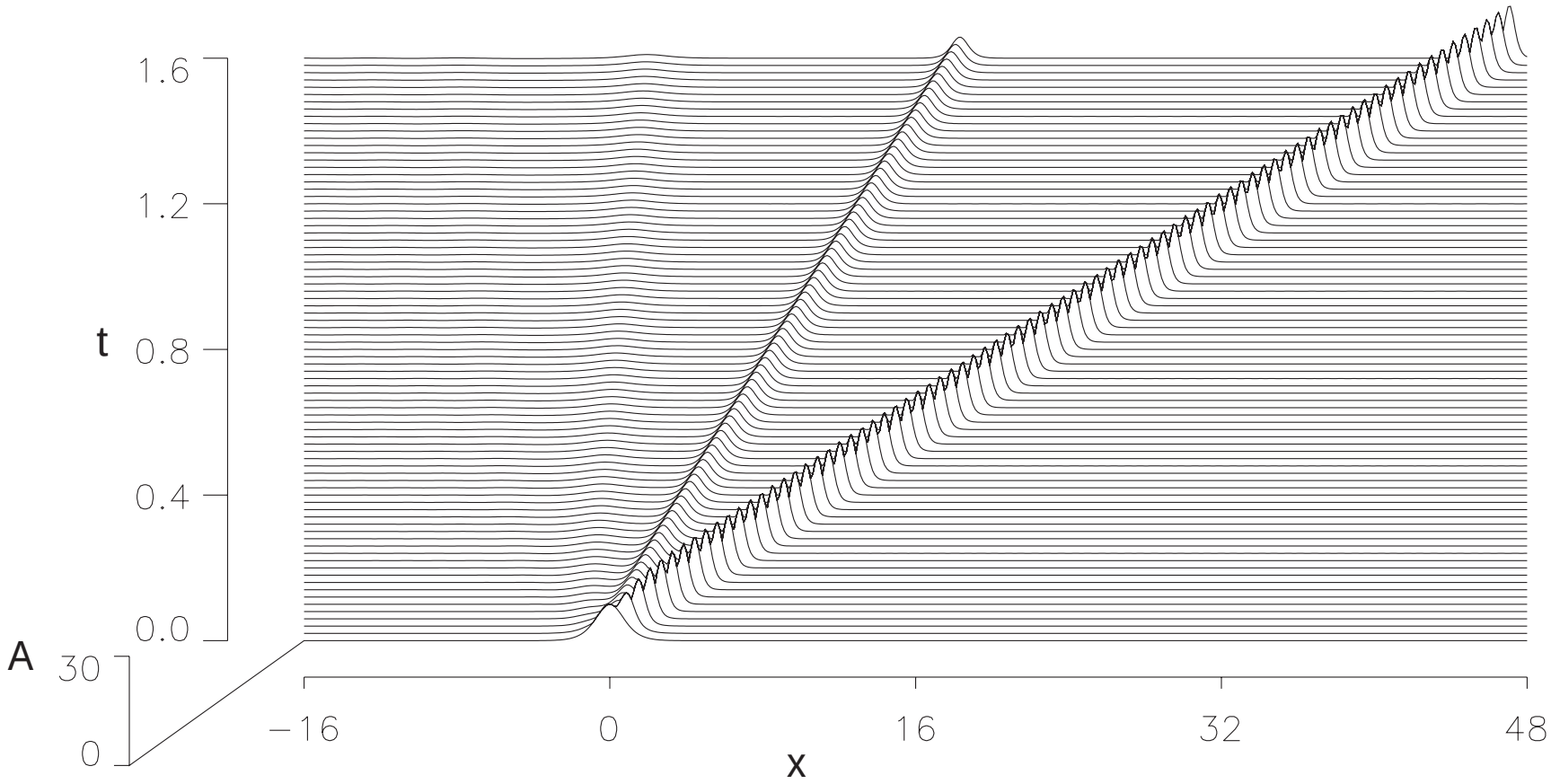
$$u_t + \mu u u_x + \beta u_{xxx} = 0\,. \tag{9}$$

This is an **integrable** equation,a result first established in the 1960's by Kruskal and collaborators. Its principal solutions are **solitons**. A single soliton is the **solitary wave**, an isolated and steadily-propagating pulse, given by

$$u = a\,\mathrm{sech}^2(\gamma(x - Vt))\,, \quad V = \frac{\mu a}{3} = 4\beta\gamma^2\,. \tag{10}$$

This is a one-parameter family of solutions, parametrized by the amplitude a say. The speed V is proportional to the amplitude a and is positive (negative) as $\beta > (<)0$, and is also proportional to the square of the wavenumber γ; thus large waves are thinner and travel faster. They are waves of elevation (depression) when $\mu\beta > (<)0$ Integrability means that the general initial-value problem for a localized initial condition can be solved through the **Inverse Scattering Transform** (IST), with the generic outcome of a finite number of solitons propagating in the positive x-direction, and some dispersing radiation, propagating in the negative x-direction (when $\mu\beta > 0$).

1.6: KdV equation, solitons



The generation of three solitons from a localized initial condition for the KdV equation

$$A_t + 6AA_x + A_{xxx} = 0\,.$$

1.7: Nonlinear Schrödinger Equation

Here we assume that the solution is a **narrow-band wave packet**, where the wave energy in Fourier space is concentrated around a dominant wavenumber k_0. The dispersion relation $\omega = \omega(k)$ can then be approximated for $k \approx k_0$ by

$$\omega - \omega_0 = c_{g0}\,(k - k_0) + \delta(k - k_0)^2\,, \tag{11}$$

where $\omega_0 = \omega(k_0)$, $c_g = c_g(k_0)$ and $\delta = c_{gk}(k_0)/2$, and we recall that $c_g(k) = d\omega/dk$, so that $c_{gk} = \omega_{kk}$. This translates to an evolution equation for the wave amplitude

$$i(A_t + c_{g0}A_x) + \delta A_{xx} = 0\,, \quad \text{where} \quad u = Real[A\exp(ikx - i\omega t)]\,. \tag{12}$$

Here it assumed that the **envelope** function $A(x,t)$ is slowly-varying with respect to the carrier phase $kx - \omega t$. The dominant term is $A_t + c_{g0}A_x \approx 0$, showing that the wave envelope propagates with the group velocity c_{g0}, modified by the effect of weak dispersion due to the term A_{xx}. The result is well-known in quantum mechanics as the Schrödinger equation.

The small dispersion effect needs to be balanced by nonlinearity, and in many physical systems this has the typical **cubic** form $\nu|A|^2A$, for some constant coefficient ν.

1.8: Nonlinear Schrödinger equation

Thus the model evolution equation for the wave envelope is the **nonlinear Schrödinger equation** (NLS), expressed here in the reference frame moving with speed c_{g0} (transform $x \longrightarrow x - c_{g0}t$),

$$iA_t + \nu|A|^2A + \delta A_{xx} = 0\,. \tag{13}$$

Like the KdV equation it is a valid model for many physical systems, including notably water waves and nonlinear optics, a result first realized in the late 1960's. Remarkably, like the KdV equation, it as an **integrable** equation through the IST, first established by Zakharov and collaborators in 1972. It also has soliton solutions, and the single soliton or solitary wave solution is

$$A = a\,\mathrm{sech}(\gamma(x - Vt))\,\exp\left(iKx - i\Omega t\right), \tag{14}$$

$$\nu a^2 = 2\delta\gamma^2\,, \quad \Omega = \delta(K^2 - \gamma^2)\,, \quad V = 2\delta K\,. \tag{15}$$

This solution exists only when $\delta\nu > 0$, the so-called focussing case. It forms a two-parameter family, the parameters being the amplitude a and "chirp" wavenumber K; however, K amounts to a perturbation of the carrier wavenumber k to $k + K$, $|K| << |k|$, and so can be removed by a gauge transformation.

1.9: Higher space dimensions

In two space dimensions the wavenumber becomes a vector $\mathbf{k} = (\mathbf{k}, \mathbf{l})$ and the dispersion relation is then

$$\omega = \omega(\mathbf{k}) = \omega(k, l)\,, \tag{16}$$

where the wave phase is now $\mathbf{k}\,.\mathbf{x} - \omega\mathbf{t} = \mathbf{kx} + \mathbf{ly} - \omega\mathbf{t}$. The phase velocity is $\mathbf{c} = \omega\mathbf{k}/\kappa^{\mathbf{2}}$, where $\kappa = |\mathbf{k}|$. The group velocity becomes the vector

$$\mathbf{c_g} = \nabla_{\mathbf{k}}\,.\omega = \left(\frac{\partial\omega}{\partial\mathbf{k}}\,, \frac{\partial\omega}{\partial\mathbf{l}}\right). \tag{17}$$

In general the group velocity and the phase velocity differ in both **magnitude** and **direction**. For water waves the dispersion relation is

$$\omega = g\kappa \tanh \kappa h\,. \tag{18}$$

This is an example of an **isotropic** medium, as the wave frequency depends only on the wavenumber magnitude, and not its direction. In this case the group velocity is parallel to the wavenumber $\mathbf{k}$, and hence parallel to the phase velocity, with a magnitude $c_g = |\mathbf{c_g}| = \mathbf{d}\omega/\mathbf{d}\kappa$.

1.10: Kadomtsev-Petviashvili equation

The KP equation is the two-dimensional extension of the KdV equation for isotropic systems, and is given by, in the reference frame moving with the linear long-wave speed c_0 in the x-direction,

$$(u_t + \mu u u_x + \beta u_{xxx})_x + \frac{c_0}{2} u_{yy} = 0\,. \tag{19}$$

This equation assumes that there is weak diffraction in the y-direction, that is $\partial/\partial y << \partial/\partial x$. The linear terms can be deduced from the linear dispersion relation $\omega = \omega(\kappa), \kappa = (k^2 + l^2)^{1/2}$, where it assumed that $l^2 << k^2$. Thus in the long-wave limit, since $\kappa \approx k + l^2/2k$,

$$\omega \approx c_0 k - \beta k^3 + \frac{c_0 l^2}{2k} \cdots .$$

Recalling that $-i\omega \sim \partial/\partial t, ik \sim \partial/\partial x, il \sim \partial/\partial y$, we se that (19) follows. When $c_0\beta > 0$ holds in (19), this is the KPII equation, and it can be shown that then the solitary wave (10) is stable to transverse disturbances. This is the case for water waves. On the other hand if $\beta c_0 < 0$ holds, this is the KPI equation for which the solitary wave is unstable; instead this equation supports "lump" solitons. Like the KdV equation, both KPI and KPII are integrable equations.

1.11: Benney-Roskes equation

For systems with an isotropic dispersion relation, the two-dimensional extension of the NLS equation is, in the reference frame moving with the x-component c_{g0} of the group velocity in the x-direction

$$iA_t + \nu|A|^2 A + \delta A_{xx} + \delta_1 A_{yy} + QA = 0\,. \tag{20}$$

Here Q is a wave-induced mean flow expression, which satisfies a forced long-wave equation. The precise form depends on the particular physical system being considered. For water waves, where $c_0^2 = gh$, it is

$$(1 - \frac{c_{g0}^2}{c_0^2})Q_{xx} + Q_{yy} + \nu_1|A|^2_{yy} = 0\,. \tag{21}$$

The resulting system (20, 21) is the Benney-Roskes equations, also known as the Davey-Stewartson equations. The linear terms in (20) can be found by expanding the dispersion relation as in the one-dimensional case (11), so that for $k \approx k_0, l \approx 0$,

$$\omega - \omega_0 = c_{g0}\,(k - k_0) + \delta(k - k_0)^2 + \delta_1 l^2\,,$$

where, as before $\delta = \omega_{kk}(k_0, 0)/2 = c_{gk}(k_0, 0)/2$ and $\delta_1 = \omega_{ll}(k_0, 0) = c_{g0}/2k_0$. For water waves $\delta < 0, \delta_1 > 0$ and $c_{g0} < c_0$, so that (20) is hyperbolic, but (21) is elliptic.

Lecture1: References

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