Viscoelastic Catenary

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1 Introduction

This paper seeks to determine the shape of a thin viscoelastic fluid filament as it sags under its own weight. The problem is an extension of the viscous catenary [1] and we refer to this problem as "viscoelastic catenary". Viscoelastic filaments appear in applications such as fiber processing from melts and solutions, extensional rheometry etc. An understanding of the dynamics of the viscoelastic catenary will therefore aid in better design of such applications.

2 Experimental Observations

We investigated a Boger fluid composed of 0.025% w/w Polystyrene of molecular weight 1.877 x 10^6 dissolved in styrene oil. The relaxation time for this fluid is around 4 seconds and its zero-shear viscosity, η_0, is 50 Pa.s. There is no shear thinning in the fluid over several decades of strain rate, and expecially in the regime of our experiments. We took some fluid between two plates and stretched out in the horizontal direction to shape it into a thin filament, $h << L$, where h is the thickness of the filament and L is the length to which it is stretched. Figure (1) shows a snapshot of one such experiment with $h = 0.002$ m, $L = 0.025$ m.

Two problems emerged out of this experiment that need to be understood. First is the problem of the viscoelastic catenary, wherein the fluid filament sags under its own weight and its shape evolves with time. Second, is what we refer to as the *chewing-gum* problem. In this problem, fluid between two plates is stretched out into a thin filament and then instantaneously, the two plates are brought closer together. This makes the filament buckle in the direction of gravity, thereby making a viscoelastic catenary to begin with. What happens then is, to our knowledge, a phenomenon unique to viscoelastic fluids only - the catenary starts moving upwards against gravity, like a recoil. However, if the plates are brought together at a rate equivalent to the inverse of the relaxation time of the fluid, we do not see this recoil effect. We refer to this effect as the *chewing-gum* problem because we observed the effect for the first time in a chewing-gum.

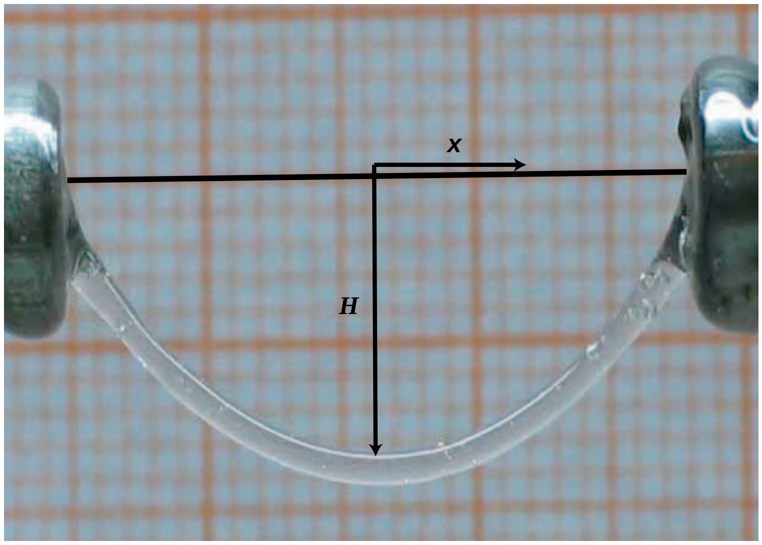

Figure 1: Snapshot of a viscelastic catenary.In this case, the fluid used is a mixture of Polystyrene (MW 1.877 x 10^6) in styrene oil - a Boger fluid. The zero shear viscosity is approximately 50 Pa.s and there is no shear thinning for the shear rates under consideration.Squares in the background are 1 mm in dimension.

3 Governing Equations

All the dynamics of a viscoelastic filament can be understood by considering the simpler problem of a 2D sheet. The analysis that we will present is applicable to many viscoelastic systems that do not shear-thin. However, for now consider a solution of polymer molecules in a viscous solvent. The governing equations for the fluid are conservation of mass, conservation of momentum and the closure model to describe the polymer stress within the fluid:

$$\nabla . u = 0 \tag{1}$$

$$\rho \left(u_t + u . \nabla u \right) = -\nabla p + \mu \nabla^2 u + \nabla . \tau - \rho g \tag{2}$$

$$\tau_t + u . \nabla \tau - \left(\nabla u \right)^T . \tau - \tau . \left(\nabla u \right) = -\frac{1}{\lambda} \left(\tau - G \right) \tag{3}$$

where, the subscripts represent differentiation with respect to the subscripted variables. τ is the polymer stress tensor, G is the equilibrium polymer stress and λ is the relaxation time of the polymer molecules. Note that eqs. (1) and (2) not closed without eq. (3) which describes the evolution of polymer stress in the flow, referred to as the Oldroyd-B constitutive model.

In order to make the above equations dimensionless, we choose a velocity scale U and a length scale L. Then, the scaling for time is L/U, where L is the length of the sheet between the clamps. We scale the pressure and polymer stress with $\mu \frac{U}{L}$. We perform the following expansion:

$$\begin{aligned}
u &= u_0 + \epsilon^2 u_2 + O(\epsilon^4) \\
\epsilon v &= v_0 + \epsilon^2 v_2 + O(\epsilon^4) \\
H &= \epsilon H_0 + \epsilon^3 H_2 + O(\epsilon^4) \\
h &= \epsilon h_0 + \epsilon^3 h_2 + O(\epsilon^4) \\
p &= p_0 + \epsilon^2 p_2 + O(\epsilon^4) \\
\tau &= \tau_0 + \epsilon^2 \tau_2 + O(\epsilon^4)
\end{aligned} \tag{4}$$

Then the governing equations for a 2D sheet become,

$$\epsilon^2 u_x + v_y = 0 \tag{5}$$

$$\epsilon^4 Re\left(u_t + \epsilon^2 u u_x + v u_y\right) = -\epsilon^2 p_x + \epsilon^2 u_{xx} + u_{yy} + \epsilon^2 \tau_x \tag{6}$$

$$\epsilon^4 Re\left(v_t + \epsilon^2 u v_x + v v_y\right) = -\epsilon^2 p_y + \epsilon^2 v_{xx} + v_{yy} - \epsilon^4 \varpi \tag{7}$$

where, $Re = \rho UL/\mu$ is the Reynolds number and $\varpi = \frac{\rho g}{\mu U/L}$ is the dimensionless weight variable. In the limit of $Ca = \mu U/\gamma >> 1$, the effects due to surface tension can be ignored. So, we consider traction-free boundaries. At $y = H \pm h/2$, the kinematic boundary condition is:

$$v = \left(H \pm \frac{h}{2}\right)_t + \epsilon^2 u \left(H \pm \frac{h}{2}\right)_x \tag{8}$$

.

The stress boundary condition results in the following two equations:

$$-\epsilon^2\left(-p + 2u_x + \tau\right)\left(H_x \pm \frac{h_x}{2}\right) + \left(u_y + v_x\right) = 0 \tag{9}$$

$$-\epsilon^2\left(u_y + v_x\right)\left(H_x \pm \frac{h_x}{2}\right) - \epsilon^2 p + 2v_y = 0 \tag{10}$$

At leading order, $O(1)$, the incompressibility equation reduces to:

$$v_{0y} = 0 \tag{11}$$

The x-momentum and y-momentum balances are respectively:

$$u_{0yy} = 0 \tag{12}$$

$$v_{0yy} = 0 \tag{13}$$

We assume that the polymer stress tensor has only one non-zero component, τ^{xx}, where the superscript refers to the component of the stress tensor. Here onwards, we drop the superscript and refer to τ^{xx} as τ. The Oldroyd-B equation for τ at this order is:

$$\tau_{0t} + v_0 \tau_{0y} = -\frac{1}{Wi}\left(\tau_0 - G\right) \tag{14}$$

We assume that $\tau_{0y} = 0$. Stretching the filament embeds a stress within the fluid, $\tau_0(0,t) = \tau_0(0)$. Boundary conditions at $y = H \pm h/2$ are:

$$v_0 = \left(H_0 \pm \frac{h_0}{2}\right)_t \tag{15}$$

$$u_{0y} + v_{0x} = 0 \tag{16}$$

$$v_{0y} = 0 \tag{17}$$

Then we conclude that $h_{0t} = 0$. Also

$$v_0 = H_{0t} \tag{18}$$

$$u_0 = H_{0xt}(H - y) + \overline{u_0}(x) \tag{19}$$

where, $\overline{u_0}(x)$ is the velocity of the centerline of the filament, i.e. $y = H_0$. Integrating Eq. (14) gives the equation for the leading order polymer stress that decays with time.

$$\tau_0 = G + (\tau_0(0) - G)\, e^{-t/Wi} \tag{20}$$

At second order, $O(\epsilon^2)$, the incompressibility equation and momentum balances yield,

$$v_{2y} = -u_{0x} \tag{21}$$

$$p_{0x} - u_{2yy} = u_{0xx} + \tau_{0x} \tag{22}$$

$$p_{0y} = v_{0xx} - u_{0xy} \tag{23}$$

The Oldroyd-B equation becomes

$$\tau_{2t} + v_0\tau_{2y} + \frac{\tau_2}{Wi} = 2\tau_0 u_{0x} - u_0\tau_{0x} \tag{24}$$

The kinematic and stress boundary conditions at this order are:

$$v_2 = \left(H_2 \pm \frac{h_2}{2}\right)_t + u_0\left(H_{0x} \pm \frac{h_{0x}}{2}\right) \tag{25}$$

$$u_{2y} + v_{2x} + p_0\left(H_{0x} \pm \frac{h_{0x}}{2}\right) = (2u_{0x} + \tau_0)\left(H_{0x} \pm \frac{h_{0x}}{2}\right) \tag{26}$$

$$p_0 = 2v_{2y} \tag{27}$$

Integrating the y-momemtum balance, Eq. (23) and applying the appropriate boundary condition, Eq. (27), we can evaluate the leading order pressure.

$$p_0 = -2u_{0x} \tag{28}$$

To calculate v_2, we integrate the second order incompressibility equation, Eq. (21),

$$v_2 = \frac{H_{0xxt}}{2}(y - H)^2 - T(y - H) + \tilde{v}_2 \tag{29}$$

where, $\tilde{v}_2 = H_{2t} + \overline{u_0} H_{0x}$ and $T = \overline{u}_{0x} + H_{0x} H_{0xt}$. Note that T is the viscous contribution to the dimensionless tension in the viscoelastic sheet. The second order horizontal velocity then is

$$u_2 = \frac{H_{0xxxt}}{2}(y-H)^3 - \frac{3}{2}\left(H_{0x}H_{0xxt} + T_x\right)(y-H)^2 - \frac{\tau_{0x}}{2}(y-H)^2 + k(x)(y-H) + \tilde{u}_2(x) \quad (30)$$

where $k(x) = -H_{0xxxt}h_0^2/2 + (3T + \tau_0)H_{0x} - \tilde{v}_{2x}$ and $\tilde{u}_2(x)$ is the constant of integration.

At this order, the boundary conditions and the equations impose the following solvability conditions:

$$h_{2t} = -\left(\bar{u}_0 h_0\right)_x \quad (31)$$

$$\left[\left(4T + \tau_0\right) h_0\right]_x = 0 \quad (32)$$

Eq. (32) is a statement of tension balance. Inertia is too small to appear at this order. So the catenary is in a quasi-static balance. At the next order, $O(\epsilon^4)$ the incompressibility, momentum balances and the Oldroyd-B equation are as follows.

$$v_{4y} = -u_{2x} \quad (33)$$

$$-p_{2x} + u_{4yy} = Re\left(u_{0t} + v_0 u_{0y}\right) - u_{2xx} - \tau_{2x} \quad (34)$$

$$p_{2y} = -Re\left(v_{0t}\right) + v_{2xx} + v_{4yy} - \varpi_0 \quad (35)$$

The boundary conditions are,

$$v_4 + v_{2y}\left(H_2 \pm \frac{h_2}{2}\right) = \left(H_4 \pm \frac{h_4}{2}\right)_t + u_2\left(H_{0x} \pm \frac{h_{0x}}{2}\right) + u_0\left(H_{2x} \pm \frac{h_{2x}}{2}\right) \quad (36)$$

$$-\left(-p_0 + 2u_{0x} + \tau_0\right)\left(H_{2x} \pm \frac{h_{2x}}{2}\right) - \left(-p_2 + 2u_{2x} + \tau_2\right)\left(H_{0x} \pm \frac{h_{0x}}{2}\right) + u_{4y} + v_{4x} = 0 \quad (37)$$

$$-\left(u_{2y} + v_{2x}\right)\left(H_{0x} \pm \frac{h_{0x}}{2}\right) - \left[p_2 + p_{0y}\left(H_2 \pm \frac{h_2}{2}\right)\right] + 2\left[v_{4y} + v_{2yy}\left(H_2 \pm \frac{h_2}{2}\right)\right] = 0 \quad (38)$$

Integrating the y-momentum balance, we get another solvability condition. For the sake of simplicity, we assume that $h_{0x} = 0$ and that $\tau_{0x} = 0$. Then,

$$h_0 Re H_{0tt} + \frac{h_0^3}{3} H_{0xxxxt} = (4T + \tau_0) h_0 H_{0xx} - \varpi h_0 \quad (39)$$

We can now rescale Eq. (39) to gain more insight into the problem. All lengths are scaled with L and time with $6\mu/\rho g h$. The centerline velocity at $x = 0$ and the ends of the catenary, $x = \pm\ 1/2$ is zero. So integrating the first solvability condition, we have that

$$4T + \tau_0 = \frac{8}{L}\left(\int_0^{L/2} \left(H_x\right)_t^2\, dx + \tau_0\right) \quad (40)$$

which is a statement of tension balance. The second solvability condition becomes,

$$Re_g H_{tt} + \frac{\epsilon^3}{32} H_{xxxxt} = \left(\int_0^{1/2} (H_x)_t^2 \, dx + \Lambda \tau_0 \right) \epsilon H_{xx} - 1 \tag{41}$$

where we have dropped the subscript "0" from the equation. $Re_g = \left(\frac{\rho g h}{6\mu}\right)^2 \frac{L}{g}$ is the appropriate Reynolds number, often referred to as the Galileo number in engineering circles. $\Lambda = \frac{G}{\rho g h}$, where G is the equilibrium polymer stress. Eq. (41), along with the boundary conditions $H(\pm 1/2, t) = 0$ and $H_x(\pm 1/2, t) = 0$, describes the shape of the viscoelastic catenary as it sags under its own weight. The second term on the left hand side, H_{xxxxt} is the contribution from torque balance and is referred to as the beding term.

4 Results and discussion

The final equation to be solved, Eq (41) is not, apparently amenable to analytical solutions.However, some simplifcations are in order. For the fluid filaments that we constructed, $Re_g \sim 10^{-5}$. So we can entirely neglect the inertial term. Also, at early times, the straight filament must first bend to begin the formation of a catenary. Neglecting the non-linear viscous stretching term, we have

$$\frac{\epsilon^3}{32} H_{xxxxt} = (\Lambda \tau_0) \, \epsilon H_{xx} - 1 \tag{42}$$

where, the parameter $\Lambda \sim 5$ and τ_0 can be evaluated from Eq.(20). As the catenary evolves, stretching will result in tension due to viscous stresses and the non-linear stretching term can no longer be ignored. At present, we present only these hypotheses. We hope to examine them in the process of solving Eq.(41) numerically.

We intend to attack *chewing-gum* problem using the framework that we have developed for the viscoelastic catenary. It appears to be a special case of the catenary - one in which the initial state of the filament is a catenary to begin with.

5 Acknowledgements

I sincerely thank Jean-Luc Thiffeault and L. Mahadevan for all their help and guidance. This project would have been impossible without them.

References

[1] J. Teichman and L. Mahadevan. Viscous catenary. *J. Fluid Mech.*, 478:71–80, 2003.