Internal Wave Generation and Scattering from Rough Topography

Kurt L. Polzin *

*Corresponding author address: Kurt L. Polzin, MS#21 WHOI Woods Hole MA, 02543.
E-mail: kpolzin@whoi.edu

Abstract

Several claims have been made about the efficiency of internal wave scattering from rough topography with consequent implications for the locality of the energy balance. In the limit that the topographic height is smaller than the vertical wavelength and the topographic slope is smaller than the wave slope, the scattering and generation problems are virtually equivalent when the internal tide kinetic energy frequency spectrum is regarded as the barotropic tidal energy. A direct consequence is that the energy balance of the internal tide is essentially local to within a single bottom bounce of the baroclinic tide above rough topography over the Mid-Atlantic Ridge. In the absence of other sources (near-inertial forcing by winds, internal lee waves or coupling with the geostrophic flow field), the observed internal wavefield results from a nonlinear equilibration process in which the boundary conditions figure prominently and interior inhomogeneity of the background buoyancy profile can be important in determining the global evolution of the wavefield.

1. Introduction

Various claims have been made regarding the efficiency of scattering from rough topography on the Mid-Atlantic Ridge. One scenario (St.Laurent and Garrett 2002) claims an $O(10\%)$ efficiency for the scattering process and regards the energy balance as inherently nonlocal. The bulk of the energy resulting from the conversion from barotropic to baroclinic energy is viewed as being able to propagate away from the ridge and ultimately dissipates elsewhere. A second scenario (Polzin 2004) presents a scale analysis of the spatial energy flux divergence terms to argue that the energy balance of the internal tide is one-dimensional (vertical). The two works differ in their respective interpretations of the first principle's derivation of a scattering transform presented in Müller and Xu (1992). St.Laurent and Garrett (2002) claim their 'second generation' calculation of an $O(10\%)$ efficiency is consistent with Müller and Xu (1992), while Polzin (2004) claims that Müller and Xu (1992) should return an O(1) transformation when applied to Mid-Atlantic Ridge topography. This work seeks to resolve those differing opinions.

Section 2 discusses a spectral representation of the internal tide generation process. Section 3 discusses a spectral representation of the internal wave scattering problem. A Summary section discusses the implications.

2. Internal Tide Generation

The vertical velocity associated with barotropic tidal flow impinging upon a corrugated boundary will excite internal waves in a stratified fluid. A representation of the radiating internal wavefield is (Bell 1975):

$$E_{flux} = \sum_{n=1}^{n_o} \frac{n\omega_1}{2\pi^2}[(N^2 - n^2\omega_1^2)(n^2\omega_1^2 - f^2)]^{1/2} \int\int_{-\infty}^{\infty} k_h^{-1}\ H(k,l)\ J_n^2([(k^2U_o^2 + l^2V_o^2)/\omega_1^2]^{1/2})\ dk\ dl \tag{1}$$

In (1), E_{flux} is the vertical energy flux, f is the Coriolis frequency, ω_1 is the fundamental frequency of the barotropic tide (M_2), n an integer restricted so wave frequencies are smaller than the buoyancy frequency N ($n\omega_1 < N$ with $\omega_n = n\omega_1$ the n^{th} harmonic), the horizontal wavevector is (k, l) with magnitude k_h, ω is the wave frequency and the vertical wavenumber m follows from a linear dispersion relation. The function J_n is a Bessel function of order n and the factors U_o and V_o in its argument represent the barotropic tide amplitude. The factor $H(k, l)$ is the topographic spectrum. The convention that the dimensions of the spectrum are indicated by its arguments will be adopted here, i.e. $E(\omega, k, l)$ is the frequency and 2-d horizontal wavenumber energy density. Subscripts will be used to indicate specific moments of the total energy, i.e. E_k represents the horizontal kinetic energy density.

The description of the energy flux in the horizontal wavenumber – frequency domain is:

$$E_{flux}(k,l,\omega_n) = 2n\omega_1[(N^2 - n^2\omega_1^2)(n^2\omega_1^2 - f^2)]^{1/2} k_h^{-1}\ H(k,l)\ J_n^2([(k^2U_o^2 + l^2V_o^2)/\omega_1^2]^{1/2}). \tag{2}$$

Bell (1975)'s quasi-linear model assumes infinitesimal amplitude bathymetry but includes the barotropic tidal advection in the momentum equations. In the linear limit that horizontal tidal excursions are smaller than the topographic scales, the internal wave energy density is proportional to the topographic slope spectrum. In the advective ($kU_o/\omega \gg 1$, small-horizontal-scale) limit, the energy density of the internal tide is no longer proportional the topographic slope, but rather proportional to the amplitude of the topographic perturbations. The appearance of this roll-off is key. It avoids an ultraviolet catastrophe of linear models (infinitesimal or finite amplitude bathymetry) that lead to an unphysical prediction of infinite shear variance and energy when realistic (i.e. fractal) topographic descriptions are used. The roll-off also enables a parametric assessment of the vertical profile of turbulent dissipation, Polzin (2004).

In the linear limit that tidal excursions are much smaller than the topographic scales, the harmonics can be neglected and the remaining Bessel function expressed in terms of its small argument expansion, $J_1(z) = z/2 +$ higher order terms, so that (2) becomes:

$$E_{flux}(k, l, \omega_n) = \frac{\omega_1}{2}[(N^2 - \omega_1^2)(\omega_1^2 - f^2)]^{1/2} k_h^{-1}\ H(k, l)\ [(k^2 U_o^2 + l^2 V_o^2)/\omega_1^2]. \qquad (3)$$

If we further assume that the tide is horizontally isotropic and represent the velocity variance of the tide as twice the barotropic kinetic energy, $U_o^2 + V_o^2 = 2E_k^o$:

$$E_{flux}(k, l, \omega_n) = \frac{1}{2\omega_1}[(N^2 - \omega_1^2)(\omega_1^2 - f^2)]^{1/2} k_h\ H(k, l)\ E_k^o. \qquad (4)$$

3. Internal Wave Scattering

The bottom boundary condition for a radiation balance equation takes the form:

$$C_{gz} E^+ = C_{gz} \Phi(E^-) + E_{flux} \qquad (5)$$

where $\Phi(E^-)$ represents a scattering transform, E^- is the energy spectrum of the downward propagating wavefield, E^+ represents the upward propagating wavefield and E_{flux} is prescribed by a model of internal wave generation at the bottom. The scattering transform describes the spectral redistribution of energy as a downgoing wave impinges upon a rough boundary. The simplest representation views scattering as a linear process from infinitesimal amplitude topography. That is, the topographic height is assumed to be small relative to the vertical scale of the incident wave and the topographic slope is assumed everywhere smaller than the ray characteristic slope.

The linear, infinitesimal amplitude scattering transform describing the redistribution of energy at the bottom boundary is given by (Müller and Xu 1992):

$$\Phi[E^-(k, l, \omega)] = \frac{(N^2 - \omega^2)(\omega^2 + f^2)}{2\omega^2(\omega^2 - f^2)}\ E^-(\omega)(k^2 + l^2)\ H(k, l) - \qquad (6)$$

$$\frac{(N^2 - \omega^2)(\omega^2 + f^2)}{2\omega^2(\omega^2 - f^2)}\ E^-(k, l, \omega)(k^2 + l^2)^{1/2} \int\int (k^2 + l^2)^{1/2} H(k, l)\ dkdl$$

The scattered spectrum is dominated by the first term on the right-hand-side of (6) (Müller and Xu 1992) and thus is proportional to the topographic slope spectrum and the down-going internal tide frequency spectrum. This response characterizes *generation* in the linear (non-advective) limit with the barotropic tidal amplitude being replaced by the $E^-(\omega)$ factor. Invoking the dominance of this first term and using linear kinematics to express the bottom boundary condition in terms of the horizontal kinetic (E_k^-), rather than total (E^-), energy spectrum, (6) becomes:

$$C_{gz} \Phi[E^-(k, l, \omega)] \cong \frac{1}{\omega}[(N^2 - \omega^2)(\omega^2 - f^2)]^{1/2}\ k_h\ H(k, l)\ E_k^-(\omega). \qquad (7)$$

Apart from a factor of 2, (7) is identical to (4). The factor of two appears to arise from the internal wavefield consisting of kinetic *and* potential energy in the scattering problem, whereas conversion of barotropic horizontal kinetic energy is considered in the generation problem.

4. Discussion

Müller and Xu (1992) characterize the scattering transform as resulting in an $O(1)$ rearrangement of internal wave energy from large scales to small when internal wave ray characteristic slopes are similar to topographic slopes associated with the dominant bathymetric scales, as is the case in the Brazil Basin for semi-diurnal frequencies [e.g. see Fig. 8 of Polzin (2004)]. Direct evaluation of (6) using a parametric representation of $H(k,l)$ presented in Polzin (2004) confirms Müller and Xu (1992)'s characterization. The scattering assists in preventing energy from escaping horizontally beyond one bottom bounce. This conclusion is at odds with St.Laurent and Garrett (2002)'s characterization of $O(10\%)$ scattering efficiency with a "second generation" calculation. Their calculation does not recognize that waves of a given frequency are phase locked for the purpose of estimating the kinematic boundary condition that the velocity be normal to the topographic slope with the attendant implication that wave scattering is linear in the frequency domain energy density. St.Laurent and Garrett (2002) treat wave scattering as being linear in the vertical wavenumber - frequency energy density.

The strong similarities between baroclinic tide generation and the scattering problem point to possible extensions of the infinitesimal amplitude scattering transform. In particular, the concerns expressed in Polzin (2004) about 2-D, finite amplitude topography and an ultraviolet catastrophe in the generation problem carry over to the scattering problem. Comparison of observed spectra returned good agreement with predictions based upon (2) in that no dramatic enhancement of the small scale wavefield associated with finite amplitude bottom boundary conditions was in evidence. A justification presented for this was that, for 2-D finite amplitude anisotropic topography, flow blocking in the minor axis (steep) direction may result in flow in the major axis (shallow) direction. Comparison of observed spectra and dissipation data were also deemed to be consistent with (2) in that the transition to the advective limit $(kU_o, lV_o) >> 1$ leads to a prediction of finite energy and shear variance, whereas (3) returns energy and shear spectra whose integrals do not converge as the topographic spectra are not sufficiently red. Inclusion of advection in the momentum equations in the scattering problem may provide a physical rationale for effectively truncating the topographic slope spectrum, as opposed to the *ad hoc* truncation invoked in Müller and Xu (1992) to avoid the issue. Following the logic behind (1) and (6), I suspect a simple reinforcement of the downgoing internal tide and the barotropic tide in terms of affecting the advective regime roll-off.

The implications of a local balance are significant. In the absence of other external energy inputs (e.g. near-inertial inputs of energy by the wind, the generation of lee waves and energy exchange associated with internal wave - mean flow interactions), the boundary conditions are mediated by spectral transfers associated with buoyancy scaling and nonlinearity. Documenting this spectrum could give important clues to how the background internal wave spectrum is formed.

Acknowledgments.

Salary support for this analysis was provided by Woods Hole Oceanographic Institution bridge support funds.

References

Bell, T. H., 1975: Topographically-generated internal waves in the open ocean. *J. Geophys. Res.*, **80**, 320–327.

Müller, P. and N. Xu, 1992: Scattering of oceanic internal gravity waves off random bottom topography. *J. Phys. Oceanogr.*, **22**, 7474–488.

Polzin, K., 2004: Idealized solutions for the energy balance of the finescale internal wavefield. *J. Phys. Oceanogr.*, **34**, 231–246.

St.Laurent, L. and C. Garrett, 2002: The role of internal tides in mixing the deep ocean. *J. Phys. Oceanogr.*, **32**, 2882–2899.