Boundary Layers: Homogeneous Ocean Circulation

Lecture 8 by Henrik van Lengerich

1 Vorticity Balance

Vorticity is conserved along streamlines and this allows us to extract information from the governing equations without the use of the solution.

1.1 Balance along x - direction

We perform an integral from $x = 0$ to $x = x_e$ about the vorticity equation (Eq. **??**), this is also known as the dissipation balance. We neglect u_y because it is small compared to v_x in the boundary layer and we ignore all inertial and viscous terms in the interior, as well as use the no slip condition on v at $x = 0$, to obtain

$$\int_0^{x_e} [\psi_x = w_e - \delta_s \nabla^2 \psi + \delta_M^3 \nabla^4 \psi] dx. \tag{1}$$

ψ is conserved so the left hand side of Eq. 1 is zero. The stream function multiplied by the boundary layer thickness is negligible close to the right hand side. This gives, for $w_e = w_e(y)$,

$$0 = x_e w_e + \delta_s \psi_x(0) - \delta_m^3 \psi_{xxx}(0). \tag{2}$$

The first derivative of ψ is zero at the left boundary due to the no slip condition. This gives

$$0 = x_e w_e + \delta_m^3 \psi_{xxx}(0), \tag{3}$$

which means that the vorticity inserted by the Ekman pumping must be dissipated by the sublayer. We verify that (1.20) is a solution to Eq. 3

$$x_e w_e = x_e w_e \delta_m^3 (\frac{\delta_m}{\delta_s})^{3/2} \frac{1}{\delta_s^3 (\delta_m/\delta_s)^{9/2}}. \tag{4}$$

We can also look at the streamlines that go through the Stommel layer. Performing an integral around the vorticity from the Stommel layer at 0_+ to the right edge at x_e of the Stommel solution (ϕ) similar to Eq. 1, gives

$$0 = x_e w_e(0_+) + \delta_s \phi_x(0_+). \tag{5}$$

We can verify that the solution previously obtained matches this condition

$$0 = x_e w_e(0_+) - \delta_s x_e w_e(0_+)/\delta_S. \tag{6}$$

This means that for the total solution of the stream function obtained the vorticity of the streamlines that pass through the Stommel layer are balanced by bottom friction.

1.2 Vorticity Balance along y - direction

We perform an integral of the vorticity equation (Eq. ??) for an area R of the boundary layer from two arbitrary latitudes y_1 to y_2.

$$\int_R \delta_I^2 \nabla \cdot \overrightarrow{u} \zeta dA + \int_R v dA + \int_R \nabla \cdot \overrightarrow{u} h_b dA = -\delta_s \int_R \zeta dA + \int_R \delta_m^3 \nabla^2 \zeta dA. \tag{7}$$

Because the velocity in the x-direction does not vary much with y, the local vorticity can be approximated as

$$\zeta \approx v_x. \tag{8}$$

We assume that the bottom is flat, so that the term with h_b is zero, then Eq. 7 becomes

$$\frac{1}{2}\delta_I^2[v^2(0,y_1) - v^2(0,y_2)] + \int_{y_1}^{y_2} \psi_l(0,y)dy = \delta_s \int_{y_1}^{y_2} v(0,y)dy - \delta_m^3 \int_{y_1}^{y_2} \zeta_x(0,y)dy. \tag{9}$$

Using the no-slip condition at $x = 0$, this simplifies to

$$\int_{y_1}^{y_2} \psi_I(0,y)dy = -\delta_m^3 \int_{y_1}^{y_2} \zeta_x(0,y)dy. \tag{10}$$

The term on the left is the vorticity added due to the wind and the term on the right is the dissipation of vorticity due to viscosity in the viscous sublayer. Because we have not fixed the bounds on y, the vorticity added on any latitude is dissipated in the boundary layer at that latitude. It should be noted that this interpretation is only valid under the assumption that $v_x \gg u_y$ as stated at the onset.

2 Inertial Boundary Layers

Previously we have assumed the δ_I term was small, but this is pretty unrealistic considering the Reynolds number of ocean flows. We focus on a parameter region where inertial effects become important, that is $1 \gg \delta_I \gg \delta_m \gg \delta_s$. To retain the inertial terms of highest order we re-scale the x variable such that

$$\xi = x/\delta_I. \tag{11}$$

To order $1/\delta_I$ the vorticity equation (Eq. ??) governs the inertial boundary layer, and is given by

$$\psi_\xi \psi_{\xi\xi y} - \psi_y \psi_{\xi\xi\xi} + \psi_\xi = 0. \tag{12}$$

Note that the left hand side is the same as the substantial derivative, so we write

$$\frac{D}{Dt}(\psi_{\xi\xi} + y) = 0 \tag{13}$$

$$\psi_{\xi\xi} + y = Q(\psi). \tag{14}$$

$$\tag{15}$$

This means that the vorticity is conserved along streamlines.

2.1 Example

Assume that the velocity u is constant and ψ is independent of x far from the boundary. Then $\psi_{\xi\xi}$=0 and we can solve Eq. 14 far from the boundary to get

$$y = Q(\psi). \tag{16}$$

Using the definition of the stream function we find that

$$\psi = -uy \tag{17}$$

far from the boundary. Here the integration constant is arbitrary and set to zero.

Now we apply Eq. 17 for ψ in the boundary region to obtain

$$\begin{aligned} Q(\psi) &= -\psi/u & (18)\\ \psi_{\xi\xi} + y &= -\psi/u & (19)\\ \psi &= A(y)e^{\psi/\sqrt{-u}} + B(y)e^{-\psi/\sqrt{-u}} + uy. & (20) \end{aligned}$$

We eliminate the $A(y)$ term because we need ψ to be bounded in x in order to match it to an inner solution where ξ goes to infinity. We use the no penetration condition on u, but allow the fluid to slip along the $x = 0$ edge. Again, setting the integration constant to zero gives

$$\psi = uy(1 - e^{-\xi/\sqrt{-u}}). \tag{21}$$

We know that the interior flow needs to be westward, so this expression cannot close the circulation; it also does not satisfy the no slip condition at $x = 0$.

2.2 Inertial Sub-layer Thickness

Looking at the balance of vorticity of an inertial sub-layer solution it can be seen that the vorticity input by the wind needs to be balanced by the viscous sub-layer; however, most streamlines do not go through the viscous sub-layer, therefore there is an accumulation of vorticity. We define a re-scaled Reynolds number as $Re = UL/A{*}\delta_I = \delta_I^3/\delta_m^3$, then numerical simulations by Fox-Kemper Fox-Kemper (2003) show that for Re = 1.95 the solution is stable, but at Re = 4.29 there is an inertial runaway.

3 Enhanced Sub-layer

Fox-Kemper and Pedlosky's Fox-Kemper and Pedlosky (2004) solution to the inertial runaway is to modify the momentum mixing viscosity such that it captures two dissipation mechanisms. The first is the effect of unresolved eddies in the interior and the boundary layers. The second is the interaction of the fluid with the boundary. These effects were incorporated into the Munk layer as

$$\delta_m^3 = \frac{\delta_I^3}{Re_i} + (\frac{\delta_I^3}{Re_b} - \frac{\delta_I^3}{Re_i})(e^{-x/\delta_d} + e^{-(1-x)/\delta_d}) \tag{22}$$

such that the effect is continuous as x is varied. The first term in the summation represents the unresolved eddies, the second term is the interaction with the boundary (which is at x=1). The two Reynolds numbers and the thickness of the region where the boundary viscosity is enhanced are given by

$$Re_i = (\frac{\delta_I}{\delta_m})^3_{interior} \quad (23)$$

$$Re_b = (\frac{\delta_I}{\delta_m})^3_{boundary} \quad (24)$$

$$\delta_d = \frac{\delta_I}{\sqrt{Re_I}}. \quad (25)$$

The effect of this enhanced dissipation mechanism can be seen in Figure 1. As the boundary layer Reynolds number is decreased the vorticity decreases due to dissipation in the boundary layer. The same is true of the interior Reynolds number. The energy of the system also decreases as either of the Reynolds numbers are decreased. Shown in the lower right hand corner of Figure 1 is a situation with a large internal vorticity (larger than what was unstable in section 2.2), but this vorticity is dissipated to the boundary region.

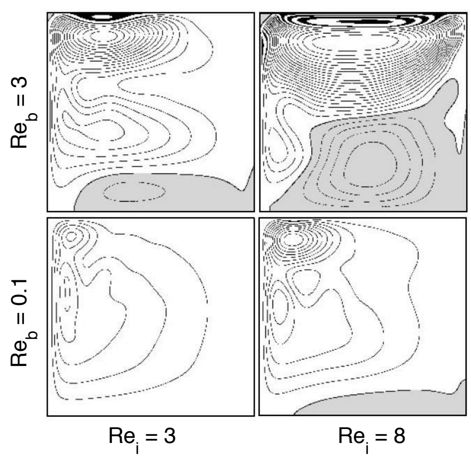

Figure 1: Streamlines for various Re_i and Re_b. Shaded regions are of negative vorticity. Figure taken from Fox-Kemper (2003).

References

B Fox-Kemper. *Eddies and Friction: Removal of vorticity from the Eddies and Friction: Removal of vorticity from the wind-driven gyre.* PhD thesis, MIT/WHOI Joint Program, 2003.

B Fox-Kemper and J. Pedlosky. Wind-driven barotropic gyre i: Circulation control by eddy vorticity fluxes to an enhanced removal region. *Journal of Marine Research*, 62:169–193, 2004.