Lecture 3: Simple Flows

E. J. Hinch

In this lecture, we will study some simple flow phenomena for the non-Newtonian fluid. From analyzing these simple phenomena, we will find that non-Newtonian fluid has many unique properties and can be quite different from the Newtonian fluid in some aspects.

1 Pipe flow of a power-law fluid

The pipe flow of Newtonian fluid has been widely studied and well understood. Here, we will analyze the pipe flow of non-Newtonian fluid and compare the phenomenon with that of the Newtonian fluid.

We consider a cylindrical pipe, where the radius of the pipe is R and the length is L. The pressure drop across the pipe is Δp and the flux of the fluid though the pipe is Q (shown in Figure 1). Also, we assume that the flow in the pipe is steady, uni-directional and uniform in z.

Thus, the axial momentum of the fluid satisfies:

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial(r\sigma_{zr})}{\partial r}. \tag{1}$$

Integrate this equation with respect to r and scale $\frac{dp}{dz}$ with $\frac{\Delta p}{L}$, we have:

$$\sigma_{zr} = \frac{r}{R}\sigma_{wall}, \tag{2}$$

where σ_{wall} represents the stress on the wall and is given by $\sigma_{wall} = \frac{\Delta p R}{2L}$. For the power-law fluid, the constitutive equation is:

$$\sigma_{zr} = k\dot{\gamma}^n, \tag{3}$$

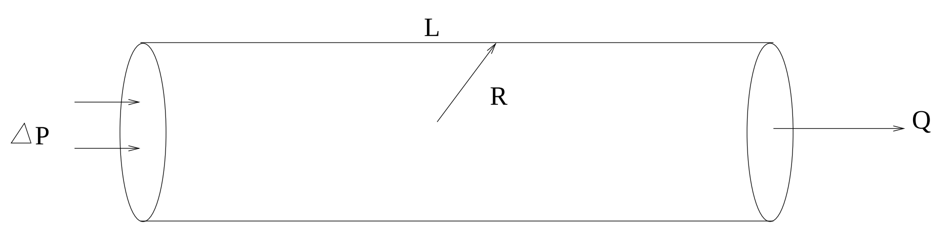

Figure 1: Pipe flow

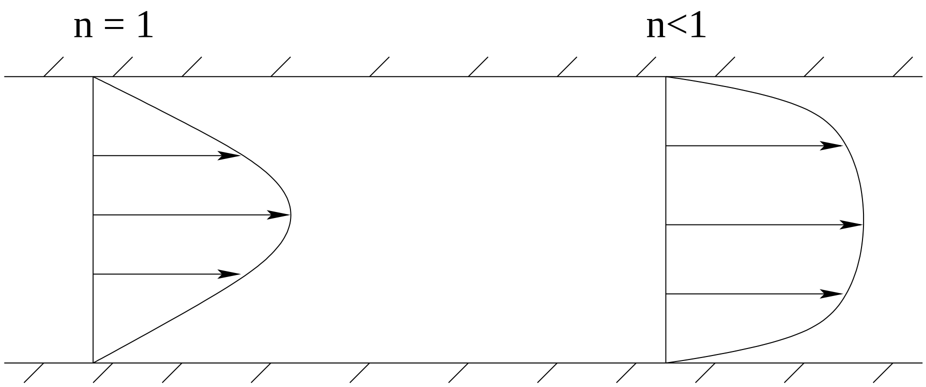

Figure 2: Flow profile

where k is a constant and $\dot{\gamma}$ is the scalar strain rate. For the pipe flow, the strain rate can be expressed as:

$$\dot{\gamma} = -\frac{dw}{dr}, \tag{4}$$

where w is the axial velocity. Substitute the expression for the strain rate into the constitutive equation, then put the constitutive equation into the equation (2); we find

$$-k\left(\frac{dw}{dr}\right)^n = \frac{r}{R}\sigma_{wall}. \tag{5}$$

Integrate this equation with respect to r; we have:

$$w = \left(\frac{\sigma_w}{kR}\right)^{\frac{1}{n}} \frac{R^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}}{\frac{1}{n}+1}. \tag{6}$$

For the different choices of n, the relation of the axial velocity to the radius is different(see Fig. 2). For Newtonian flow (Poiseuille flow) with $n = 1$, the structure of the flow is quadratic ($w \propto r^2$). For shear thinning fluid with $n < 1$, the profile of the flow is flatter in the middle and decays faster towards the wall. Near the middle of the pipe, the stress σ is low, and the viscosity μ is high; near the boundary of the pipe, the stress σ is high, and the viscosity μ is low.

The volume flux Q of the flow through the pipe is given by:

$$Q = \int_0^R w 2\pi r dr = \frac{\pi R^3}{\frac{1}{n}+3}\left(\frac{\Delta P R}{2Lk}\right)^{\frac{1}{n}}. \tag{7}$$

The volume flux Q of the shear thinning non-Newtonian fluid increases more quickly with pressure gradient than the Newtonian fluid. The flow of a power law fluid along pipe has common applications in wire coating, film draining and drop spreading.

2 Capillary rheometry

In Capillary rheometry, the shear viscosity of a fluid can be determined by shear rate. Assuming the flux through the pipe to be Q and the axial velocity to be w, the flux of the fluid can be expressed as:

$$Q = \int_0^R w 2\pi r dr = w\pi r^2|_0^R - \int_0^R \frac{dw}{dr}\pi r^2 dr. \tag{8}$$

Since the axial velocity is zero at the boundary (no slip boundary conditions), then $w\pi r^2|_0^R = 0$. Also, the strain rate $\dot{\gamma}$ can be expressed as $\dot{\gamma} = \frac{dw}{dr}$. Thus,

$$Q = -\int_0^R \dot{\gamma}\pi r^2 dr. \tag{9}$$

From the calculation of the flow through the pipe in Section 1, we know that the stress can be expressed as:

$$\sigma_{zr} = \frac{r}{R}\sigma_{wall}. \tag{10}$$

This relation does not depend on the constitutive equation, and the stess σ_{zr} linearly depends on the stress on the wall σ_{wall}. Therefore, the radius can also be expressed as a function of the stress: $r = \frac{R\sigma_{rz}}{\sigma_{wall}}$

Changing the integration variable from the radius r to the stress σ, we may express the flux through the pipe:

$$Q = -\frac{\pi R^3}{\sigma_{wall}^3}\int_0^{\sigma_{wall}} \dot{\gamma}(\sigma)\sigma^2 d\sigma. \tag{11}$$

Before the change of variables, the strain rate is a function of the radius, which describes the geometric property of the strain rate. After the change the variable, the strain rate is a function of the stress, which describes the material property of the strain rate. If we differentiate the above equation with respect to the stress on the wall σ_{wall}, we have:

$$\dot{\gamma}_{wall} = -\frac{1}{\sigma_{wall}^2}\frac{d}{d\sigma_{wall}}\left(\frac{\sigma_{wall}^3 Q}{\pi R^3}\right) = -\frac{1}{\pi R^3}\left(3Q + \sigma_{wall}\frac{dQ}{d\sigma_{wall}}\right). \tag{12}$$

Since the stress on the wall σ_w is linearly proportional to the pressure gradient Δp, we may simplify this equation as:

$$\dot{\gamma}_{wall} = -\frac{1}{\pi R^3}\left(3 + \frac{d\ln Q}{d\ln \Delta p}\right). \tag{13}$$

Here $\frac{d\ln Q}{d\ln \Delta p}$ is the slope of the $\ln Q$ and $\ln \Delta p$ plot. For a Newtonian flow (Poisseuille fluid), we have $\frac{d\ln Q}{d\ln \Delta p} = 1$. For a shear thinning fluid with $n = 1/4$, the relationship will be: $\frac{d\ln Q}{d\ln \Delta p} = 4$. But for any fluid, by measuring the volume flux through the pipe Q and the pressure gradient Δp, the value of the strain rate on the wall can be calculated from the above equation. Thus the viscosity on the wall can be expressed as:

$$\mu_{wall} = \frac{\sigma_{wall}}{\dot{\gamma}_{wall}} = \frac{\Delta p R}{2L\dot{\gamma}_{wall}}, \tag{14}$$

which gives the rheological law.

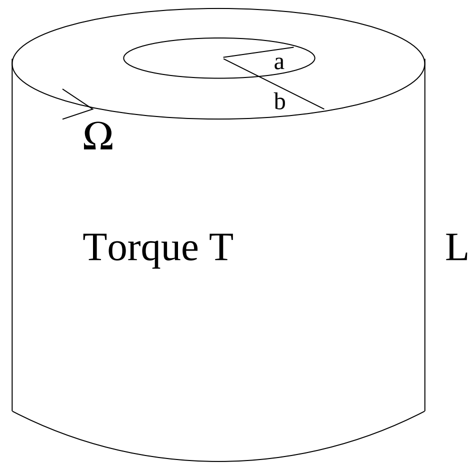

Figure 3: Couette device

3 Bingham yield fluid in a Couette device

For a Bingham fluid, if the applied stress σ is smaller than the yield stress, the fluid will remains rigid. But if the applied stress σ is larger than the yield stress, the fluid yields and shears. For a Couette device containing Bingham fluid, we assume that the inner radius is a and the outer radius is b and apply a torque T on the outer cylinder (see Figure 3). For an axisymmetric steady flow with velocity field $(0, u_\theta(r), 0)$, conservation of momentum in the θ direction demands that:

$$0 = \frac{1}{r^2}\frac{d}{dr}(r^2\sigma_{\theta r}). \tag{15}$$

Integrating this equation with respect to r, we have:

$$\sigma_{r\theta} = \frac{T}{2\pi L r^2}. \tag{16}$$

The constitutive equation for the Bingham fluid is:

$$\dot{\gamma} = 0 \qquad \text{if } \sigma < \sigma_y \tag{17}$$

$$\sigma_{r\theta} = \sigma_y + \mu\dot{\gamma} \qquad \text{if } \sigma > \sigma_y, \tag{18}$$

where σ_y is the yield stress and $\dot{\gamma}$ is the only non-zero component of the strain rate tensor: $\dot{\gamma} = r\frac{d}{dr}\left(\frac{u_\theta}{r}\right)$. Therefore, we could get the yield radius r_y in the device to be:

$$r_y = \sqrt{\frac{T}{2\pi L\sigma_y}}. \tag{19}$$

Therefore, the different positions for r_y correspond to different flow types of the Bingham fluid in the Couette device:

$$\text{Yield throughout} \qquad \text{if } r_y > b, \tag{20}$$

$$\text{Yield nowhere} \qquad \text{if } r_y < a, \tag{21}$$

$$\text{Intermediate situation} \qquad \text{if } a < r_y < b. \tag{22}$$

In the intermediate state,for $a < r_y < b$, the material yields and shears; but for $r_y < r < b$, the material rotates rigidly. Now we will calculate these two regions separately. For $a < r < r_y$, the shear rate is expressed as:

$$\dot{\gamma} = r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{\sigma_y}{\mu}\left(\frac{r_y^2}{r^2} - 1\right). \tag{23}$$

Integrating this equation with respect to r, we find:

$$\frac{u_\theta}{r} = \frac{\sigma_y}{\mu}\left(\frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right) - \ln\left(\frac{r}{a}\right)\right). \tag{24}$$

For the region $r_y < r < b$, the solid body rotation can be expressed as:

$$\frac{u_\theta}{r} = \Omega, \tag{25}$$

where Ω is the constant rotation rate. At $r = r_y$, we match the velocity in these two regions to find that

$$\Omega = \frac{\sigma_y}{\mu}\left(\frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right) - \ln\left(\frac{r}{a}\right)\right). \tag{26}$$

Since the radius for the yield surface r_y is also a function of the torque T, the angular velocity of the solid body rotation Ω is a function of the applied torque T.

4 Rod climbing of a second order fluid

As we have seen in earlier lectures, a well-documented phenomenon of non-Newtonian fluids is rod climbing. When a rod is rotated in a non-Newtonian fluid, the fluid is forced towards the center instead of being thrown away from the rod as in a Newtonian fluid. In a non-Newtonian fluid, tension in the streamlines, or the hoop stress, has a radial force inward that forces the fluid towards and then up the rod. In this section we determine the free surface of a weakly non-linear second-order fluid caused by a rod of radius a rotating at a frequency Ω in the fluid (figure 4).

The flow is weakly non-linear so to leading order, the fluid behaves like a Newtonian fluid. The velocity, which has only an azimuthal component, is determined from the azimuthal component of the momentum equation,

$$\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right). \tag{27}$$

Integrating this equation and applying boundary condition of $r = a$ and $r = \infty$ yields

$$u_\theta = \frac{\Omega a^2}{r}, \tag{28}$$

and the strain rate is

$$\dot{\gamma} = r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = -\frac{2\Omega a^2}{r^2}. \tag{29}$$

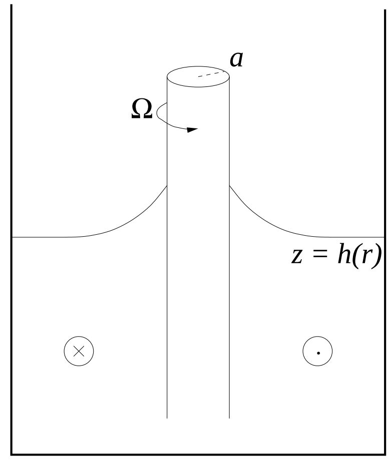

Figure 4: Geometry of rod climbing problem. A cylinder of radius a is rotating with frequency Ω, which causes a displacement of the free surface which is described by $h(r)$.

The constitutive equation of a second-order fluid is given by

$$\sigma = -pI + 2\mu E - 2\alpha \overset{\triangledown}{E} + \beta E \cdot E, \tag{30}$$

where $\overset{\triangledown}{E}$ is the upper convective derivative of the strain tensor E, and the last two terms of equation (30) are small compared to the others. Then, because $E_{r\theta} = E_{\theta r} = \frac{1}{2}\dot{\gamma}$, the components of stress are given by

$$\sigma_{\theta r} = \mu\dot{\gamma}, \tag{31}$$

$$\sigma_{rr} = -p + \frac{1}{4}\beta\dot{\gamma}^2, \tag{32}$$

$$\sigma_{\theta\theta} = -p + \left(2\alpha + \frac{1}{4}\beta\right)\dot{\gamma}^2, \tag{33}$$

$$\sigma_{zz} = -p. \tag{34}$$

In order to determine the pressure dependence in the radial direction, we apply the steady state Stokes equation and consider the radial component in cylindrical coordinates to find

$$0 = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}, \tag{35}$$

where the last term is the normal stress N_1 or the hoop stress. Using equations (32) and (33) we can write the normal stresses in terms of r. Integrating with respect to r yields

$$\sigma_{rr} + 2\alpha\frac{\Omega^2 a^4}{r^4} = -p + \frac{1}{4}(2\alpha + \beta)\dot{\gamma}^2 = f(z), \tag{36}$$

where f is an arbitrary function of z. The axial or vertical component of the Stokes equation gives a hydrostatic balance

$$0 = \frac{\partial \sigma_{zz}}{\partial z} + \rho g. \tag{37}$$

Integrating with respect to z and applying the boundary condition $p = 0$ at the free surface, $z = h(r)$, we find

$$p = \rho g \left(h(r) - z\right). \tag{38}$$

Finally, we choose $f(z) = \rho g z$ to cancel the z dependence in the pressure to find that the free surface height is described by

$$h(r) = \frac{1}{\rho g}\left(2\alpha + \beta\right)\frac{\Omega^2 a^4}{r^4}. \tag{39}$$

Therefore as long as $2\alpha + \beta > 0$ rod climbing occurs in the fluid. If this term is negative, though, it implies that the fluid is plunging down the rod rather than climbing up the rod. This solution seems unphysical and may be due to an improper force balance or improper definition of the constitutive equation.

5 Unchanged flow field for some second order fluids

In the previous section we have assumed that there is no small non-linear correction to the flow field u_θ as there is in the constitutive equation (30). In this section we sketch how to show that this is a valid assumption for some flows. At the end of the section we will see that the method described here does not necessarily verify an unchanged flow field in the rod climbing problem. The following proof will require the use of the identity,

$$\nabla \cdot \left(\overset{\triangledown}{E} + 4E \cdot E\right) = \frac{D}{Dt}\nabla^2 u + \nabla u \cdot \nabla^2 u + \nabla(E : E). \tag{40}$$

As we only touched on this topic briefly in lecture, we will not prove the identity here, but leave it as an exercise for the interested reader.

We define p_1 and p_2 as the pressure that satisfies the Stokes flow and the pressure related to the non-linear elastic effect, respectively. Then u and p_1 satisfy the equation for Stokes flow,

$$0 = \nabla p_1 + \mu \nabla^2 u. \tag{41}$$

Since u does not have an elastic correction, we must be able to show that u and p_2 satisfy

$$\nabla \cdot \sigma = 0, \tag{42}$$

$$\sigma = -p_2 I + 2\mu E - 2\alpha \overset{\triangledown}{E} - 4\alpha E \cdot E \tag{43}$$

with

$$p_2 = p_1 - \frac{\alpha}{\mu}\frac{Dp_1}{Dt} + \alpha E : E. \tag{44}$$

We can show that this is indeed true by using the identity given in (40). (Details are again left for the interested reader). Note that in (43) β has the specific value -4α. Recall that for rod climbing to occur $2\alpha + \beta > 0$, and therefore the method sketched here does not apply to the rod climbing problem. This analysis does show that for some flows, the equations of motion and the constitutive equation are satisfied without a small elastic correction to the flow field u. This holds in planar and uni-directional flows without a restriction on the relationship between α and β.

6 Anisotropic converging channel flow of a suspension of rigid rods

The constitutive equation for a suspension of rigid rods is given by

$$\sigma = -pI + 2\mu_{\text{shear}} E + 2\mu_{\text{ext.}} \hat{\mathrm{p}}\hat{\mathrm{p}} \left(\hat{\mathrm{p}} \cdot E \cdot \hat{\mathrm{p}} \right), \tag{45}$$

where $\hat{\mathrm{p}}$ is the unit vector in the direction of the rods. The shear and extensional viscosities, μ_{shear} and $\mu_{\text{ext.}}$, are constants with $\mu_{\text{ext.}}$ representing an additional viscosity over μ_{shear} in the direction of the rods. In this problem it is assumed that the rods are fixed in the direction of the flow, and we will describe the formation of recirculating eddies (cf. lecture 2) around a point sink.

The flow field is described by a two-dimensional sink flow in the half plane given by

$$\mathbf{u} = \left(\frac{f(\theta)}{r}, 0 \right), \qquad \hat{\mathrm{p}} = (1, 0) \tag{46}$$

in polar coordinates, where the radial direction represents the distance for the point sink and the azimuthal direction varies between 0 and π in the upper half plane. Therefore the velocity is purely in the radial direction and satisfies incompressibility.

Using the constitutive equation given in (45), we can determine the different components of the stress using

$$E_{rr} = \frac{\partial u_r}{\partial r}, \qquad E_{r\theta} = \frac{1}{2r}\frac{\partial u_r}{\partial \theta}, \qquad E_{\theta\theta} = \frac{u_r}{r}. \tag{47}$$

Note that the extensional viscosity only makes a contribution to σ_{rr}, so

$$\begin{aligned} \sigma_{rr} &= -\frac{g(\theta)}{r^2} - 2(\mu_{\text{shear}} + \mu_{\text{ext.}})\frac{f}{r^2}, & (48)\\ \sigma_{r\theta} &= \mu_{\text{shear}}\frac{f'}{r^2}, & (49)\\ \sigma_{\theta\theta} &= -\frac{g(\theta)}{r^2} + 2\mu_{\text{shear}}\frac{f}{r^2}, & (50) \end{aligned}$$

where $g(\theta)/r^2$ is the pressure, which is an unknown function of θ, and primes here denote differentiation with respect to θ. Now the two unknows, $f(\theta)$ and $g(\theta)$, can solved for by applying the two components of momentum conservation.

Assuming steady state and taking the azimuthal component of the steady Stokes equation $\nabla \cdot \sigma = 0$ (see table in Bird *et al.*),

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} = 0. \tag{51}$$

Using equations (49) and (50), this can then be written in terms of f and g by

$$g' = 2\mu_{\text{shear}} f'. \tag{52}$$

Similarly, the radial component of $\nabla \cdot \sigma = 0$ gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{53}$$

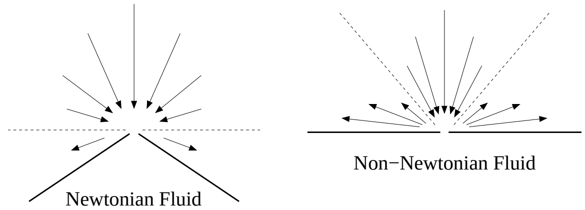

Figure 5: Diagrams of point sink converging channel flow. The diagram on the left shows that for a Newtonian fluid, recirculating vortices only appear when the plane is bent past horizontal. Non-Newtonian fluids can generate recirculating vortices even with a flat plate because of the additional extensional viscosity. The streamlines shown here are not valid close to the point sink as the flow is no longer purely radial in this region.

or in terms of f and g

$$f'' + \left(4 + 2\frac{\mu_{\text{ext.}}}{\mu_{\text{shear}}}\right) f = \text{const.} \tag{54}$$

The equation for f is a harmonic oscillator equation. We require that f goes to zero on the boundaries $\theta = 0, \pi$ to satisfy the condition of no slip, and we expect the maximum flow rate to occur in the middle, or $\theta = \pi/2$. For a Newtonian fluid, the additional extensional viscosity $\mu_{\text{ext.}} = 0$ and by applying the no-slip boundary condition we find to within a multiplicative constant that

$$f = 1 - \cos(2\theta), \tag{55}$$

and there are no recirculating eddies. If the plate is bent at an angle larger than π then recirculating eddies are present even in a Newtonian fluid. For a non-Newtonian fluid, $\mu_{\text{ext.}} \neq 0$ and we can solve equation (54) to find,

$$f = \cos\frac{\lambda\pi}{2} - \cos\lambda\left(\theta + \frac{\pi}{2}\right), \tag{56}$$

where $\lambda = sqrt4 + 2\mu_{\text{shear}}/\mu_{\text{ext.}}$. In this case there are always recirculating eddies on both sides of the sink (figure 5). Note that these equations approximate the flow far from the sink so that the streamlines are always in the radial direction. Close to the sink the streamlines must turn to form the vortices. From (56) we can see that the angle of the flow into the sink, and thus the size of the vortices, is determined by the ratio $\sqrt{\mu_{\text{shear}}/\mu_{\text{ext.}}}$.

7 Spinning of an Oldroyd B fluid

Finally, the last simple flow we will consider is fiber spinning of an Oldroyd B fluid. The constitutive equation of an Oldroyd B fluid is given by

$$\sigma = -pI + 2\mu E + GA, \tag{57}$$

where GA is the elastic stress, and A is a measure of the deformation of the microstructure. The time dependence of A is described by

$$\frac{DA}{Dt} = A \cdot \nabla U + (\nabla U)^T \cdot A - \frac{1}{\tau}(A - 1). \tag{58}$$

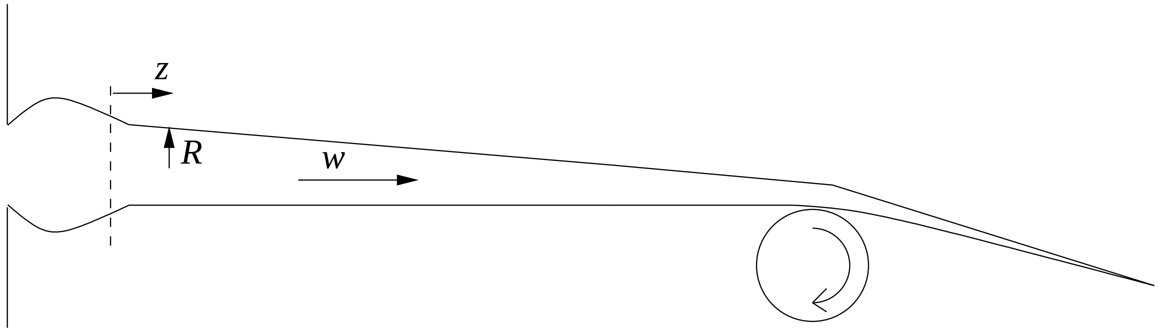

Figure 6: Geometry of fiber spinning of an Oldroyd B fluid. The fluid is pulled from a reservoir by a spinning wheel. After a brief initial transient region, the fluid is pulled into a cylinder whose radius decreases and velocity increases in the direction of positive z.

The first two terms on the right hand side describe the stretching of the microstructure while the last term describes relaxation back to the steady state, $A = 1$, on a time scale τ.

The geometry for this problem is shown in figure 6. Fluid begins in a reservoir, and after a brief initial transient region is pulled out into a cylinder beginning at $z = 0$ by a spinning wheel. The fluid is stretched and accelerated so that the velocity w increases and the radius R decreases moving in the direction of positive z. The flux and tension are both constant and are given by

$$Q = \pi R^2 w \tag{59}$$

and

$$F = \pi R^2 \sigma_{zz} \tag{60}$$

respectively.

Considering the constitutive equation and applying the boundary condition of no radial stress on the free surface $\sigma_{rr} = 0$ (57) gives

$$p = -\mu \frac{dw}{dz} + GA_{rr}. \tag{61}$$

Then, considering the axial component of stress and using the expression for pressure given above, we find

$$\sigma_{zz} = 3\mu \frac{dw}{dz} + G\left(A_{zz} - A_{rr}\right) = \frac{Fw}{Q}, \tag{62}$$

where we have made use of the expressions for the tension and the flux in the cylinder. Finally, from equation (58) the steady state equations for the radial and axial components of the deformation of A can be written as

$$w\frac{dA_{rr}}{dz} = -A_{rr}\frac{dw}{dz} - \frac{1}{\tau}\left(A_{rr} - 1\right) \tag{63}$$

$$w\frac{dA_{zz}}{dz} = 2A_{zz}\frac{dw}{dz} - \frac{1}{\tau}\left(A_{zz} - 1\right). \tag{64}$$

In general, for a given Deborah number, $\mathrm{De} = \tau\frac{dw}{dz}$, the equations above must be solved numerically. However, we will consider two limits where the equations are analytically tractable. In the Newtonian limit of small Deborah number, De $\ll 1$, the two terms on the right hand side of equations (63) and (64) balance to give

$$A_{rr} \sim 1 - \tau \frac{dw}{dz}, \qquad A_{zz} \sim 1 + 2\tau \frac{dw}{dz}. \tag{65}$$

Plugging these expressions for A_{rr} and A_{zz} into (62) gives a simple linear equation

$$3(\mu + G\tau)\frac{dw}{dz} = \frac{Fw}{Q}, \tag{66}$$

which has the solution

$$w(z) = w(0) \exp\left[\frac{Fz}{3Q(\mu + G\tau)}\right]. \tag{67}$$

Therefore in the Newtonian regime the velocity increases exponentially until the elastic limit is reached, in which the relaxation rate is comparable to the shear, or $\tau \frac{dw}{dz} \sim 1$.

In the elastic limit the stretching term in the axial direction dominates radial stretching and viscosity, $\mu \frac{dw}{dz} \ll GA_{rr} \ll GA_{zz}$, and therefore

$$\sigma_{zz} \sim GA_{zz} = \frac{Fw}{Q}. \tag{68}$$

From (64) we have

$$w\frac{dA_{zz}}{dz} = 2A_{zz}\frac{dw}{dz} - \frac{1}{\tau}(A_{zz}) \tag{69}$$

since $A_{zz} \gg 1$. Plugging $A_{zz} = Fw/Q$ into this equation yields a simple linear differential equation which has the solution

$$w = w_1 + \frac{z}{\tau}. \tag{70}$$

Upon reaching the elastic regime the velocity only grows linearly. Numerical results indicate that there is an abrupt change in the dynamics of the thread as the fluid transitions between the Newtonian and elastic regimes. This transition affects not only the velocity, but also the radius of the thread as given from the thread's flux and tension relationships (59) and (60).

Notes by Junjun Liu and Andrew Thompson