The Stability of Boundary Layer Flows

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Abstract

This study aims to answer the question: Are stably stratified boundary layer flows marginally unstable? Using the Taylor Goldstein equation, we analyse the linear stability of a number of observed mean flows in stably stratified boundary layers. We find that although Kelvin Helmholtz instability may occur the growth rates of unstable modes are small compared to the time scales of fluctuation in the flow and in all cases where unstable or stable modes are found a change in velocity shear of no more than 20% is required to stabilise or destabilise the flow, respectively. The implications of these results and potential for further studies are discussed.

1 Introduction

In his book *Buoyancy Effect in Fluids* J.S. Turner (1973) makes the following conjecture regarding gravity driven flows down a uniform slope: 'While turbulence is present the drag on the layers increases and the velocity falls, but when it is suppressed the flow is accelerated again by gravity'. The mean flow is consequently self-controlled close to a state at which turbulence sets in and is one of marginal stability. This study presents evidence in support of Turner's conjecture that boundary layer flows are maintained in a marginal state of stability, which we shall assume is that in which small disturbances to the mean flow have zero growth rate.

Our method for analysing the stability of such flows shall be to use the Taylor-Goldstein equation

$$\frac{\partial^2 \phi}{\partial \hat{z}^2} + \left(\frac{\hat{N}^2}{(U-C)^2} - k^2 - \frac{\frac{\partial^2 U}{\partial \hat{z}^2}}{U-C} \right) \phi = 0 \tag{1}$$

where the perturbation streamfunction is $\psi(\hat{x}, \hat{z}, \hat{t}) = \phi(\hat{z}) e^{[ik(\hat{x}-C\hat{t})]}$ with wave number k, phase speed $C = C_r + iC_i$ and mean velocity U and the bouncy frequency $\hat{N}^2 = -g\rho_z/\rho$ (variables with a hat are in their dimensional form and will later be nondimensionalised). The Taylor-Goldstein equation is derived from the mass and momentum conservation equations for a stratified fluid without rotation and describes the evolution of an initial disturbance to the steady, inviscid and unidirectional stratified flow under a velocity shear. In using (1) we implicitly disregard the stresses and eddy diffusivities associated with turbulent motion. The common dimensionless number used to characterise the flow is the Richardson number $Ri = \hat{N}^2/(U_z)^2$.

The canonical theorem of Miles and Howard (Miles, 1961) (Howard, 1961) shows, using 1, that steady, inviscid and unidirectional flows with Richardson numbers above a quarter everywhere in the flow are *stable* to small perturbations (Drazin and Reid, 1981). Although it is commonly espoused that flows are unstable below this critical value it is in fact (theoretically at least) not the case. For particular flows with U and ρ prescribed, Hazel (1972) has shown that when the boundary of a stratified flow is not at infinity the instability can be inhibited and the critical value of minimum Richardson number (called J) is reduced. As is shown in figure (1) as the distance from the boundary increases the region where growing waves exist and perturbations can grow decreases. Indeed where the boundaries are within 1.5 H (H being the typical height scale) of the shear the flow is stable for Richardson numbers above 0.125 and almost always for 1.25H. Indeed, even if the stratification is increased (increasing Ri) the stability is not necessarily increased, Thorpe (1969) and Miles (1963) both show cases where the presence or increase in stratification can reduce the stability of a flow. Such examples suggest that the stability of a shear flow can not accurately be described by the Richardson number at an isolated point alone but solutions to the Taylor-Goldstein equation involving representation of U and $\hat{\rho}$ as functions of $\hat{z}$ over the entire flow should be considered.

The body of evidence in support of Turner's conjecture is small and not yet convincing. Indeed the original statements are made with reference to the laboratory studies of Mittendorf (1961) who showed how the Kelvin-Helmholtz mechanism reduces the shear leading to a maintenance of a gravity driven flow in a state of marginal stability. Thorpe and Hall discussed such a concept in their study of a wind driven flow in Loch Ness, Scotland (1977). Small perturbations were shown to be likely to grow if the Richardson number were increased from that observed by only 10% (i.e. an increase in the mean shear of only 5%). Merrill (1977) examined the linear stability of an airflow near the ground and although he did not estimate whether the flow was 'marginally' unstable he found that for a boundary flow with $J = 0.15$, the growth rates of the most unstable modes were small. Nielsen (1991) looked at instability on a frontal inversion in the Atmosphere and although unstable modes where not found for the observed profiles, he was able to extrapolate

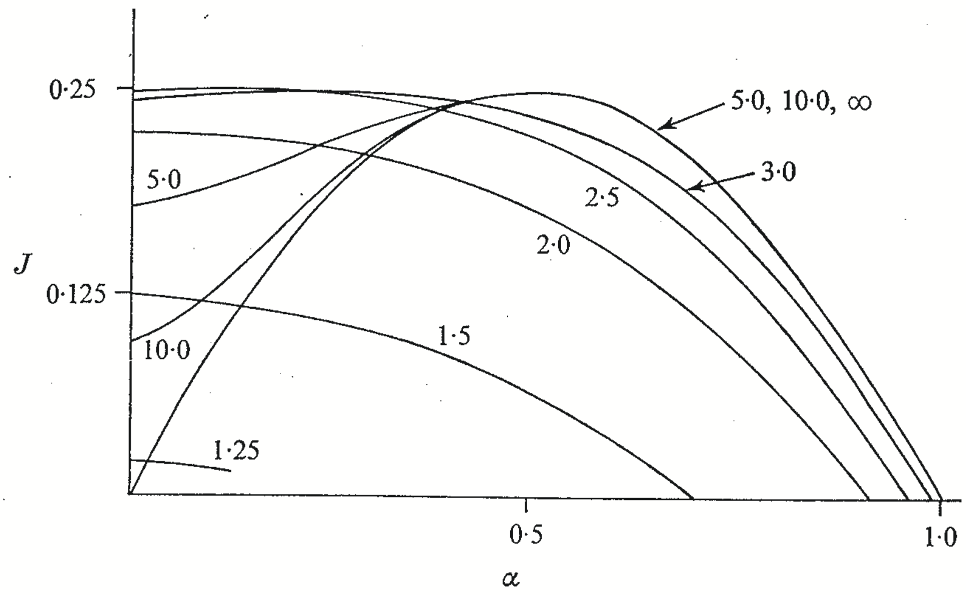

Figure 1: Stability boundaries for 'tanh' profiles at nondimensional distances marked. The vertical axis J represents the minimum Richardson number (found at $z = 0$) and the dimensionless wavenumber $\alpha = kH$ for some typical length H. (Taken from Hazel (1972)).

from unstable modes found by reducing the Richardson number of the flow and found that the observed profiles were at or very near a state of marginal stability. As in this study numerical problem often arise when such stability analysis is conducted on actual profiles of velocity and density. A recent study by Thorpe and Ozen (2007) (hereafter TO07) has asked the question of whether boundary flows are marginally stable. They look at a cascading flow in Lake Geneva and find that both a functional fit to the data and the data itself in a canonical case, are unstable but only marginally so.

Boundary currents and how they influence the dynamics of the Ocean and Atmosphere is currently the subject of great interest in geophysical research. Two examples of stably stratified boundary currents were mixing and turbulence are known to be of great importance in the Ocean are wind driven flows in the presence of surface heating and gravity driven boundary currents that feed dense water into the major ocean basins. Correctly describing such flows and the instabilities that can arise from them is extremely important for numerical models of the Climate System.

The consequences of boundary flows being generally in a state of marginal stability (if that can be shown) is important from the point of view of numerical models. It is customary in many numerical models, to mix and entrain fluid in a stratified flow when the Richardson number drops below the canonical 1/4 value or to adopt an empirical entrainment coefficient. As we have discussed, boundaries may act to stabilise the flow despite such a shear existing. It would preferential to use linear stability analysis, rather than simply the Ri condition to assess the accuracy, and thus constrain, a numerical model and test whether numerically predicted flows are, like those observed, close to marginal stability.

The following section of this report, Section (2), establishes the theoretical framework under which we will conduct our stability analysis. Section (3) discusses the data collected from a cascading flow in Lake Geneva and section (4) describes the results of our linear stability theory. A discussion of the implications of these results and ongoing research are discussed in section (5).

2 Nondimensionalisation and Boundary Conditions

We follow the same nondimensionalisation as that of TO07 where h is defined by the thickness of the current, $z = \hat{z}/h$, $u = U/U_{max}$, $c = \hat{C}/U_{max}$, $g\Delta$ is the reduced gravity ($\Delta = (\rho(z = 0) - \rho(z = 1))/(\rho(z = 0) + \rho(z = 1)))$ and U_{max} is the maximum velocity difference in the flow (see Figure (2)) and $N^2 = g\Delta\hat{N}^2/h$ and the Taylor-Goldstein equation (1) becomes

$$\frac{\partial^2\phi}{\partial z^2} + \left(\frac{N^2}{Fr^2(u-c)^2} - \alpha^2 - \frac{\frac{\partial^2 u}{\partial z^2}}{u-c}\right)\phi = 0. \tag{2}$$

where the Froude number $Fr = U_{max}/\sqrt{g\Delta h}$

The data we will use to assess the linear stability of boundary layer flows will, in the majority of cases, be limited to the region of the boundary flow itself. In previous treatments of flow using the Taylor-Goldstein equation it is assumed that there is some point in the flow where a solid boundary exists such that $\phi = 0$. In cases where we have only data in the flow region it is more preferable to assign some mean stratification and flow to the region distant from the flow or interior (i.e. for $z > 1$). For our purposes we will assume the interior has the following properties: the mean flow ($\overline{u}$,$\overline{v}$) and density ($\overline{\rho}$) are constant in x, y, z and t, the mean flow is hydrostatic such that $\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g$ and

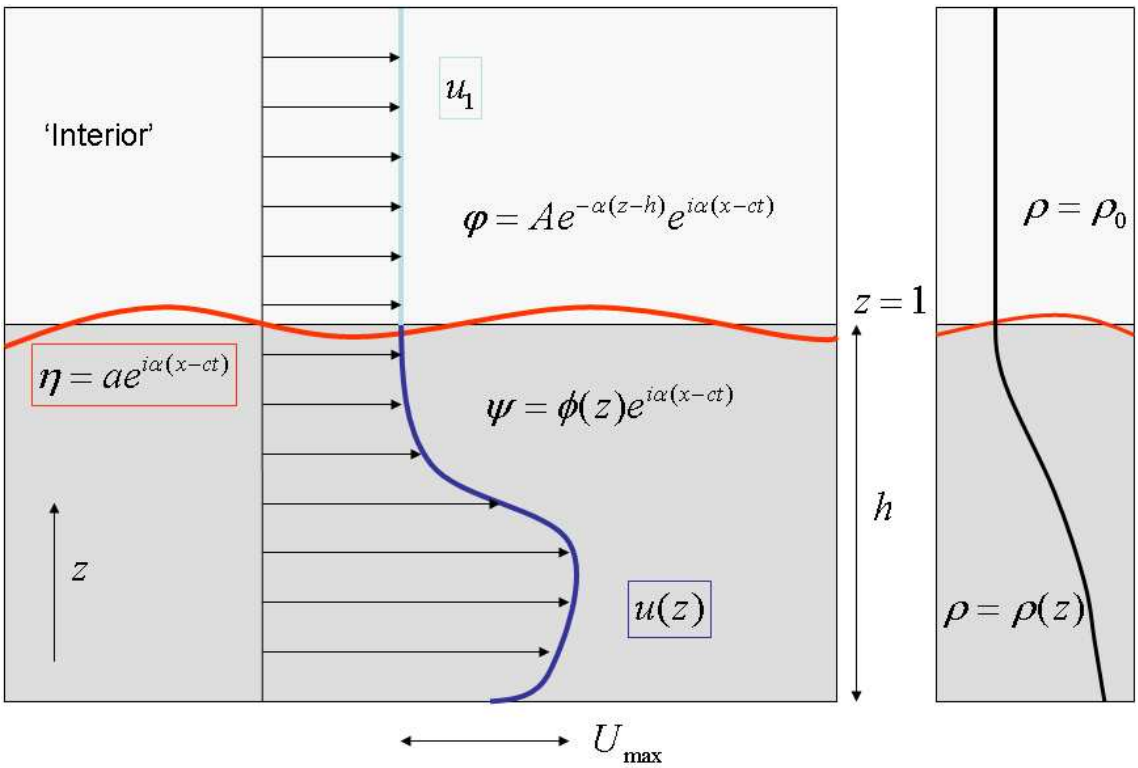

Figure 2: A schematic showing how the scales U_{max}, h, the perturbation streamfunction Ψ, potential flow φ and free surface displacement in the matching region η are defined for a typical flow profile.

it is irrotational. In the upper region ($z > 1$) there exists a potential flow $\varphi = Ae^{k(z-1)}e^{i\alpha(x-ct)}$ which will be matched to the perturbation streamfunction close to $z = 1$ where the free surface displacement is $\eta = ae^{i\alpha(x-ct)}$. Here A and a are unknown constants. Assuming the flow is two dimensional (for the time being) we write a linearized equation for η (eliminating $u'\frac{\partial \eta}{\partial x}$ terms) at the interface between the two regions ($\hat{z} = 1 + \eta$)

$$\frac{\partial \eta}{\partial t} + \overline{u}\frac{\partial \eta}{\partial x} = w \tag{3}$$

where $w = w' = \frac{\partial \varphi}{\partial z} = -\frac{\partial \Psi}{\partial x}$ and $\overline{u}$ is at the interface. Substituting our potential φ into the above equation we get $a = \varphi/(c - \overline{u})$ and $A = i\varphi$. Below $z = 1 + \eta$ and we may describe the pressure as having a mean and wave component such that $p = P(z) + p(z)e^{i\alpha(x-ct)}$ and we may invoke the hydrostatic approximation also ($P(z) = P_0 + g\int_z^1 \rho dz$). The momentum equation for the perturbation in this region is

$$\frac{\partial u}{\partial t} + \overline{u}\frac{\partial u}{\partial x} + w\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial x}. \tag{4}$$

Recalling that $\frac{\partial \Psi}{\partial z} = u$ and $\frac{\partial \Psi}{\partial x} = -w$ we have

$$p = \overline{\rho}\left[(c - \overline{u})\frac{\partial \phi}{\partial z} + \phi\frac{\partial \overline{u}}{\partial z}\right] \tag{5}$$

and therefore the total pressure at $z = 1$ is

$$P = P_0 + \left\{-ga\rho(h) + \overline{\rho}\left[(c - \overline{u})\frac{\partial \phi}{\partial z} + \phi\frac{\partial u}{\partial z}\right]e^{i\alpha(x-ct)}\right\}. \tag{6}$$

Bernoulli's equation for the upper region is

$$\frac{P}{\rho} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}(u + \overline{u})^2 + \frac{1}{2}(w + \overline{w})^2 + gz = B \tag{7}$$

where B is a constant and thus the pressure at $z = 1$ (determined from the upper side) is also

$$P = \overline{\rho} \left\{ C - \frac{1}{2}\overline{u}^2 - gh + [ik(c - \overline{u})A - ga]e^{i\alpha(x-ct)} \right\}. \tag{8}$$

Matching the pressure described by (5) and (8) and assuming the velocity and density are continuous at $z = 1+\eta$ it follows that the boundary condition at $z = 1$ is

$$(c - \overline{u})\frac{\partial \phi}{\partial z} + \frac{\partial u}{\partial z}\phi + \alpha(c - \overline{u})\phi = 0. \tag{9}$$

3 Boundary Flow Data from Lake Geneva

The data we shall use to investigate the stability of cascading boundary flows comes primarily from Lake Geneva. In winter, during cold nights, shallow regions of the Lake are cooled and these form cascading gravity currents which flow down the boundaries of Lake Geneva (Fer *et al.*, 2001). Profiles of density ($\overline{\rho}$), downslope velocity (U) and across slope velocity (V) have been collected from the bottom to 25m above equating to about a third of the overall water column. An example of such a flow is shown in Figure (3) with a density section taken down the slope.

The flow displays hallmarks of many forms of instability. The flows are punctuated by pulses of water consistent with roll waves in steep open channel flows (Fer *et al.*, 2001). Hydraulic jumps can occur in the fluid and the stability of such events is discussed in Thorpe and Ozen's Study. A time series of both U and V along with temperature is shown in figure (4). The flow is clearly unsteady, and probably turbulent, although data are not presently available to characterise the variability at frequencies less than about 0.01Hz. The data we shall consider in this study are 2hr averages taken of both the

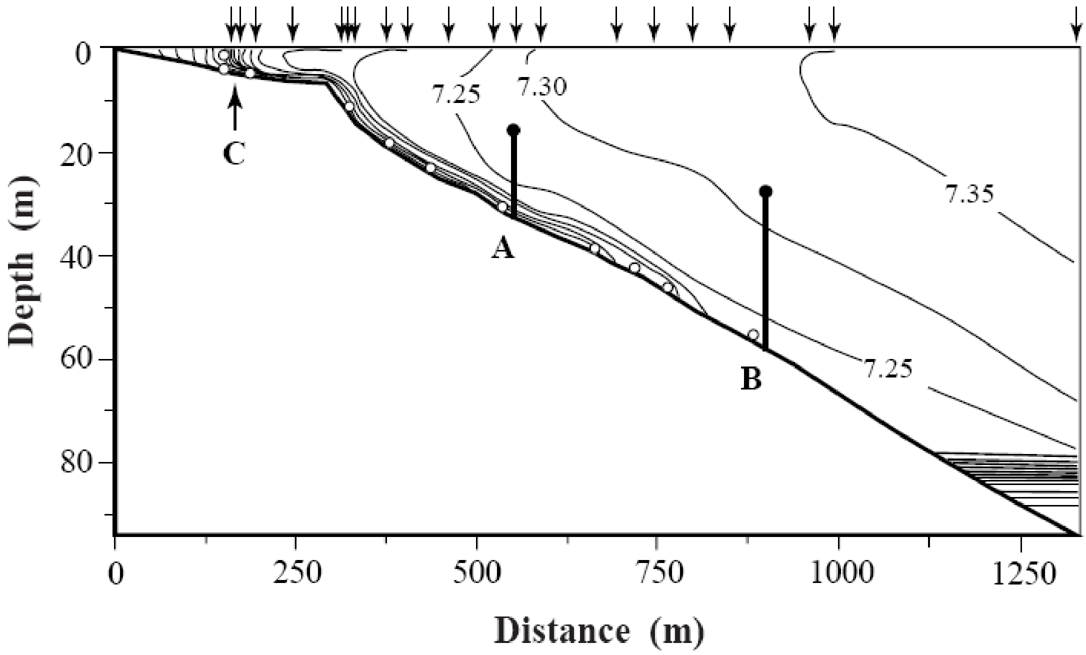

Figure 3: Cold water spilling down the sloping side of Lake Geneva from shallow water. Temperature contours at 0.05^{o}C from a CTD section made on the northern side of Lake Geneva between 1030-1330 hrs local time, 23 December 1998 after a period of nocturnal cooling with positive surface buoyancy flux. Station positions are marked by arrows at the top. Circles show the positions of temperature miniloggers .A and B mark positions of vertical arrays and C marks a warm front in shallow water. Taken from Fer *et. al* (2001).

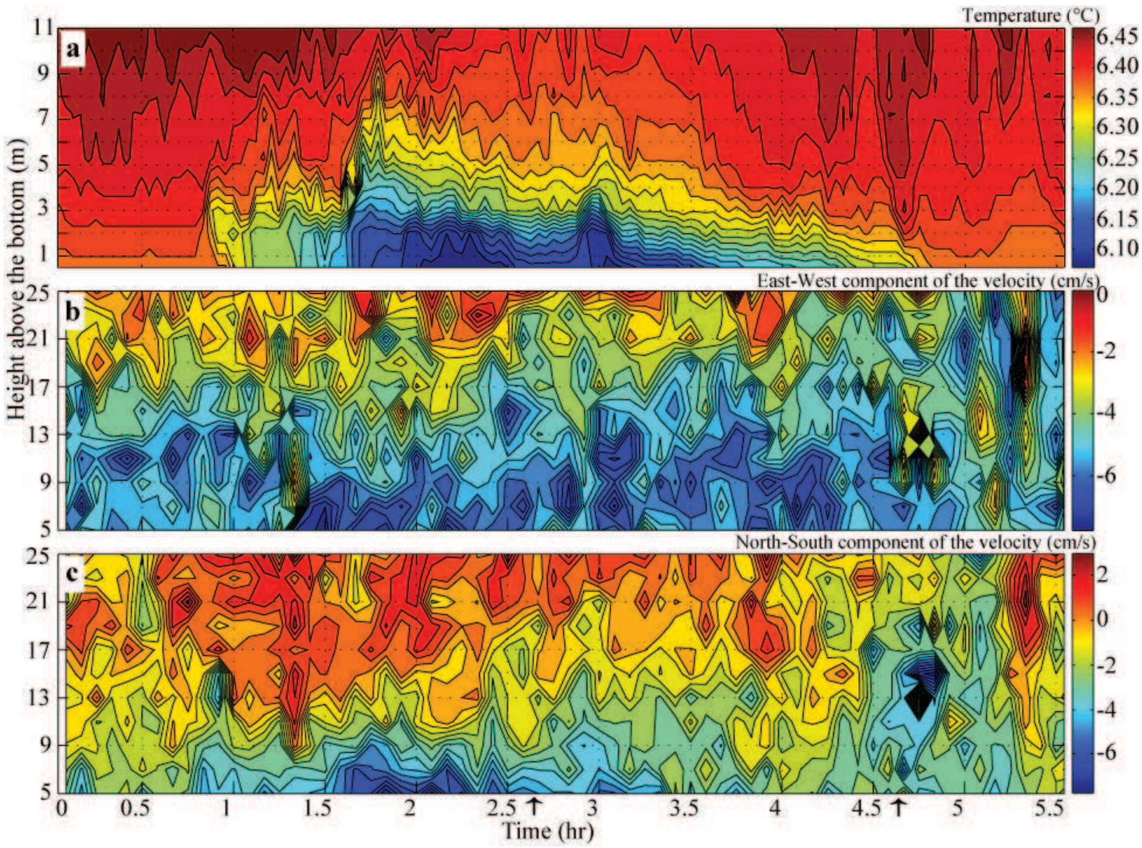

Figure 4: Contour plots of the recorded temperature and velocity data during an interval of 5.5, starting from 22 January 2004, 09:00 PM. Panel (a) shows temperatures, (b) East-West and (c) North-South components of the velocity. Data for case one is a 2 hour average taken between times marked on panel (c) (Ozin private communication).

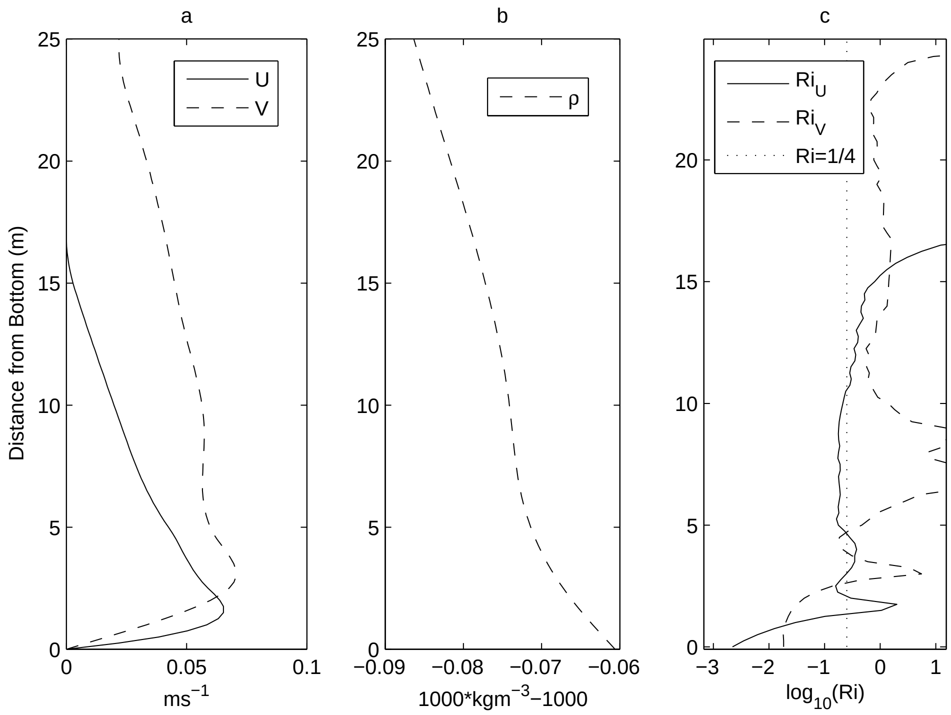

Figure 5: Density (panel a), down-slope (solid) and along-slope (dashed) velocity (panel b) and Richardson number (panel c) for case 1.

down slope and along slope components of the velocity. Measurements are taken on an incline which is typically of similar order to that shown in figure (2) and are chosen specifically to be those with steady slopes. The quality of the data reduces close to 25m from the bottom and the shear becomes weak. We thus match the data smoothly to constant profiles of density and velocity close to this height taken as $z = 1$.

The two examples we shall consider are characterised as follows: Case 1, shown in figure (5), the flow has a canonical shape in the down-slope direction, similar to those discussed in TO07 but there exists a significant mean flow in the along-slope direction. The flow around Lake Geneva is often cyclonic, the along-slope flow in this case being toward the West (the

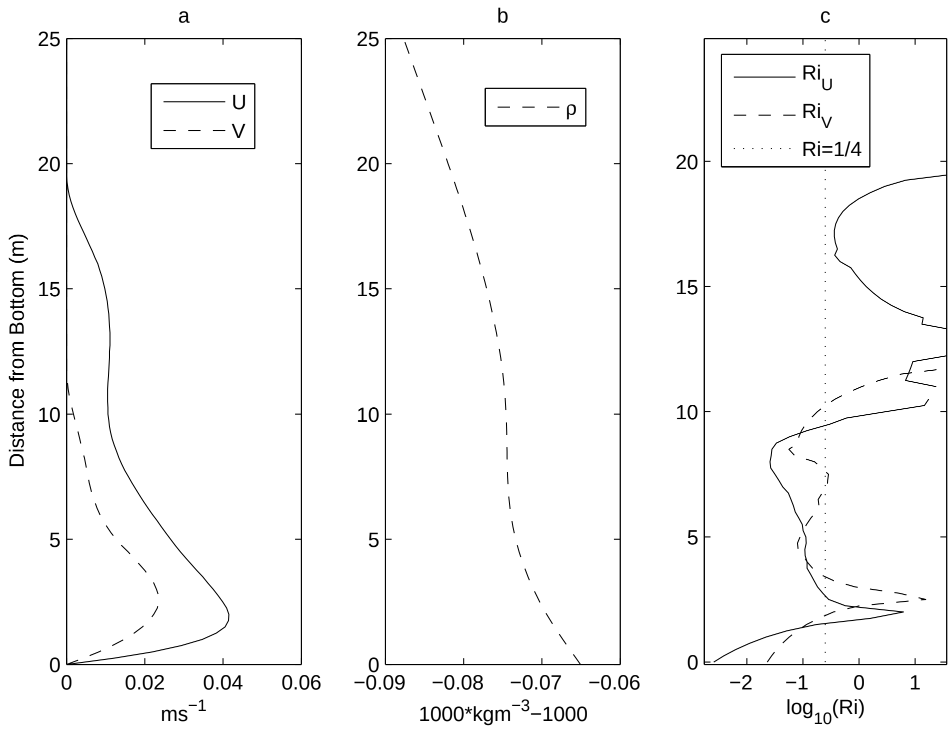

Figure 6: Density (panel a), down-slope (solid) and along-slope (dashed) velocity (panel b) and Richardson number (panel c) for case 2.

measurements are taken on the northern side of the lake). It is possible that Ekman effects are occurring in the boundary layer driving flow up the slope (i.e. to the right of the along slope flow) competing with the gravity driven cascade. We do not consider such effects in this study. The flow in case 2 (figure 6) is largely down-slope but displays a curious double hump structure perhaps due to the fluid mixing in various layers as it moves down the slope. The presence of two inflection points may allow the development of multiple modes of instability. In both case 1 and case 2 the Richardson number falls well below the 'critical' value of 1/4 and so the Miles-Howard theorem tells us that instabilities 'may' occur and linear stability to the K-H mechanism is not assured.

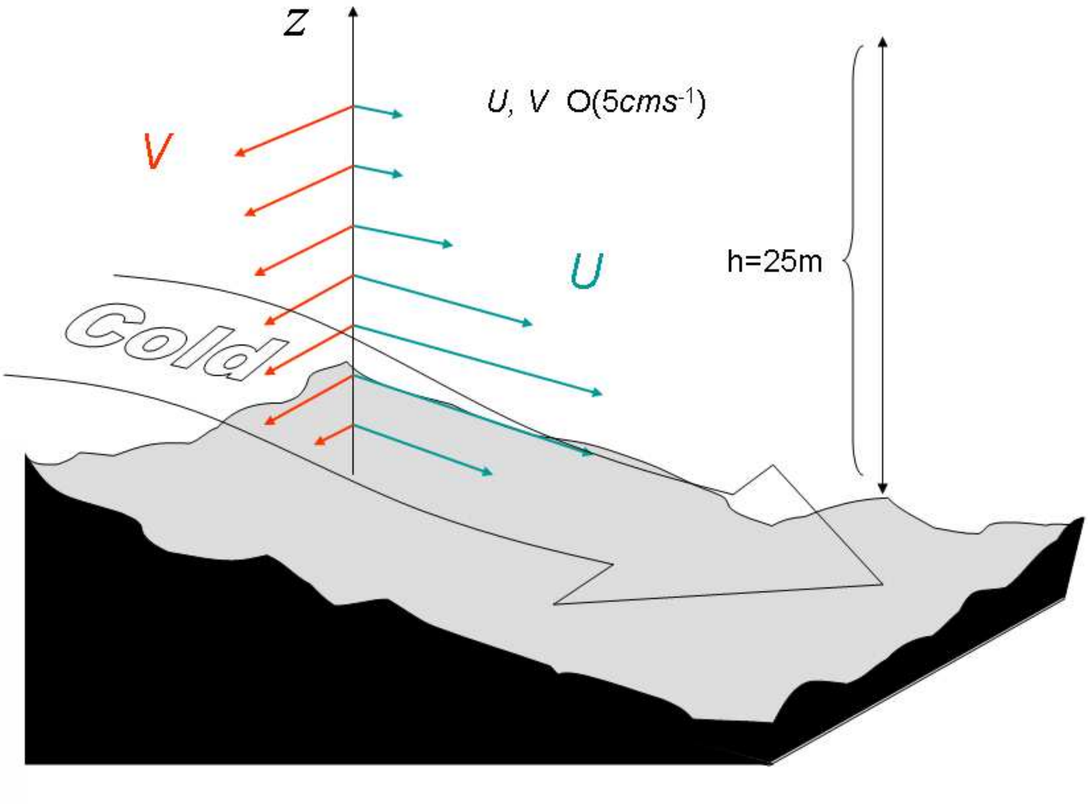

Figure 7: Image showing typical flow of cold water down slope (U) along slope mean flow (V) both of order $5cms^{-1}$ and measured over a vertical scale of 25m

4 Stability analysis of stratified shear flow

We wish to solve the Taylor Goldstein equation (2) searching for the Eigenvalue c of the fastest growing modes. As well as the condition on the upper boundary of the flow described in section (2), the solid boundary beneath the flow allows us to impose a no normal flow condition ($\phi(z=0)=0$). The Howard semicircle theorem states that $[C_r - 1/2(U_{max}+U_{min})^2] + C_i^2 \leq [1/2(U_{max}-U_{min})]^2$, where U_{max} and U_{min} are the maximum and minimum values of the velocity. Initial attempts to solve (2) were made using a shooting method similar to those of Merrill (1977), and TO07. For the measured values of U and ρ, coherent solutions to the Eigenvalue problem are not found and the solution is dominated by modes associated with numerical instabilities. Many techniques are attempted to abate such difficulties including, changing the grid resolution, constraining the modes possible and fitting the data with smoother cubic spline functions but no method yields coherent results for the observed velocity and density profiles. The scheme (which we will call M1), despite being checked against 'synthetic cases' generated from the hyperbolic tan profiles used by Hazel (see figure 1), is only able to resolve stability curves for the observed velocity and density profiles for increased Fr (discussed later) .

In order to avoid stability problems at small c_i and to allow more efficient search for eigenvalues, a second method (M2) is developed. In M2 we write (2) in the following form

$$(D^2 + F(c))\psi = 0 \tag{10}$$

where D^2 is the second derivative operator and $F(c)$ is function of the known mean velocity and density profiles and c. Eigenvalues c exist when the determinant of $D^2 + F(c)$ is zero. An advanced nonlinear root finder is used to find the roots of this expression and far more accuracy is found close to the observed Froude number than M2 but the same results are yielded for the cases shown in this report.

Our aim is to determine how far from being at the margin between stability and instability the observed profiles are or more precisely, how much would we need to increase or decrease the mean velocity (or change the stratification) to have a flow regime where infitesimal disturbances grow. In order to do this we introduce a factor ϵ by which we shall divide the Froude number. A value of $\epsilon = 0$ translates to an infinite mean velocity, no stratification or some combination of the two extremes while $\epsilon = 1$ translates to the observed velocity and density profiles. By starting at $\epsilon = 0$ and iterating toward $\epsilon = 1$ we can observe the trend in the maximum growth rates of the most unstable modes. This technique is similar to that used by Nielsen (1991) and TO07 increasing the bulk Froude number of the flow by the factor $1/\epsilon$ and extrapolating to a value of the Froude number such that the flow becomes marginally stable. Figure (8) shows such curves for case 1 for epsilon=0 to 0.5 in steps of 0.1.

As we have both components of the velocity we may continue this 2 dimensional analysis, but as in (Thorpe, 1999) we may orient the 2d disturbance in each direction θ where the velocity in that direction is $u_\theta = usin(\theta) - vcos(\theta)$ and $\psi(x, z, t)$ has the same meaning but the x direction is that of θ (see figure 9). We plot the maximum growth rates for each angle θ, again for $\epsilon = 0$ to 0.5 in figure (12). Focusing on angles between $\Theta = -90^o$ and $\theta = 0^o$ rotating between the along slope and down slope direction we estimate the critical values of the bulk Froude number and the critical value of the maximum Ri of the profile. We do this by extrapolating from the approximately linear re-

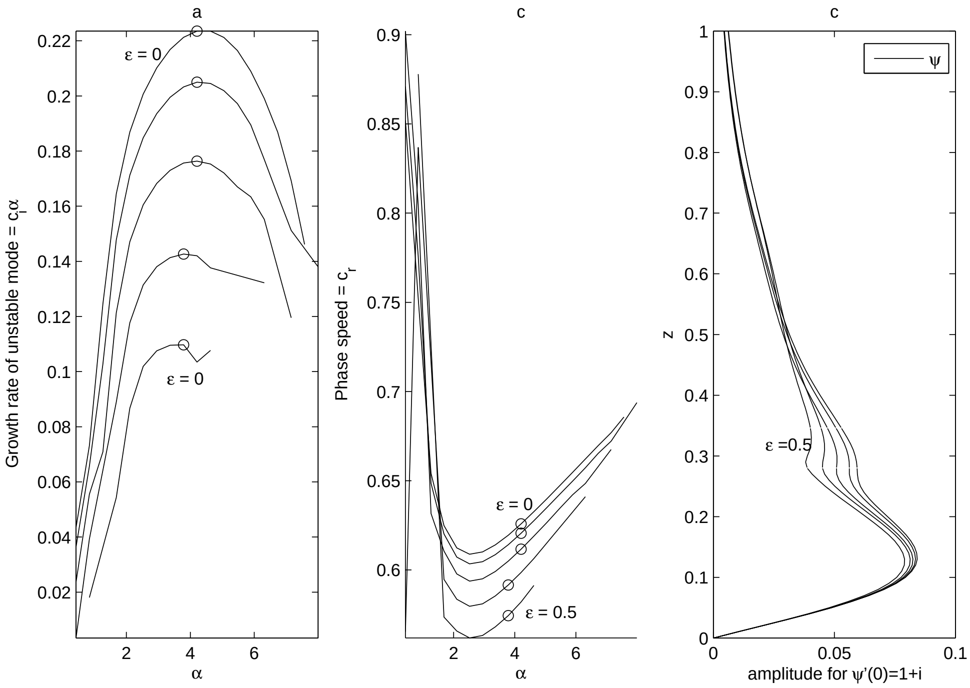

Figure 8: Nonsimensionalised Growth Rate (panel a), Phase Speed (panel b) and (panel c) for various values of ϵ for case 1.

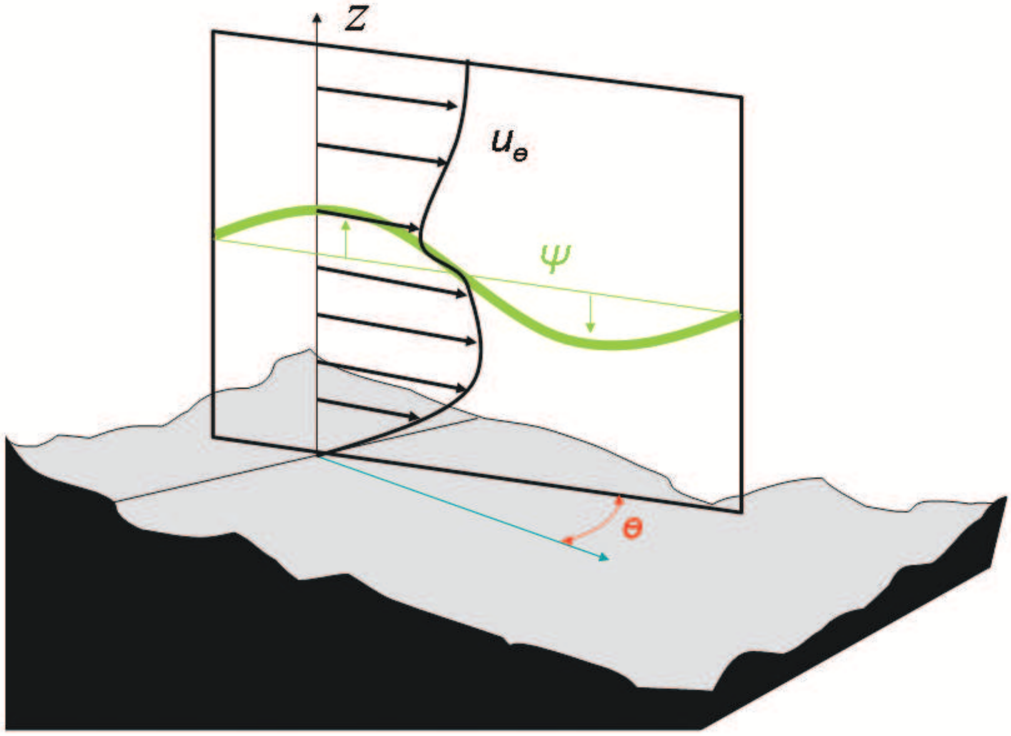

Figure 9: A cartoon showing how we may orient disturbances ψ angle θ to the down slope flow and use the velocity in that direction to conduct the stability analysis.

lationship between ϵ and $C_i\alpha$ to a point where the growth rate would change sign. An example of this linear relationship for the down slope flow of case 1 is shown in figure (10).

The manner in which such extrapolation is conducted is highly subjective and such a method is only used to give an estimate of whether the observed flow is 'close' (i.e. within some range of 10%-20%) to being stable/unstable.

We may follow the same process for case 2. Figure (13) shows the stability curves, phase speeds and growing modes for different values of ϵ for disturbances oriented in the down slope direction. Figure (14) again shows the result of rotating the 2d analysis over various angles θ. We see in figure (14)

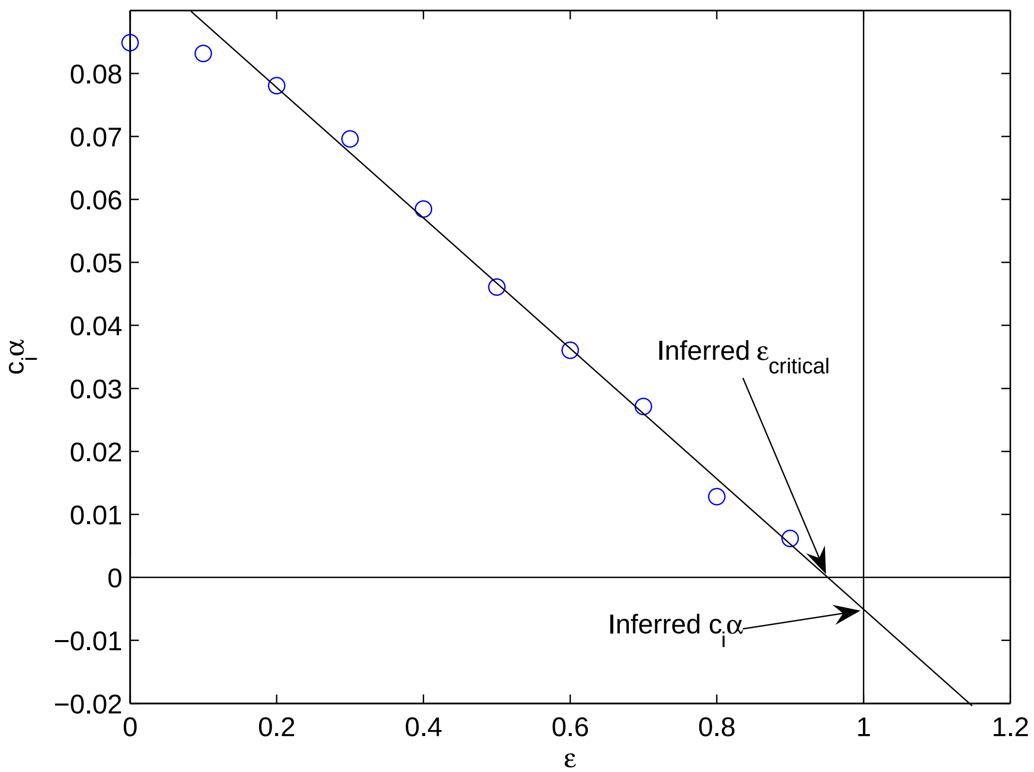

Figure 10: Case 1: Example of inference of $\epsilon_{critical}$ at which the flow is likely to become stable. In this case $\epsilon_{critical}$ is less than 1 and disturbances to the mean flow are not predicted to grow.

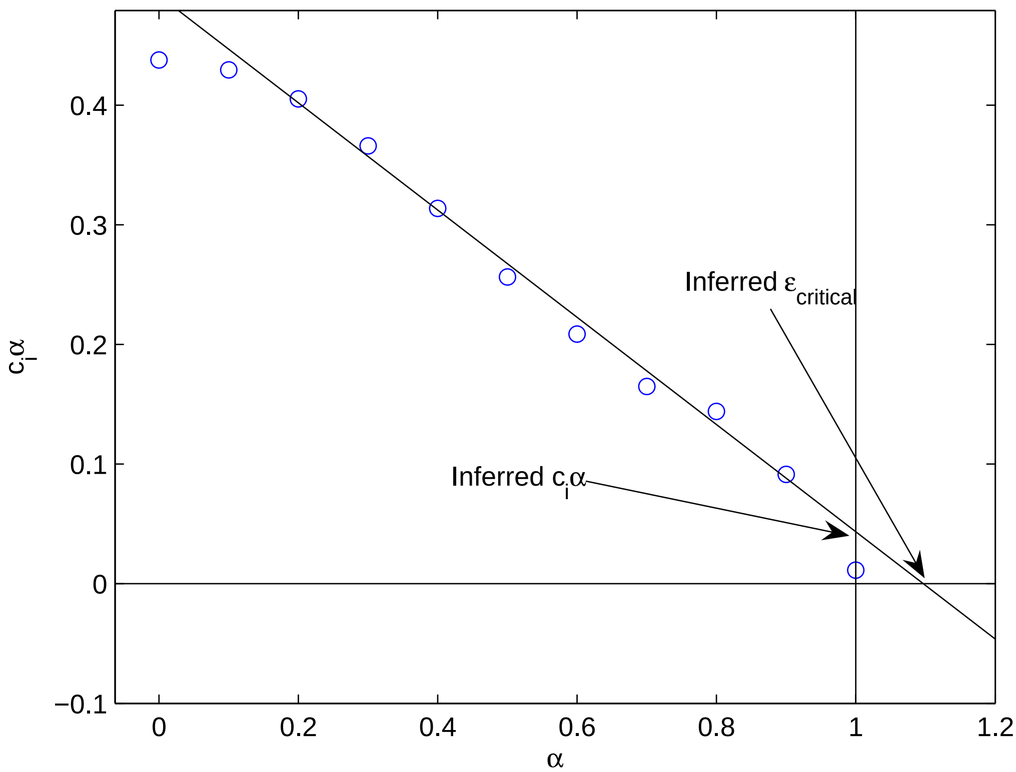

Figure 11: Case 2: Example of inference of $\epsilon_{critical}$ at which the flow marginally stable. In this case $\epsilon_{critical}$ is greater than 1 and disturbances to the mean flow are not predicted to grow at a rate $c_i\alpha$.

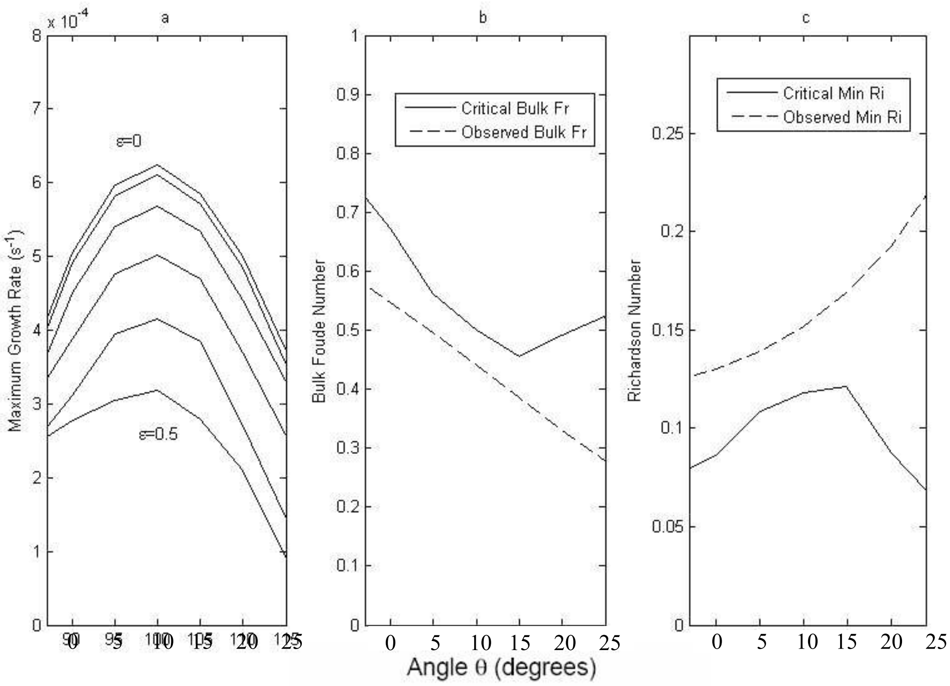

Figure 12: Case 1: Greatest Growth Rate at each angle (panel a) and for increasing $\epsilon * Fr$. Panel b shows a comparison between the observed bulk Froude number of the flow and the critical value inferred from extrapolating with increasing ϵ while panel c shows the same but for the peak Richardson number in that direction.

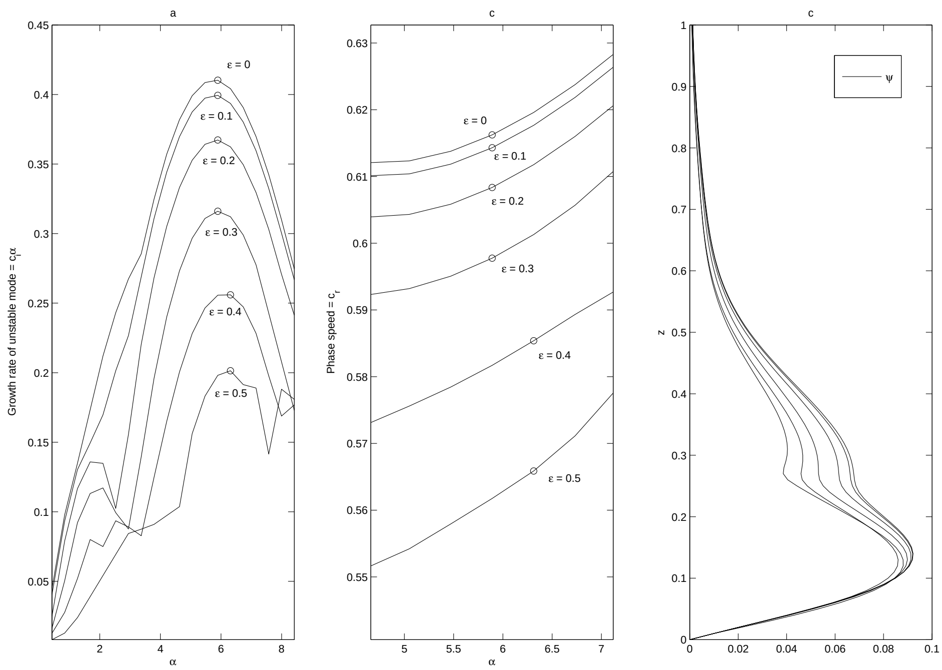

Figure 13: Nonsimensionalised Growth Rate (panel a), Phase Speed (panel b) and (panel c) for various values of ϵ for case 2.

that a decrease in the mean velocity of case 2 reducing the bulk Froude number from the observed 0.44 to only 0.38, a reduction in the maximum velocity of only 10-15%, would be enough to prevent perturbations from growing for all wavenumbers. The stability analysis of case 1 suggests the flow is stable as the extrapolation predicts a critical Froude number above that of the profile, however an increase in the maximum velocity difference of only 10% in the down stream direction would bring the flow into the estimated critical region.

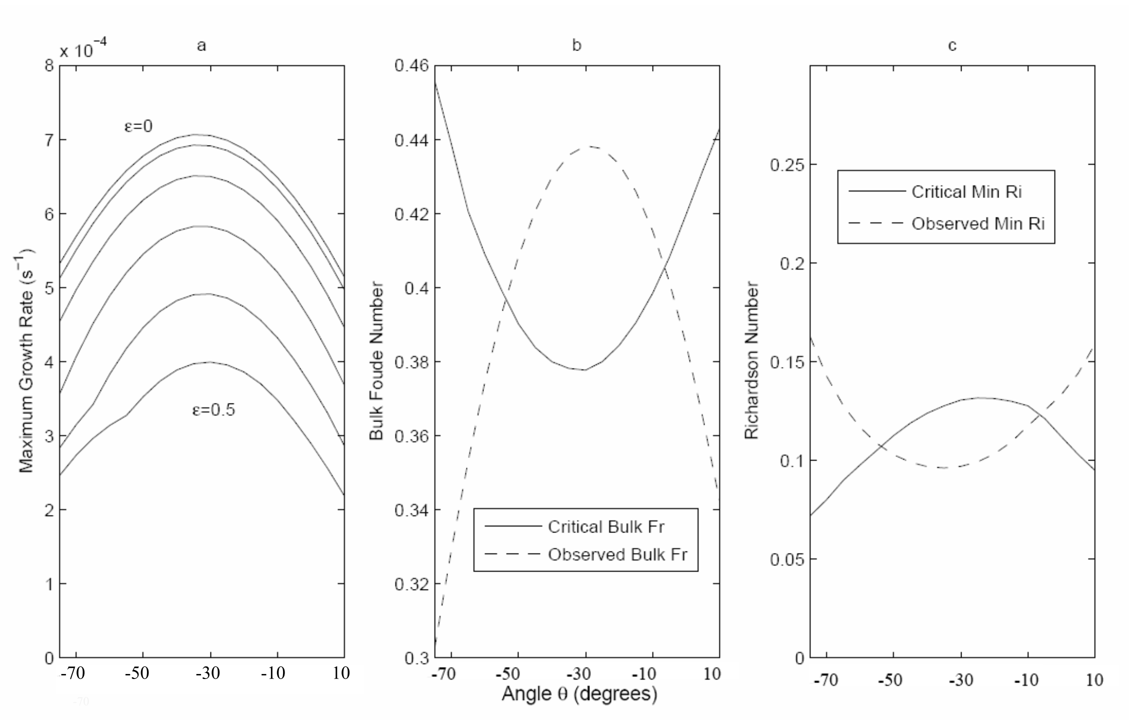

Figure 14: Case 2: Panel a shows the greatest growth rate at each angle and for increasing $\epsilon * Fr$. Panel b shows a comparison between the observed bulk Froude number of the flow and the critical value inferred from extrapolating with increasing ϵ while panel c shows the same but for the peak Richardson number in that direction.

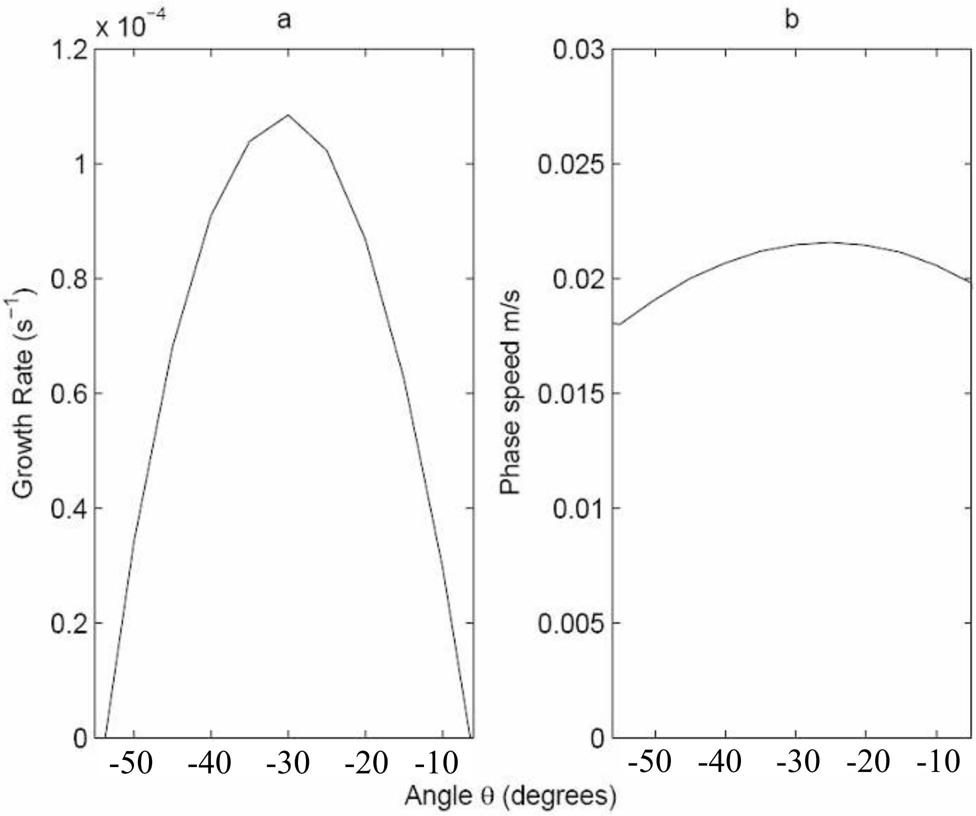

Figure 15: Predicted growth rates (panel a) and phase speeds (b) of most unstable modes at given orientations θ, of the perturbation. For all values shown the wavenumber k is approximately $0.24m^{-1}$ ($\lambda \approx 4.17m$)

5 Discussion and ongoing work

In this study we have looked at two stratified boundary flows and find them both to be close to marginally stable to Kelvin-Helmholtz instability. We define the flows as marginal if an increase or decrease of the observed mean velocity of around 10% would render the flow unstable or stable (respectively) to linear (infinitesimally small) perturbations.

We observe, in case 2 of the cascading flows from Lake Geneva, that instabilities are likely to occur in an orientation 60^o downslope of the along slope direction and with growth rates of order $10^{-4}s^{-1}$ (i.e. the perturbations would grow by a factor of $e = 2.71$ over a period of 2.5-3hr) phase speeds of order 0.02ms^{-1}, and a wavelength of approximately 4.2m. The velocity and Richardson number profile for this angle θ are shown in figure (16). Case 1 is predicted to be stable to K-H instability but an increase in flow velocity of approximately 10% would be likely, according to our analysis, to allow waves of lengths of order 6m and phase speeds around 4cms^{-1} to grow in a direction down the slope.

As was discussed in section (1) the presence of a boundary can greatly inhibit the growth of unstable modes and allow smaller Richardson numbers (i.e. larger shears) to exist without instabilities occurring. In the observed mean profiles, only when the minimum Richardson number is below 0.1 for 6-7m of the profile is it unstable. This may seem surprising to those who take $Ri = 1/4$ to be the critical point below which instability occurs but there exist analytical examples such as those of exponential u and ρ against a rigid boundary, where the flow is stable to all Richardson numbers. Reinforcing the assertion that the entire profile should be considered when conducting such stability analysis.

In the two cases discussed the along slope component of the flow had

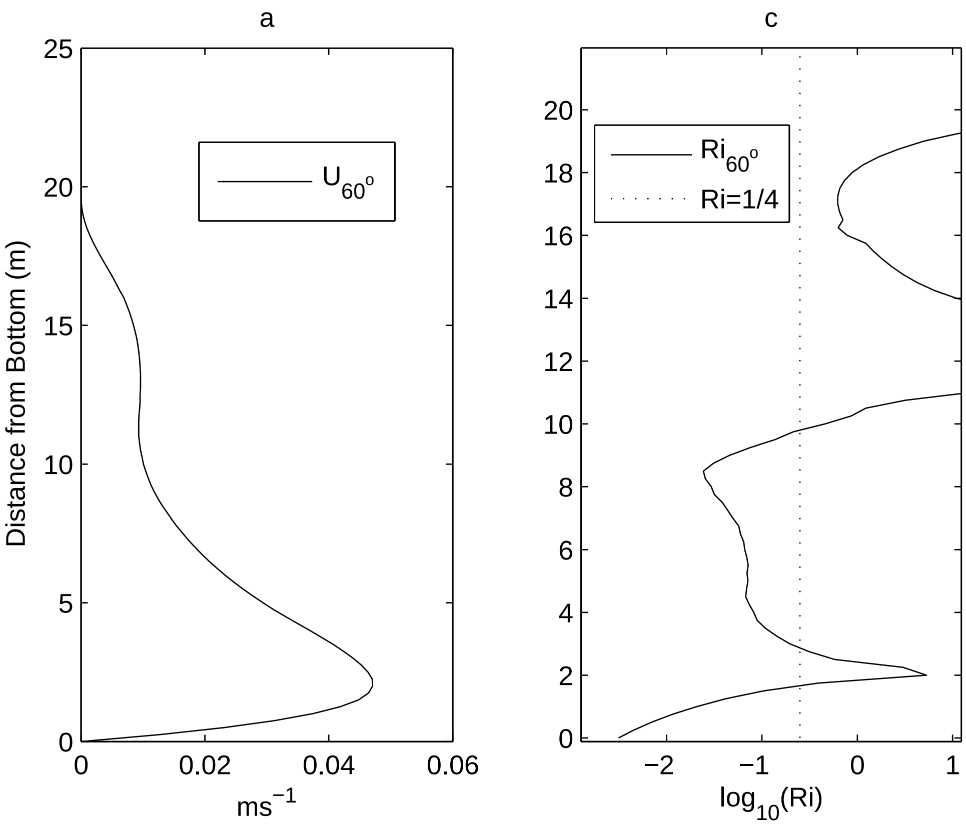

Figure 16: Velocity and Richardson number of flow oriented 30^o to the down slope direction, the orientation in which fastest growing disturbances are found.

a nontrivial influence on the stability. Indeed, although in case 2 the along slope component of the flow was much smaller than the down slope, the effect on the Richardson number was significant in both profiles displaying different minimum Ris at different depths (figure (6)) leading to a slightly different, and apparently more unstable profile at the angle of 30^o to the right of the down slope direction.

Although many cases remain to be explored, this work has presented evidence in support of Turner's conjecture that the mean state of stably stratified boundary flows is maintained in a state of marginal stability. It remains to be seen if deep ocean overflows and wind driven surface flows, can be shown to be stable in a similar way and whether simple techniques can then be applied to the output of numerical model data to asses its stability and constrain such a model.

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