

Elastic Critical Layers

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Abstract

We consider the weakly nonlinear growth of instabilities of a submerged elastic jet. We look at the large Weissenberg and Reynolds number cases with small and moderate elasticity. As in inviscid Newtonian shear flows, critical layers develop, but they are affected by the elastic properties of the fluid. At small elasticity, the early development of the critical layer is not significantly changed. At moderate elasticity, the critical layer splits into two different layers, whose location depends on the elasticity. The resulting amplitude equation is significantly altered from the Newtonian case.

1 Introduction

Rallison and Hinch [1] studied the inertial instability of a submerged elastic jet having a parabolic velocity profile. They used a large Weissenberg and Reynolds number limit and concentrated on the effects of elasticity on the instability. At the end of their paper, they found hints of a critical layer for certain parameter values. This critical layer disappears when elasticity is removed from the equations and so depends on elastic effects.

As in [1], we consider a jet of an elastic fluid entering into a motionless fluid. We can think of the motionless fluid as being the same material as the jet, or we can consider it to be Newtonian without any change in the governing equations. Because it is motionless, its elastic properties will not affect the dynamics.

The jet itself is 2 dimensional, rectilinear (i.e., the fluid particles all travel parallel), symmetric about $y = 0$ and bounded between $y = -L$ and $y = L$. We are primarily interested in the large Weissenberg and Reynolds number limit of this jet. Hence the relaxation time of the elastic fluid will be large in comparison to the shear rate and inertia will dominate viscosity.

The velocity profile of the jet which we use is similar to the Rallison and Hinch profile. It is $U(y) = V(L^2 - y^2)^2/L^4$. The choice of this profile will be explained in more detail later (and our results are not strongly dependent on the particular profile), but it is chosen so that U' is continuous between the jet and the ambient fluid. We find that critical layers will exist, and we concentrate on the influence these critical layers have on the weakly nonlinear evolution of the instability.

In section 2 we give a brief description of related results in Newtonian fluids and magneto-hydrodynamics. In Section 3 introduces the equations that govern the motion of our elastic fluid. Section 4 describes the linear problem to be solved assuming large Weissenberg number, derives some results about neutrally stable modes and discusses the

influence of those results on the weakly nonlinear analysis. Sections 5 and 6 look at the influence of an elasticity parameter $E \ll 1$ and $E \sim 1$ respectively, performing both the linear and weakly nonlinear analysis. Section 7 concludes this work and suggests future lines of attack.

2 Rayleigh's Equation

An inviscid Newtonian fluid can have a 2-dimensional flow profile $\mathbf{U} = (U(y), 0)$ where $U(y)$ is any function. If we restrict U to twice continuously differentiable functions, and look for linear disturbances $\mathbf{u} \exp[ik(x - ct)]$ then the linear stability is governed by Rayleigh's equation

$$\left[(U - c)^2 \left(\frac{\psi}{U - c} \right)' \right]' = k^2 (U - c) \psi$$

where ψ is the stream function for $\mathbf{u}$.

If c has positive imaginary part, then the disturbance will grow — the system is unstable. It was shown by Rayleigh that a necessary condition for instability is that $U'' = 0$ for some y .

In order to go a step beyond the linear analysis into a weakly nonlinear theory, we generally start from a mode which is neutrally stable, that is, c has zero real part. We then try to understand what happens as the growth rate is increased from zero to $\mathcal{O}(\epsilon)$. In the case of Rayleigh's equation, it can be shown that if c is real, then $U(y_c) = c$ for some y_c satisfying $U''(y_c) = 0$. In this case, a critical layer develops about where $U = c$, which is where the background flow is equal to the movement of the instability. It can be shown that although there is an apparent singularity in the differential equation, the solution for ψ is continuously differentiable.

A large amount of research has been done into this problem, as well as the effect that weak viscosity has (see [2] and references therein). In the presence of viscosity, we can no longer use an arbitrary flow $\mathbf{U}$. Generally, people will use a flow profile which does not satisfy the equations of motion, but justify it either by arguing that the time scale that the viscosity acts on is slower than the time scale of the instability or by explicitly adding a body force.

A paper by Hughes and Tobias [3] studies the linear stability of magneto-hydrodynamic shear flows. The linear stability has been studied by others as well (see references in [4]). Some papers by Shukhman [4, 5] have analyzed the weakly nonlinear problem in the presence of a magnetic field parallel to the flow. They used a modified Rayleigh equation

$$\left[[(U - c)^2 - c_A^2] \left(\frac{\psi}{U - c} \right)' \right]' = [(U - c)^2 - c_A^2] \frac{\psi}{U - c}.$$

Rather than the critical layer occurring where the background flow is as fast as the instability, the critical layer will occur here where the speed of the instability relative to the background flow is equal to the Alfvén wave speed.

The magneto-hydrodynamic version of the Rayleigh equation is similar to that which

we will derive for an elastic fluid (previously derived by [6, 1])

$$\left[[(U - c)^2 - 2EU'^2] \left(\frac{\psi}{U - c} \right)' \right]' = [(U - c)^2 - 2EU'^2] \frac{\psi}{U - c}.$$

Here the critical layer occurs where the speed of the instability relative to the background flow equals the elastic wave speed.

3 Basic equations

Because we are interested in an elastic fluid, we cannot use the Navier-Stokes equations. There are a wide variety of equations developed to describe elastic fluids. Many of them are applicable in different regimes, and none seem to be universally valid. The principal lectures for this year discuss them more completely.

We use the Oldroyd-B equations

$$\begin{aligned} \rho \frac{D\mathbf{U}}{Dt} &= -\nabla P + \mu \nabla^2 \mathbf{U} + G \nabla \cdot \mathbf{A} \\ \overset{\nabla}{\mathbf{A}} &= \frac{1}{\tau} (\mathbf{I} - \mathbf{A}) \\ \nabla \cdot \mathbf{U} &= 0. \end{aligned}$$

where $\mathbf{U}$ is the fluid velocity, P the pressure, ρ the density μ the viscosity, τ the relaxation time of the fluid and t time. Often G is considered to be C/τ where C is proportional to the concentration of a polymer in the fluid. It measures the strength of the fluid's response to stretching, while $\mathbf{A}$ measures the amount the fluid is stretched. The upper convected derivative is defined by $\overset{\nabla}{\mathbf{A}} = \frac{D\mathbf{A}}{Dt} - \mathbf{A} \cdot (\nabla \mathbf{U}) - (\nabla \mathbf{U})^T \cdot \mathbf{A}$.

We non-dimensionalize with a typical length scale L equal to the half-width of the jet and velocity scale V equal to the center-line velocity. Then using asterisks to denote the new non-dimensionalized variables, $\nabla^* = \frac{1}{L} \nabla$, $\mathbf{U}^* = V \mathbf{U}$, $\mu^* = \mu / \rho V L$, $P^* = P / \rho V^2$ and $t^* = \frac{V}{L} t$. In the base flow, A_{11} will be $1 + 2\tau^2 U_y^2$, and a characteristic value for U_y is V/L . Defining $\lambda = \text{Wi}^{-1} = L/V\tau$ we will normalize $\mathbf{A}$ by $\mathbf{A}^* = \lambda^2 \mathbf{A}$. Setting $E = G\tau^2 / \rho L^2$ and dropping the asterisks, we arrive at

$$\frac{D\mathbf{U}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{U} + E \nabla \cdot \mathbf{A} \quad (1)$$

$$\overset{\nabla}{\mathbf{A}} = \lambda^3 \mathbf{I} - \lambda \mathbf{A} \quad (2)$$

$$\nabla \cdot \mathbf{U} = 0. \quad (3)$$

Because of the length rescaling, the jet is now bounded between $y = -1$ and $y = 1$. The elasticity parameter E is independent of the speed of the base flow. It depends entirely on geometrical and material properties. We will take the Newtonian viscosity μ to be small. We are interested in the influence of the elasticity and the inverse Weissenberg number, λ on the growth of instabilities.

In the presence of nonzero viscosity or elasticity, the momentum equation (1) will not allow $\mathbf{U} = (U(y), 0)$ to be a solution. A body force $\mathbf{b}(y)$ may be added to the right

hand side to maintain this base flow. Alternatively, we can assume that the instability investigated develops over a short enough time scale that the base flow is effectively steady.

For a steady rectilinear flow, the elastic stress will reach a steady state, so $\frac{D\mathbf{A}}{Dt} = 0$. Expanding the upper convected derivative in equation (2) yields $-\mathbf{A} \cdot (\nabla \mathbf{U}) - (\nabla \mathbf{U})^T \cdot \mathbf{A} = \lambda^3 \mathbf{I} - \lambda \mathbf{A}$. Solving this we get

$$\mathbf{A} = \begin{pmatrix} 2U'^2 + \lambda^2 & \lambda U' \\ \lambda U' & \lambda^2 \end{pmatrix}.$$

We allow perturbations to the base flow so that the velocity is $\hat{\mathbf{U}} = \mathbf{U} + \mathbf{u}$ and the elastic stress is $\hat{\mathbf{A}} = \mathbf{A} + \mathbf{a}$. We substitute $\hat{\mathbf{U}}$ and $\hat{\mathbf{A}}$ into equations (1) and (2). Since the flow is two dimensional and incompressible, we introduce a streamfunction ψ such that $\mathbf{u} = (\psi_y, -\psi_x)$. We eliminate pressure by taking the curl of the momentum equation (1) yielding an equation for the vorticity ω and we expand the constitutive equation (2) giving

$$\nabla^2 \psi = -\omega \quad (4)$$

$$\omega_t + U\omega_x + U''\psi_x - J(\psi, \omega) = \mu \nabla^2 \omega + E[-\partial_{xy}a_{11} + (\partial_{xx} + \partial_{yy})a_{12} + \partial_{xy}a_{22}] \quad (5)$$

$$\mathbf{a}_t + U\mathbf{a}_x - J(\psi, \mathbf{a}) - \psi_x \mathbf{A}' - U' \begin{pmatrix} 2a_{12} & a_{22} \\ a_{22} & 0 \end{pmatrix} - \mathbf{F} - \mathbf{f} = -\lambda \mathbf{a}. \quad (6)$$

The Jacobian J satisfies $J(q, r) = q_x r_y - q_y r_x$. The tensors $\mathbf{F} = \mathbf{A} \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \cdot \mathbf{A}$ and $\mathbf{f} = \mathbf{a} \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \cdot \mathbf{a}$ are given by

$$\mathbf{F} = \begin{pmatrix} 2A_{11}\psi_{xy} + 2A_{12}\psi_{yy} & A_{22}\psi_{yy} - A_{11}\psi_{xx} \\ A_{22}\psi_{yy} - A_{11}\psi_{xx} & -2A_{12}\psi_{xx} - 2A_{22}\psi_{xy} \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} 2a_{11}\psi_{xy} + 2a_{12}\psi_{yy} & a_{22}\psi_{yy} - a_{11}\psi_{xx} \\ a_{22}\psi_{yy} - a_{11}\psi_{xx} & -2a_{12}\psi_{xx} - 2a_{22}\psi_{xy} \end{pmatrix}.$$

4 Linear Problem

We shall make the assumption that λ and μ are negligibly small, and so we reach a simpler expression for $\mathbf{F}$

$$\mathbf{F} = 2U'^2 \begin{bmatrix} 2\psi_{xy} & -\psi_{xx} \\ -\psi_{xx} & 0 \end{bmatrix}.$$

We now linearize the perturbation equations (4)–(6), holding on to the leading order linear terms. We seek solutions proportional to $\exp[ik(x - ct)]$. If c has positive imaginary part, then this mode will grow in time at a rate of $\Re[c]k$. It is referred to as *unstable*. If c has negative imaginary part, then the mode will decay in time and is called *stable*. The resulting relation between $\Re[c]k$ and k is a *dispersion relation*. It gives the growth rate as a function of the wavenumber k .

Equation (6) shows us that a_{22} and a_{12} are both much less than a_{11} and that

$$a_{11} = [4U'^2\psi_{xy} + 2U'^2\psi_x]/[ik(U - c)].$$

Substituting this into the vorticity equation (5) and dropping nonlinear terms provides

$$\frac{\partial}{\partial y} \left(\Gamma \frac{\partial}{\partial y} \eta \right) = k^2 \Gamma \eta \quad (7)$$

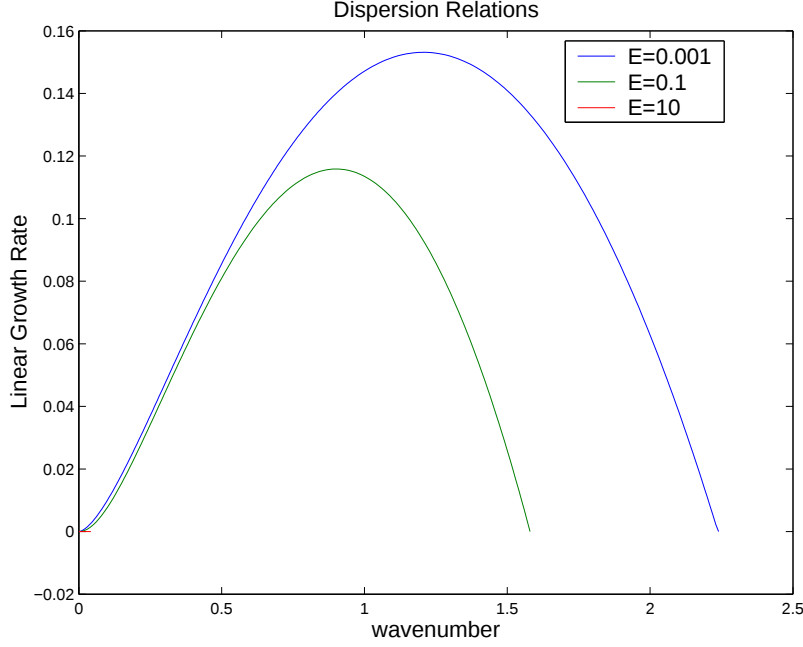


Figure 1: The dispersion relation for varicose modes with different values of E . The elasticity tends to stabilize the mode. For all cases, the growth rate is positive for sufficiently small k , and the mode disappears at some finite value of k .

where $\Gamma = (U - c)^2 - 2EU'^2$ and $\eta = \psi/(U - c)$. Note that Γ is continuous.

We will be looking for varicose and sinuous modes. Because of the symmetries of these modes, we can restrict our computations to just looking at half of the jet. For a varicose mode, the perturbation has no flow across the center line of the jet. Consequently, $\psi_x = 0$ at $y = 0$. This means that ψ is constant on the center line. Since ψ can have an arbitrary constant added to it, we choose that constant to make $\psi(0) = 0$ for a varicose mode.

Conversely, for a sinuous mode, the perturbation has no flow along the center line of the jet. Consequently for a sinuous mode $\psi_y(0) = 0$.

To find the boundary conditions at $y = \pm 1$, we observe that for $|y| > 1$, the value of Γ is c^2 . Thus η solves

$$c^2 \eta'' = k^2 c^2 \eta$$

and so $\eta = C_1^+ \exp(ky) + C_2^+ \exp(-ky)$ for $y \geq 1$ and $\eta = C_1^- \exp(ky) + C_2^- \exp(-ky)$ for $y \leq -1$. We assume that η decays as $|y| \rightarrow \infty$ so $C_1^+ = C_2^- = 0$. We use this to choose boundary conditions at $y = \pm 1$. The boundary condition we apply at 1 is that $\Gamma \eta' = -kc^2 \eta$ and at $y = -1$, $\Gamma \eta' = kc^2 \eta$. Either of these conditions along with the conditions previously discussed at $y = 0$ will suffice to determine the solutions. However, for what immediately follows, it is easier to use the conditions at ± 1 .

We are interested in conditions under which we can have a marginally stable mode, that is, a solution where c is real. Clearly if c is such that $\Gamma = 0$ for some value of y , then c is real and the differential equation will be singular. The values of c which allow this will form

a continuous set, the continuous spectrum of the problem. We will now show that if c is not in the continuous spectrum, then c has nonzero imaginary part. For generality, we do not make many assumptions on U here. We take only that U is continuously differentiable and that $U(-1) = U(1) = 0$.

Assume that c is real but outside of the continuous spectrum. This means Γ is nowhere 0. We multiply equation (7) by η^* , the complex conjugate of η , and integrate from -1 to 1 . One integration by parts gives

$$[\Gamma\eta'\bar{\eta}]_{-1}^1 - \int_{-1}^1 \Gamma|\eta'|^2 dy = k^2 \int_{-1}^1 \Gamma|\eta|^2 dy.$$

The boundary term evaluates to $-kc^2(|\eta(-1)|^2 + |\eta(1)|^2) \leq 0$. Since $U(-1) = U(1)$, the mean value theorem can be used to prove that $U' = 0$ at some point in the interior. At this point $\Gamma = (U - c)^2 - 2EU'^2 \geq 0$. By assumption, $\Gamma \neq 0$, hence Γ is positive at some $y \in (-1, 1)$. Since Γ is continuous in this interval, and is nowhere 0, it is positive throughout. Both integrals are positive, and thus the left hand side is negative while the right hand side is positive, a contradiction.

We have shown that all real eigenvalues c lie within the continuous spectrum. Thus if an unstable mode stabilizes, the eigenvalue is actually entering the continuous spectrum. This will substantially complicate the nonlinear analysis. Generally when we attempt a weakly nonlinear analysis, we separate the dynamics into a small number of slowly growing or neutrally stable modes on which we focus along with some quickly decaying modes which are ignored. We then get coupled ODEs relating the amplitudes of these modes. Here there is a continuum of slow modes, so we cannot reduce the problem to even a finite set of modes, much less a small number. Consequently we will arrive at a PDE rather than the ODEs.

It has been shown [7, 8] that a jump in first normal stress (A_{11}) can lead to an instability at zero Reynolds number. If there were a discontinuity in U' , then there would be such a jump, and we might expect it to play a significant role in the dynamics. To simplify our analysis, we will not investigate that effect. To prevent this from occurring, we need U' continuous everywhere, including $y = \pm 1$. This is why we have chosen $U(y) = (1 - y^2)^2$. The theoretical results we obtain here do not depend strongly on this form. Our numerical work has shown qualitatively similar behavior for other flow profiles.

We expect to find a neutrally stable mode proportional to $\exp[ik(x - ct)]$ which goes unstable. We will attempt a weakly nonlinear analysis of this mode, looking for modulations over a long time scale $T = \epsilon^{-1}t$ where $\epsilon \ll 1$.

We change to a frame moving with the disturbance, and so ∂_t is replaced by $-c\partial_x + \epsilon\partial_T$. The equations (4)–(6) become

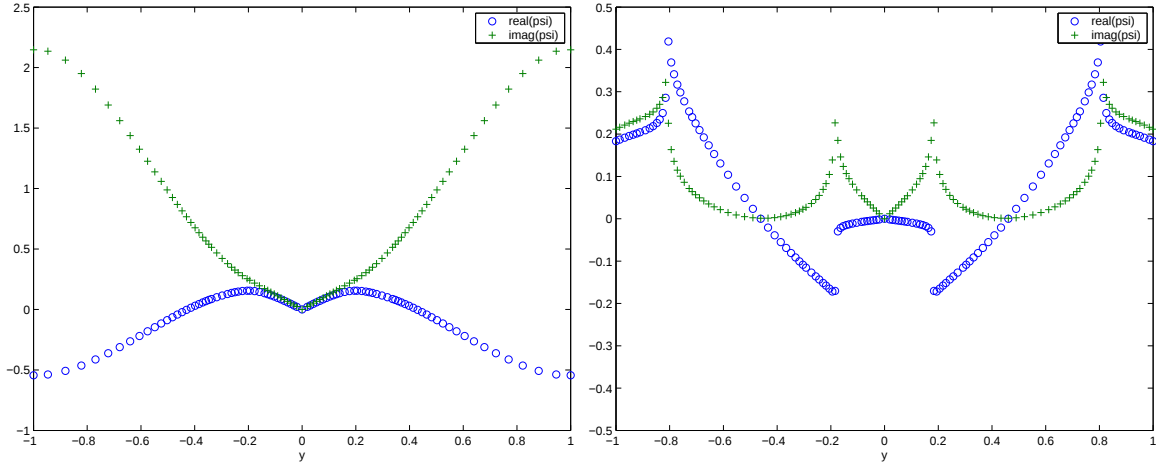
$$\nabla^2\psi = -\omega \tag{8}$$

$$\epsilon\omega_T + (U - c)\omega_x + U''\psi_x - J(\psi, \omega) = \mu\nabla^2\omega + E[-\partial_{xy}a_{11} + (\partial_{xx} + \partial_{yy})a_{12} + \partial_{xy}a_{22}] \tag{9}$$

$$\epsilon\mathbf{a}_T + (U - c)\mathbf{a}_x - J(\psi, \mathbf{a}) - \psi_x\mathbf{A}' - U' \begin{pmatrix} 2a_{12} & a_{22} \\ a_{22} & 0 \end{pmatrix} - \mathbf{F} - \mathbf{f} = -\lambda\mathbf{a}. \tag{10}$$

These are the equations we must use for the weakly nonlinear analysis.

We consider two cases



(a) Eigenfunction for a varicose mode with $E = 0.001$ with a value of k very close to where the mode disappears. Note that the solution is smooth.

(b) Eigenfunction for a varicose mode with $E = 0.1$ with a value of k very close to where the mode disappears. Note the two singularities for positive and negative y .

Figure 2: Plots of the varicose eigenfunctions for small and moderate elasticities.

- $E \ll 1$, $\lambda \ll 1$
- $E \sim 1$, $\lambda \ll 1$.

5 Small E , small λ

There are two limits which are of interest here. In the limit where $E \sim \epsilon^2$, $\lambda \sim \epsilon$ the elastic stresses appear in the leading order balance inside the critical layer. Almost all of the terms are of the same order, so this limit turns out to be quite hard. It corresponds to the scaling at which the critical layer splits into two layers whose width is comparable to the distance between them. The interaction between the two layers is important. We do not discuss this limit here.

The limit where $E \sim \epsilon^4$, $\lambda \sim \epsilon$ is more tractable. We set $E = \epsilon^4 E_4$ and $\lambda = \epsilon \lambda_1$. The elastic stresses do not appear at leading order inside the critical layer.

When the instability begins to appear, we expect ψ to be very small. As the instability develops, ψ should grow and saturate at some size ξ . We seek an appropriate relation between ξ and ϵ . The dominant terms in equation (9) are $\epsilon \omega_T$, $(U - c)\omega_x$, $U''\psi_x$ and $J(\psi, \omega)$. Inside the critical layer, U is about 0, and can be approximated by $(y - y_c)U'_c$. For small T , the balance between $\epsilon \omega_T$ and $(y - y_c)U'_c \omega_x$ tells us that $y - y_c = \mathcal{O}(\epsilon)$ and so the proper length scale inside the critical layer is $Y = \epsilon^{-1}(y - y_c)$. As T grows, we expect $J(\psi, \omega)$ to become order $\epsilon^{-1}\xi\omega$, which should be comparable in size to $\epsilon\omega$. This gives $\xi = \epsilon^2$.

We typically have a long wave instability which stabilizes at higher wave number k_0 . We make the assumption that our domain is such that the longest wave possible corresponds

to the wavelength at which the modes restabilize. A small perturbation to the domain size allows an unstable mode to develop with wavenumber $k = k_0 + \epsilon k_1$. Using our scaling for ψ , we have (for the outer solution)

$$\begin{aligned}\psi &= \psi_2 \epsilon^2 + \psi_3 \epsilon^3 + cc + hot \\ \omega &= \omega_2 \epsilon^2 + \omega_3 \epsilon^3 + cc + hot\end{aligned}$$

where cc denotes complex conjugate and hot denotes higher order terms. We expect ψ_2 and ω_2 to be proportional to $\exp(ikx)$ and separable in y and T . Let $\psi_2 = B(T)\hat{\psi}_2(y)e^{ikx}$ and $\omega_2 = B(T)\hat{\omega}_2(y)e^{ikx}$. At $\mathcal{O}(\epsilon^2)$ equation (8) becomes $\hat{\psi}_2'' - k_0^2 \hat{\psi}_2 = -\hat{\omega}_2$ and equation (9) is $(U - c)\hat{\omega}_2 + U''\hat{\psi}_2 = 0$ which combine to give

$$(U - c)[\hat{\psi}_2'' - k_0^2 \hat{\psi}_2] - U''\hat{\psi}_2 = 0 \quad (11)$$

which is identical to the Newtonian case. As in the Newtonian case, the mode of interest satisfies $U = c$ when $U'' = 0$. For future reference we define a linear operator $\mathcal{L}$ such that $\mathcal{L}[\hat{\psi}_2] = 0$

$$\mathcal{L}[\psi] := (U - c)(\psi_{yy} - k_0^2 \psi) - U''\psi.$$

Solving this linear problem gives us information about the shape of $\hat{\psi}_2$, but tells us nothing about the evolution of B . That will come from the next order.

At $\mathcal{O}(\epsilon^3)$ equations (8) and (9) become

$$\begin{aligned}\nabla^2 \psi_3 - 2k_0 k_1 \psi_2 &= -\omega_3 \\ -\omega_{2T} &= (U - c)\omega_{3x} + U''\psi_{3x}.\end{aligned}$$

The $k_0 k_1$ term comes from the fact that $\partial_x^2 \psi_2 = (-k_0^2 - \epsilon 2k_0 k_1 - \epsilon^2 k_1^2)\psi_2$. We are only interested in the part of ψ_3 proportional to $\exp(ikx)$, which we express $\hat{\psi}_3$. The x derivatives again become multiplication by ik . Combining these equations gives

$$\mathcal{L}[\hat{\psi}_3] = -iB_T \hat{\omega}_2 / k_0 + 2B(T)(U - c)k_0 k_1 \hat{\psi}_2. \quad (12)$$

We multiply this by $\hat{\psi}_2^*/(U - c)$ and integrate from $y = -1$ to 0 (the 0 to 1 contribution follows similarly). This will give us a differential equation to solve for B . A complication arises at y_c , so instead we integrate over $(-1, y_c - \delta)$ and $(y_c + \delta, 0)$ for a δ determined later, but assumed to be small. On the right hand side we approximate this integral in the limit $\delta \rightarrow 0$ with a principle value integral so that the right hand side evaluates to $iI_0 B_T + k_1 I_1 B$ where $I_0 = \int_{-1}^0 |\psi_2|^2 U'' / k_0 (U - c)^2 dy$ and $I_1 = \int_{-1}^0 2k_0 |\psi_2|^2 dy$.

When we integrate the left hand side by parts, we get

$$\begin{aligned}\int_{-1}^{y_c - \delta} \frac{\hat{\psi}_2^* \mathcal{L}[\hat{\psi}_3]}{U - c} + \int_{y_c + \delta}^0 \frac{\hat{\psi}_2^* \mathcal{L}[\hat{\psi}_3]}{U - c} &= \int_{-1}^{y_c - \delta} \frac{\hat{\psi}_3 \mathcal{L}[\hat{\psi}_2^*]}{U - c} + \int_{y_c + \delta}^0 \frac{\hat{\psi}_3 \mathcal{L}[\hat{\psi}_2^*]}{U - c} \\ &\quad + [\hat{\psi}_2^* \hat{\psi}_{3y}]_{-1}^{y_c - \delta} + [\hat{\psi}_2^* \hat{\psi}_{3y}]_{y_c + \delta}^0 - [\hat{\psi}_{2y}^* \hat{\psi}_3]_{-1}^{y_c - \delta} - [\hat{\psi}_{2y}^* \hat{\psi}_3]_{y_c + \delta}^0 \\ &\rightarrow \hat{\psi}_2^* \left[\left[\hat{\psi}_{3y} \right] \right]_c - \hat{\psi}_{2y}^* \left[\left[\hat{\psi}_3 \right] \right]_c \quad \text{as } \delta \rightarrow 0\end{aligned}$$

where we use the fact that $\hat{\psi}_2^*$ satisfies $\mathcal{L}[\psi] = 0$, that the boundary conditions at -1 and 0 cause boundary terms to disappear and that continuity in $\hat{\psi}_2^*$ across y_c means the jump in $\hat{\psi}_2^*$ is 0 . To evaluate the remaining jumps we need to know more about $\hat{\psi}_3$ close to y_c .

Close to y_c , we will use a Taylor Series approximation for $U - c$. At y_c , $U = c$ and $U'' = 0$, so $U - c = (y - y_c)U'_c + (y - y_c)^3 U'''_c / 6 + \dots$. As $y \rightarrow y_c$, $\hat{\psi}_3$ diverges and so to leading order in $y - y_c$ equation (12) becomes

$$\hat{\psi}_3'' = -\frac{iB_T\hat{\omega}_2}{k_0(y - y_c)U'_c}$$

which gives a solution of the form

$$\hat{\psi}_3 = -(y - y_c)B_TQ \ln |y - y_c| + |y - y_c|\gamma + R$$

where Q and R are regular functions of y and γ is a constant measuring the jump in $\hat{\psi}_{3y}$ across the critical layer. Note that $\hat{\psi}_3$ is continuous across the critical layer, so we arrive at

$$iI_0B_T + k_1I_1B = 2\hat{\psi}_{2c}^*\gamma. \quad (13)$$

Our solution for $\hat{\psi}_3$ gives us no information about γ . However, it does show that $\epsilon\psi_3$ becomes larger than ψ_2 as $y \rightarrow y_c$. This continues at even higher orders, and the asymptotic expansion which we have assumed for ψ will fail close to y_c . Even before determining this solution, we could see that it would fail because in obtaining an equation for ψ_3'' , we neglected terms which we expect to be small. However, we also divided by $U - c$, and so as $y \rightarrow y_c$ some discarded terms will inevitably become unbounded.

We will have to resolve the critical layer more carefully in order to retain an asymptotic solution. In the process, we will be able to determine the value of γ , which allows us to find B . We will introduce a new space variable Y satisfying $y - y_c = \epsilon Y$. Thus $\partial_y \mapsto \frac{1}{\epsilon}\partial_Y$. Inside the critical layer $U - c$ will be given by $U - c = \epsilon Y U'_c + \frac{\epsilon^3 Y^3}{6} U'''_c$ using the observation that $U_c - c = U''_c = 0$. We will match the outer solution with the inner solution at $y = \delta$ corresponding to $Y = \Delta$. Δ satisfies $1 \ll \Delta \ll \epsilon^{-1}$ so that $\delta = \epsilon\Delta \ll 1$.

The outer solution evaluated at $y_c + \delta$ is

$$\begin{aligned} \psi &= \epsilon^2\psi_{2c} + \epsilon^2\delta\psi'_{2c} + \epsilon^3\psi_3 + \dots \\ &= \epsilon^2\psi_{2c} + \epsilon^3\Delta\psi'_{2c} - \epsilon^4\Delta \ln |\epsilon| B_TQ - \epsilon^4\Delta B_TQ \ln |\Delta| + \epsilon^4|\Delta|\gamma + \epsilon^3R_c + \dots \\ \omega &= \epsilon^2\omega_{2c} + \dots \end{aligned}$$

5.1 Inner solution

To match we take an inner solution of the form

$$\begin{aligned} \psi(x, Y, T) &= \epsilon^2\Psi_2(x, T) + \epsilon^3[\Psi_3(x, T) + Y\Phi_3(x, T)] + (\epsilon^4 \ln \epsilon)Y\Psi_{31/2}(x, T) + \epsilon^4\Psi_4(x, Y, T) + \dots \\ \omega &= \epsilon^2Z_2 + \dots \end{aligned}$$

Note that the only dependence of ψ on Y is in $Y\Phi_3$ and Ψ_4 . Ψ_4 will be allowed to grow large as $Y \rightarrow \Delta$. It is immediately obvious that

$$\begin{aligned}\Psi_2(x, T) &= \psi_{2c}(x, T) \\ \Phi_3(x, T) &= \psi_{2c}'(x, T) \\ \Psi_{31/2}(x, T) &= -B_T Q \\ \Psi_3(x, T) &= R_c(x, T).\end{aligned}$$

The remaining terms from the outer solution will match with Ψ_4 . In particular $2\Delta\psi_{2c}'' + [[\psi_3']] = [\Psi_{4Y}]_{-\Delta}^\Delta$. We know ψ_{2c}'' , so we just need to find $[\Psi_4]_{-\Delta}^\Delta$ in order to get $\gamma = [[\psi_3']] / 2$.

From the $\mathcal{O}(\epsilon^2)$ component of equation (8), we get $\Psi_{2xx} + \Psi_{4YY} = -Z_2$ and so

$$[\Psi_{4Y}]_{-\Delta}^\Delta = - \left(\int_{-\Delta}^\Delta Z_2 dY + 2\Delta\Psi_{2xx} \right).$$

Substituting for $[\Psi_{4Y}]_{-\Delta}^\Delta = 2\Delta\psi_{2c}'' + 2\gamma$ we get

$$2\gamma = - \left(\int_{-\Delta}^\Delta Z_2 dY + 2\Delta(\Psi_{2xx} + \psi_{2c}'') \right).$$

Since $\Psi_2 = \psi_{2c}$, we can rewrite the final term as $2\Delta(\psi_{2xx} + \psi_{2c}'')$. L'Hôpital's rule and equation (11) tell us that this is $2\Delta U_c''' \psi_{2c} / U_c'$. Finally substituting for ψ_{2c} with Ψ_2 , we reach

$$2\gamma = - \int_{-\Delta}^\Delta \left(Z_2 + \frac{U_c'''}{U_c'} \Psi_2 \right) dY.$$

One further change of variables $\zeta = -Z_2 - U_c''' \Psi_2 / U_c'$ reduces our problem to finding ζ .

Taking equation (9) will give a PDE for ζ . The $\mathcal{O}(\epsilon^2)$ terms will be zero since $U_c - c = U_c'' = 0$. We are left

$$\zeta_T + \frac{U_c'''}{U_c'} \Psi_{2T} + Y U_c' \zeta - \Psi_{2x} \zeta_Y = \epsilon E_4 [-\epsilon^{-1} \partial_{xY} a_{11} + (\partial_{xx} - \epsilon^{-2} \partial_{YY}) a_{12} + \epsilon^{-1} \partial_{xY} a_{22}].$$

We need to determine how the stresses a_{11} , a_{12} and a_{22} scale.

Using equation (10) and the leading order approximation for $U - c$ we arrive at

$$\begin{aligned}\epsilon Y U_c' a_{22,x} + \epsilon a_{22,T} - \epsilon^{-1} J_Y(\psi, a_{22}) - F_{22} - f_{22} &= -\epsilon \lambda_1 a_{22} \\ \epsilon Y U_c' a_{12,x} + \epsilon a_{12,T} - \epsilon^{-1} J_Y(\psi, a_{12}) - U_c' a_{22} - F_{12} - f_{12} &= -\epsilon \lambda_1 a_{12} \\ \epsilon Y U_c' a_{11,x} + \epsilon a_{11,T} - \epsilon^{-1} J_Y(\psi, a_{11}) - 2U_c' a_{12} - F_{11} - f_{11} &= -\epsilon \lambda_1 a_{11}\end{aligned}$$

where $J_Y(q, r) = q_x r_Y - q_Y r_x$. Looking at the order of the driving terms in each component suggests that $a_{11} = \alpha_{11}$, $a_{12} = \epsilon \alpha_{12}$ and $a_{22} = \epsilon^2 \alpha_{22}$ is a good scaling.

This yields

$$Y U_c' \alpha_{22,x} + \alpha_{22,T} - \Psi_{2x} \alpha_{22,Y} - 2\lambda_1 U_c' \Psi_{2xx} - 2\alpha_{12} \Psi_{2xx} = -\lambda_1 \alpha_{22} \quad (14)$$

$$Y U_c' \alpha_{12,x} + \alpha_{12,T} - \Psi_{2x} \alpha_{12,Y} - U_c' \alpha_{22} + 2U_c'^2 \Psi_{2xx} + \alpha_{11} \Psi_{2xx} = -\lambda_1 \alpha_{12} \quad (15)$$

$$Y U_c' \alpha_{11,x} + \alpha_{11,T} - \Psi_{2x} \alpha_{11,Y} - 2U_c' \alpha_{12} = -\lambda_1 \alpha_{11} \quad (16)$$

coupled together with the PDE

$$\zeta_T + \frac{U_c'''}{U_c'} \Psi_{2T} + Y U_c' \zeta - \Psi_{2x} \zeta_Y = E_4 [\partial_{xY} \alpha_{11} - \partial_{Y Y} \alpha_{12}]. \quad (17)$$

Solving this system and taking the limit $\Delta \rightarrow \infty$, we can find $\gamma = \int_{-\infty}^{\infty} \hat{\zeta}/2 dY$. Then finally we have

$$i I_0 B_T + k_1 I_1 B = \hat{\psi}_{2c}^* \int_{-\infty}^{\infty} \hat{\zeta} dY \quad (18)$$

5.1.1 Linearization

The coupled system of partial differential equations is nonlinear and generally difficult. To solve it completely would demand a numerical attack. We can still manage some progress through theoretical approaches. We follow [2] and references therein.

We can linearize these equations to get some idea of the early growth of the mode prior to its saturation. The linear equations are

$$(\partial_T + \lambda_1 + ik U_c' Y) \alpha_{22} = -2\lambda_1 U_c' k^2 B \quad (19)$$

$$(\partial_T + \lambda_1 + ik U_c' Y) \alpha_{12} = U_c' \alpha_{22} + 2k^2 U_c'^2 B \quad (20)$$

$$(\partial_T + \lambda_1 + ik U_c' Y) \alpha_{11} = 2U_c' \alpha_{12} \quad (21)$$

$$(\partial_T + ik U_c' Y) \zeta = -\frac{U_c'''}{U_c'} B_T - E(\alpha_{11,x} + \alpha_{12,Y})_Y \quad (22)$$

5.1.2 Normal Modes

Taking the linear equations (19)–(22), we look for modes proportional to $\exp(\sigma T)$. So $\alpha_{22} = \hat{\alpha}_{22} \exp(\sigma T)$ and similarly for the other terms.

Then

$$\begin{aligned} \hat{\alpha}_{22} &= -\frac{2\lambda_1 U_c' k^2 \hat{B}}{\sigma + \lambda_1 + ik U_c' Y} \\ \hat{\alpha}_{12} &= -\frac{2\lambda_1 U_c'^2 k^2 \hat{B}}{(\sigma + \lambda_1 + ik U_c' Y)^2} + \frac{2k^2 U_c'^2 \hat{B}}{\sigma + \lambda_1 + ik U_c' Y} \\ \hat{\alpha}_{11} &= \frac{2U_c' \hat{\alpha}_{12}}{\sigma + \lambda_1 + ik U_c' Y} \\ \hat{\zeta} &= -\frac{U_c''' \hat{B}_T}{U_c'(\sigma + ik U_c' Y)} - \frac{4k^4 U_c'^4 E \hat{B}}{(\sigma + ik U_c' Y)(\sigma + \lambda_1 + ik U_c' Y)^3} \end{aligned}$$

We want to back out the integral $\int_{-\infty}^{\infty} \hat{\zeta} dY$. $\hat{\zeta}$ is the sum of two terms, each of which will have to be attacked separately.

The first term is not difficult

$$\begin{aligned} \int_{-\infty}^{\infty} -\frac{U_c''' \hat{B}_T}{U_c'(\sigma + ik U_c' Y)} dY &= -\frac{U_c''' \hat{B}_T}{ik U_c'^2} \int_{-\infty}^{\infty} \frac{1}{Y - \frac{i\sigma}{k U_c'}} dY \\ &= \frac{i U_c''' \hat{B}_T}{k U_c'^2} i\pi \operatorname{sign}(\Re[\sigma]/U_c'). \end{aligned}$$

Since we are looking at $-1 < y < 0$, and $U(y) = (1 - y^2)^2$, we know that $U'_c > 0$. The second term is the term that depends on the elasticity. Therefore it gives the elastic contribution.

$$\int_{-\infty}^{\infty} \frac{4k^4 U_c'^4 E \hat{B}}{(\sigma + ikU_c'Y)(\sigma + \lambda_1 + ikU_c'Y)^3} dY = -4E\hat{B} \int_{-\infty}^{\infty} \frac{1}{(Y - \frac{i\sigma}{kU_c'})(Y - \frac{i(\sigma + \lambda_1)}{kU_c'})^3} dY$$

We use contour integration. For the following we assume $U'_c > 0$, though equivalent arguments can be made if it is negative. The contour we choose is from $-R$ to R and then closing it with a semicircle. The contribution from the arc goes to 0 as R gets large [since the denominator is $\mathcal{O}(Y^4)$]. The poles are at $Y = i\sigma/kU_c'$ and $Y = i(\sigma + \lambda_1)/kU_c'$. We know $\lambda_1 > 0$. If $\Re[\sigma] > 0$, both of these poles are in the upper half plane and so we close the contour in the lower half plane. This shows that the integral is 0. In contrast, if $\sigma_R < -\lambda_1$, then both poles are in the lower half plane, and we can close the integral in the upper half plane. We get a nonzero integral only when $-\lambda_1 < \sigma_R < 0$. In this case the integral is

$$-\frac{4E2\pi}{(\lambda_1/kU_c')^3}$$

Thus elasticity only has an effect on the normal modes if the mode is decaying. From equation (18)

$$i\sigma I_0 + k_1 I_1 = \left[-\frac{U_c'''\sigma}{kU_c'^2} \pi \text{sign}(\Re[\sigma]) - \frac{4E2\pi}{(\lambda_1/kU_c')^3} \chi \right] \hat{\psi}_{2c}^*$$

where $\chi = 1$ if $-\lambda_1 < \Re[\sigma] < 0$. This can be used to solve for σ , but it is of limited value since we expect the solution to saturate at large enough values that the linearized inner equations are invalid. This can give useful information for small T .

5.1.3 Initial Value Problem

Rather than looking for a normal mode, we can alternately try to solve equations (19)–(22) as an initial value problem using Laplace transforms.

We get a very similar set of equations to the normal mode equations. Here we define $\hat{\alpha}_{22}$ such that $\hat{\alpha}_{22} = \int_0^\infty e^{-sT} \alpha_{22}(s) ds$. We similarly define the other hatted variables. As before we arrive at

$$\begin{aligned} \hat{\alpha}_{22} &= -\frac{2\lambda_1 U_c' k^2 \hat{B}}{s + \lambda_1 + ikU_c'Y} \\ \hat{\alpha}_{12} &= -\frac{2\lambda_1 U_c'^2 k^2 \hat{B}}{(s + \lambda_1 + ikU_c'Y)^2} + \frac{2k^2 U_c'^2 \hat{B}}{s + \lambda_1 + ikU_c'Y} \\ \hat{\alpha}_{11} &= \frac{2U_c' \hat{\alpha}_{12}}{s + \lambda_1 + ikU_c'Y} \\ \hat{\zeta} &= -\frac{U_c''' \hat{B}_T}{U_c'(s + ikU_c'Y)} - \frac{4k^4 U_c'^4 E \hat{B}}{(s + ikU_c'Y)(s + \lambda_1 + ikU_c'Y)^3} \end{aligned}$$

and we want $\int_{-\infty}^\infty \zeta dY$.

When we inverse transform $\hat{\zeta}$ using a Bromwich contour integral, we will take $\Re[s] > 0$. Using the arguments from the previous section, the contribution from the second term will

be identically 0 because s and $s + \lambda_1$ both have positive real part. So once again, the contribution from the elasticity disappears.

The first term gives a contribution. We end up with

$$iI_0 B_T + k_1 I_1 B = -\frac{\pi U_c''' B_T}{k U_c'^2} \hat{\psi}_{2c}^*$$

This is a linear, first order constant coefficient ODE for B . B will have exponential growth or decay. Until B becomes large, this equation should give useful information about its growth.

6 $E \sim 1$, $\lambda \ll 1$

We now consider larger values of E . To simplify the analysis in this case, we will completely ignore λ .

As before, we assume that ψ saturates at some size ξ . Balancing the leading terms, the width of the critical layer is ϵ again. The difference is that now E is order 1, so that the elastic stresses will have to appear in the leading order balance.

We still solve (8)–(10), but now when $\lambda = 0$, the value of $\mathbf{F}$ becomes

$$\mathbf{F} = \begin{pmatrix} 2A_{11}\psi_{xy} & -A_{11}\psi_{xx} \\ -A_{11}\psi_{xx} & 0 \end{pmatrix}.$$

It is convenient to use the following expression for a_{11} and a_{22} :

$$\begin{aligned} a_{11} &= \frac{4U'U''\psi}{U-c} + \frac{\partial_x^{-1}2U'a_{12}}{U-c} + \frac{4U'^2\psi_y}{U-c} + \frac{\partial_x^{-1}S}{U-c} \\ a_{12} &= -\frac{2U'^2\psi_x}{U-c} + \frac{\partial_x^{-1}R}{U-c} \end{aligned}$$

where ∂_x^{-1} denotes an integral and

$$\begin{aligned} R &= -\epsilon a_{12,T} + J(\psi, a_{12}) - a_{11}\psi_{xx} \\ S &= -\epsilon a_{11,T} + J(\psi, a_{11}) + 2a_{11}\psi_{xy} + 2a_{12}\psi_{yy}. \end{aligned}$$

This results in the equation

$$\mathcal{L}[\psi]_x = -\epsilon\omega_T + J(\psi, \omega) + E \left(\left[\frac{\partial_x^{-1}2U'R}{(U-c)^2} \right]_y + \left[\frac{S}{U-c} \right]_y - \left[\frac{\partial_x^{-1}R}{U-c} \right]_{yy} + \left[\frac{R}{U-c} \right]_x \right) \quad (23)$$

where

$$\mathcal{L}[\psi] := \frac{\partial}{\partial y} \left(\Gamma \frac{\partial}{\partial y} \frac{\psi}{U-c} \right) + \frac{\partial^2}{\partial x^2} \left(\Gamma \frac{\psi}{U-c} \right)$$

with $\Gamma = (U-c)^2 - 2EU'^2$. At leading order, this is the linear problem (7). We expect that if there is a zero of Γ , then there is likely to be another zero. Usually we anticipate that there will be two critical layers. We can generally look at the critical layers independently of each other, and we will use a subscript j to distinguish between different critical layers.

The places where $\Gamma = 0$ correspond to where the elastic wave speed equals the speed of the disturbance relative to the base flow.

We now use a different ordering for ψ following [9]. In the outer solution

$$\begin{aligned}\psi &= \epsilon^{5/2}\psi_{5/2} + \epsilon^3\psi_3 + \epsilon^{7/2}\psi_{7/2} \\ \omega &= \epsilon^{5/2}\omega_{5/2}.\end{aligned}$$

At leading order, the outer solution satisfies

$$\mathcal{L}[\psi_{5/2}] = 0.$$

If c is real, the solution to this is singular. We can use a Frobenius expansion to approximate the outer solution close to the j -th singularity as

$$\psi_{5/2} = B(T)(a_j^\pm\phi_1 + b_j^\pm\phi_2)\exp(ikx) + cc$$

where $a_j^\pm$ and $b_j^\pm$ are constants that depend on the sign of $y - y_j$ and

$$\begin{aligned}\phi_1 &= 1 + \frac{k^2(y - y_j)^2}{4} - \frac{\Gamma_j''k^2(y - y_j)^3}{18} + hot \\ \phi_2 &= -\Gamma_j''(y - y_j) - \left(\Gamma_j''' + \frac{k^2}{2} - \Gamma_j''^2\right)\frac{(y - y_j)^2}{2} + \left(\frac{2\Gamma_j'''\Gamma_j''}{3} - \frac{5\Gamma_j''k^2}{108} - \frac{\Gamma_j''^3}{3}\right)(y - y_j)^3 \\ &\quad + \phi_1 \ln|y - y_j| + hot\end{aligned}$$

We need to determine what the jumps are and how B evolves. This will require careful analysis inside the critical layer.

At order ϵ^3 , the outer solution satisfies the same problem

$$\mathcal{L}[\psi_3] = 0.$$

At the next order, $\mathcal{O}(\epsilon^{7/2})$, we get a new equation

$$\begin{aligned}\mathcal{L}[\psi_{7/2}]_x &= 2ik_0^2k_1\Gamma\frac{\psi_{5/2}}{U - c} + \psi_{5/2}yy_T - k_0^2\psi_{5/2}T \\ &\quad + E\left(\left[\frac{4U'^3\psi_{5/2}T}{(U - c)^3}\right]_y - \left[\frac{4U'U''\psi_{5/2}T}{(U - c)^2} - \frac{4U'^3\psi_{5/2}T}{(U - c)^3} + \frac{4U'^2\psi_{5/2}y_T}{(U - c)^2}\right]_y\right. \\ &\quad \left.- \left[\frac{2U'^2\psi_{5/2}T}{(U - c)^2}\right]_{yy} + \left[\frac{2U'^2\psi_{5/2}x_T}{(U - c)^2}\right]_y\right)\end{aligned}$$

As before, we multiply by $\psi_{5/2}^*/(U - c)$. We would like to integrate from -1 to 1 , but this integral will have singularities at the critical layers causing it to diverge. To accomodate this, we will have to leave out regions of width 2δ about each critical layer. On the left hand side, we will use integration by parts. After using the fact that $\mathcal{L}[\psi_{5/2}] = 0$ all that will be left is boundary terms involving $\psi_{7/2}$ and $\psi_{5/2}$. The values of those boundary terms will

have to be evaluated by solving the inner problem. On the right hand side, all the integrals will involve $\psi_{5/2}$. We can evaluate them. As δ gets small, a careful matching in orders of δ will match the portions of the integrals on each side that go to infinity. We will be left with an expression of the form

$$\text{boundary terms} = iI_0 B_T + k_1 I_1 B \quad (24)$$

We need to solve the inner problem to advance further.

Figure 3: We see that across the critical layers there is a change in phase of the solution.

6.1 Inner Solution

The inner solution will have

$$\begin{aligned} \psi &= \epsilon^{5/2} \Psi_{5/2} + \epsilon^3 \Psi_3 + \epsilon^{7/2} \Psi_{7/2} \\ \omega &= \epsilon^{1/2} Z_{1/2} \end{aligned}$$

Notice that in the previous section the leading order term of ψ in the inner solution is independent of Y . Here however it depends on Y . We know $\omega = -\nabla^2 \psi$ which has two derivatives in Y . Each derivative in Y introduces a factor of ϵ^{-1} , consequently ω is two orders larger than ψ . It is straightforward to see that

$$\begin{aligned} R &= R_{7/2} \epsilon^{7/2} + R_4 \epsilon^4 + R_{9/2} \epsilon^{9/2} + \dots \\ S &= S_{5/2} \epsilon^{5/2} + S_3 \epsilon^3 + S_{7/2} \epsilon^{7/2} + \dots \end{aligned}$$

Again we use the variable $Y = \epsilon^{-1}(y - y_j)$. The left hand side of equation (23) becomes

$$\left[\frac{1}{\epsilon} \frac{\Gamma'_j}{U_j - c} Y \psi_Y + \frac{\Gamma'_j U'_j}{(U_j - c)^2} Y^2 \psi_Y + \frac{\Gamma''_j}{U_j} Y^2 \psi_Y - \frac{\Gamma'_j U'_j}{(U_j - c)^2} Y \psi \right]_{xY} + \mathcal{O}(\epsilon^{7/2})$$

while the right hand side becomes

$$\begin{aligned} & -\epsilon \omega_T + J(\psi, \omega) \\ & + E \left[\left(\frac{2U''}{(U-c)^2} - \frac{4U'^2}{(U-c)^3} \right) \partial_x^{-1} R + \frac{2U'}{(U-c)^2} \partial_x^{-1} R_y \right] \\ & + E \left[\frac{1}{U-c} S_Y + \frac{U'}{(U-c)^2} S \right] \\ & - E \left[\left(\frac{U''}{(U-c)^2} - \frac{2U'^2}{(U-c)^3} \right) \partial_x^{-1} R - \frac{2U'}{(U-c)^2} \partial_x^{-1} R_y + \frac{1}{U-c} \partial_x^{-1} R_{yy} \right] \\ & + E \frac{R_x}{U-c} \\ & = \frac{1}{\epsilon} \psi_{YYT} - \frac{1}{\epsilon^3} J_Y(\psi, \psi_{YY}) + E \left[\frac{1}{\epsilon} \frac{2U'_j}{(U_j - c)^2} \partial_x^{-1} R_Y \right] + E \left[\frac{1}{\epsilon} \frac{1}{U_j - c} S_Y - \frac{U'_j}{(U_j - c)^2} Y S_Y + \frac{U'_j}{(U_j - c)^2} S \right] \\ & - E \left[-\frac{1}{\epsilon} \frac{2U'_j}{(U_j - c)^2} \partial_x^{-1} R_Y + \frac{1}{\epsilon^2} \frac{1}{U_j - c} \partial_x^{-1} R_{YY} + \frac{1}{\epsilon} \frac{U'_j}{(U_j - c)^2} Y \partial_x^{-1} R_{YY} \right] + \mathcal{O}(\epsilon^{7/2}) \end{aligned}$$

We want to match these equations order by order.

At $\mathcal{O}(\epsilon^{3/2})$, equation (23) becomes

$$\left[\frac{\Gamma'_j}{U_j - c} Y \Psi_{5/2Y} \right]_{xY} = \Psi_{5/2Y} Y T + \frac{E S_{5/2Y}}{U_j - c} - \frac{E \partial_x^{-1} R_{7/2Y} Y}{U_j - c}. \quad (25)$$

Order ϵ^2 gives

$$\left[\frac{\Gamma'_j}{U_j - c} Y \Psi_{3Y} \right]_{xY} = \Psi_{3Y} Y T - J_Y(\Psi_{5/2}, \Psi_{5/2Y} Y) + \frac{E S_{3Y}}{U_j - c} - \frac{E \partial_x^{-1} R_{4Y} Y}{U_j - c} \quad (26)$$

and order $\epsilon^{5/2}$ gives

$$\begin{aligned} & \left[\frac{\Gamma'_j}{U_j - c} Y \Psi_{7/2Y} + \frac{\Gamma'_j U'_j}{(U_j - c)^2} Y^2 \Psi_{5/2Y} + \frac{\Gamma''_j}{U_j} Y^2 \Psi_{5/2Y} - \frac{\Gamma'_j U'_j}{(U_j - c)^2} Y \Psi_{5/2} \right]_{xY} \\ &= \Psi_{7/2Y} Y T - J_Y(\Psi_{5/2}, \Psi_{3Y} Y) - J_Y(\Psi_3, \Psi_{5/2Y} Y) + \frac{2E U'_j \partial_x^{-1} R_{7/2Y}}{(U_j - c)^2} + \frac{E S_{7/2Y}}{U_j - c} \\ & - \frac{E U'_j Y S_{5/2Y}}{(U_j - c)^2} + \frac{E U'_j S_{5/2}}{(U_j - c)^2} + \frac{2E U'_j \partial_x^{-1} R_{7/2Y}}{(U_j - c)^2} - \frac{E \partial_x^{-1} R_{9/2Y} Y}{U_j - c} - \frac{E U'_j Y \partial_x^{-1} R_{7/2Y} Y}{(U_j - c)^2}. \end{aligned} \quad (27)$$

We need to go into detail on the expansions for ψ , R and S . Before we do this, we make the observation that if $(U_j - c)^2 - 2E U'_j{}^2 = 0$, then $2E U'_j{}^2 / (U_j - c)^2 = 1$.

Some messy algebra shows that

$$\begin{aligned} R_{7/2} &= \frac{U_j - c}{E} \Psi_{5/2xT} \\ R_4 &= \frac{U_j - c}{E} [\Psi_{3T} - (\Psi_{5/2Y} \Psi_{5/2x})]_x \\ R_{9/2} &= \frac{U_j - c}{E} [Y \Psi_{5/2T} - \Psi_{5/2TT} + \Psi_{7/2T} - (\Psi_{3Y} \Psi_{5/2x}) - (\Psi_{3x} \Psi_{5/2Y})]_x \\ S_{5/2} &= -\frac{2(U_j - c)}{E} \Psi_{5/2YT} \\ S_3 &= -\frac{2(U_j - c)}{E} [\Psi_{3YT} - \Psi_{5/2Y} \Psi_{5/2xY}] \\ S_{7/2} &= -\frac{2(U_j - c)}{E} \left[\left(\frac{U'_j}{U_j - c} + \frac{2U''_j}{(U_j - c)U'_j} \right) Y \Psi_{5/2YT} + \frac{U''_j}{U'_j} \Psi_{5/2T} + \Psi_{7/2YT} \right. \\ & \quad \left. - \frac{U'_j}{U_j - c} \Psi_{5/2T} - \frac{1}{U_j - c} \partial_x^{-1} \Psi_{5/2YTT} - (\Psi_{5/2Y} \Psi_{3Y})_x \right]. \end{aligned}$$

We are finally in a position to write down equations for the evolution of ψ inside the critical layers.

We can immediately make a perhaps remarkable observation. When these values for R and S are inserted in equations (25)–(27) E is cancelled in every term where it appears. So the only role E appears to play is in determining where the critical layers are. Other than that it does not directly affect the dynamics within the critical layers.

After substituting for R and S and integrating once in Y , equation (25) becomes

$$\Psi_{5/2YT} + \frac{\Gamma'_j}{2(U_j - c)} Y \Psi_{5/2xY} = V_1(x, T). \quad (28)$$

Substituting for R and S in equation (26) gives

$$\left[\frac{\Gamma'_j}{U_j - c} Y \Psi_{3xY} \right]_Y = -2\Psi_{3YYT} - J_Y(\Psi_{5/2}, \Psi_{5/2Y}).$$

However,

$$\begin{aligned} J_Y(\Psi_{5/2}, \Psi_{5/2Y}) &= \Psi_{5/2x} \Psi_{5/2YY} + \Psi_{5/2xY} \Psi_{5/2Y} - \Psi_{5/2xY} \Psi_{5/2Y} - \Psi_{5/2Y} \Psi_{5/2xYY} \\ &= (\Psi_{5/2x} \Psi_{5/2YY} - \Psi_{5/2xY} \Psi_{5/2Y})_Y \\ &= [J_Y(\Psi_{5/2}, \Psi_{5/2Y})]_Y \end{aligned}$$

and so integrating in Y gives

$$\Psi_{3YT} + \frac{\Gamma'_j}{2(U_j - c)} Y \Psi_{3xY} = -J_Y(\Psi_{5/2}, \Psi_{5/2Y})/2 + V_2(x, T). \quad (29)$$

After some effort, equation (27) becomes

$$\begin{aligned} &\Psi_{7/2YT} + \frac{\Gamma'_j}{2(U_j - c)} Y \Psi_{7/2xY} \\ &= -\frac{1}{2} \left(\frac{\Gamma'_j U'_j}{(U_j - c)^2} + \frac{\Gamma''_j}{U_j} \right) Y^2 \Psi_{5/2xY} + \frac{\Gamma'_j U'_j}{2(U_j - c)^2} Y \Psi_{5/2x} \\ &\quad - \left[\left(\frac{U'_j}{U_j - c} + \frac{2U''_j}{(U_j - c)U'_j} \right) - \frac{U'_j}{U_j - c} + \frac{1}{2} + \frac{U'_j}{2(U_j - c)} \right] Y \Psi_{5/2YT} \\ &\quad + \left[\frac{U'_j}{U_j - c} - \frac{U''_j}{U'_j} + \frac{U'_j}{U_j - c} \right] \Psi_{5/2T} + \left[\frac{1}{2} + \frac{1}{U_j - c} \partial_x^{-1} \right] \Psi_{5/2YTT} \\ &\quad - 2(\Psi_{5/2x} \Psi_{3YY} + \Psi_{3x} \Psi_{5/2YY}) + V_3(x, T). \end{aligned} \quad (30)$$

We can now get the jumps in a_j and b_j in the outer solution ($\psi_{5/2}$) from our inner solution for $\Psi_{5/2}$.

The large Y limit of (28) forces

$$\Psi_{5/2xY} \sim 2(U_j - c)V_1/Y\Gamma'_j$$

and so $\Psi_{5/2x} \sim 2(U_j - c)V_1(\ln|Y|)/\Gamma'_j$. However, we must be able to match this to the outer solution for ψ . At $Y = \Delta$, this term has become $\mathcal{O}(\epsilon^{5/2} \ln \Delta)$, and so this must match to $ikB(T)b_j^+ \ln|\Delta| \exp(ikx)$. Hence we can choose $b_j^+ = 2(U_j - c)/\Gamma'_j$, and $V_1 = ikB(T) \exp(ikx)$. A similar look at $Y = -\Delta$ will show that $b_j^- = b_j^+$.

Since $V_1 \propto \exp(ikx)$, equation (28) implies that $\Psi_{5/2} \propto \exp(ikx)$. Defining $\hat{\Psi}_{5/2}$ such that $\Psi_{5/2} = \hat{\Psi}_{5/2} \exp(ikx)$ yields the equation

$$\hat{\Psi}_{5/2YT} + \frac{ik}{b_j} Y \hat{\Psi}_{5/2Y} = ikB(T).$$

We integrate this in T using an integrating factor

$$\hat{\Psi}_{5/2Y} = ik \int_0^T e^{\frac{ikY(S-T)}{b_j}} B(S) dS$$

The jump in a_j can now be calculated by integrating $\hat{\Psi}_{5/2Y}$ from $-\Delta$ to Δ and dividing by $B(T)$. To simplify the calculation, we take $\Delta \rightarrow \infty$.

$$\begin{aligned} a_j^+ - a_j^- &= \frac{ik}{B(T)} \int_0^T B(S) \int_{-\infty}^{\infty} e^{\frac{ikY(S-T)}{b_j}} dY dS \\ &= \frac{2b_j\pi}{B(T)} \int_0^T B(S) \tilde{\delta}(S-T) dS \\ &= b_j\pi \end{aligned}$$

where $\tilde{\delta}$ is the Kronecker delta function. Because the outer integral only goes to $S = T$, the delta function picks out only half of the value of $B(T)$.

We now turn to equation (29). Rather than going into detail on it, we note that it can be solved with the integrating factor $\exp[\Gamma'_j Y/2(U_j - c)] = \exp(Y/b_j)$. However, we will end up with an integral from 0 to T of the nonlinear terms in the Jacobian. Ψ_3 will have a dependance on an integral of a quadratic in B .

Approaching equation (30), a similar problem occurs with the nonlinear terms that involve Ψ_3 and $\Psi_{5/2}$. $\Psi_{7/2}$ has a tangled dependence on B . It will have a double integral of a cubic term in B involving a delay. When we return to the outer solution and update the boundary terms in equation 24, we will have a delay differential equation, which we expect to diverge in finite time.

It is likely that introducing a sufficiently large λ should prevent this divergence.

7 Conclusions

We have made a significant step towards understanding critical layers in elastic fluids at high Weissenberg and high Reynolds number limits.

We have found that the presence of small elasticity does not significantly affect the early growth of the instability, though it may affect the later development. To understand the later development would require solving a nonlinear system of coupled PDEs.

In the case of moderate elasticity, the elasticity substantially affects the critical layers, changing the position and number of critical layers. The equation governing the growth of the amplitude is nonlinear, and depends on earlier times. Consequently, we expect that the solutions will grow to infinity in finite time. Decreasing the Weissenberg number from infinity (increasing λ from 0) may help to stabilize this unbounded growth.

7.1 Future Work

There is a lot of work left to do on this problem. Quite likely a PhD thesis or two's worth.

We have done part of the $E = \mathcal{O}(\epsilon^4)$, $\lambda = \mathcal{O}(\epsilon)$ case. To do more would likely require considerable computations. We have also done part of the $E = \mathcal{O}(1)$, $\lambda = 0$ case. It should not be difficult to add small λ in to this analysis.

It would be difficult to attack the $E = \mathcal{O}(\epsilon^2)$, $\lambda = \mathcal{O}(\epsilon)$ case because there are two critical layers which will interact strongly. A successful attack on this should also be straightforward to translate into the MHD community, where the corresponding case has also been neglected for being too difficult.

It would be interesting to approach the $E \gg 1$ case because in that limit we can neglect inertia. We would then be looking at the zero Reynolds number limit.

In all cases we discussed in this we used the high Weissenberg number limit ($\lambda \ll 1$). Dropping this assumption would complicate matters because we would not arrive at the same elastic Rayleigh equation. A new continuous spectrum is created at where $ik(U - c) + \lambda = 0$ (note that this has $\Re[c] < 0$). For small λ , this overlapped with the continuous spectrum of the standard Rayleigh's equation. We did the case where $E = \mathcal{O}(\epsilon^3)$, $\lambda = \mathcal{O}(1)$ though we did not report it here. This case is not difficult since the small value of E keeps λ from affecting the leading order and hence the Rayleigh equation remains unchanged. In this case, the effect of elasticity is identical to the effect of weak viscosity. It is not clear what happens as E gets larger.

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