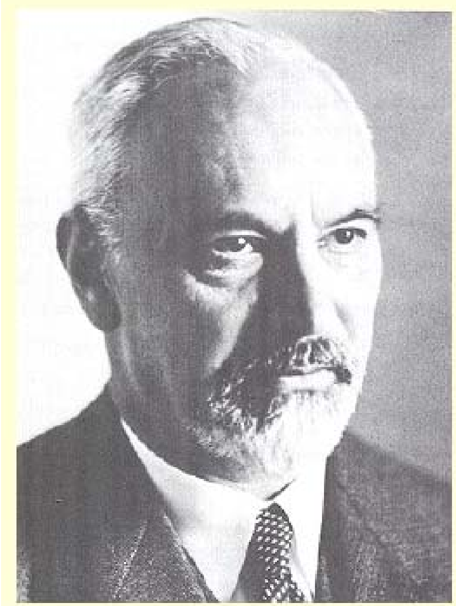
GFD 2007
Boundary Layers

The idea of the *boundary layer* dates back at least to the time of Prandtl (1904, see the article: Ludwig Prandtl's boundary layer, *Physics Today*, 2005, **58, no.12**, 42-48).

Ludwig Prandtl

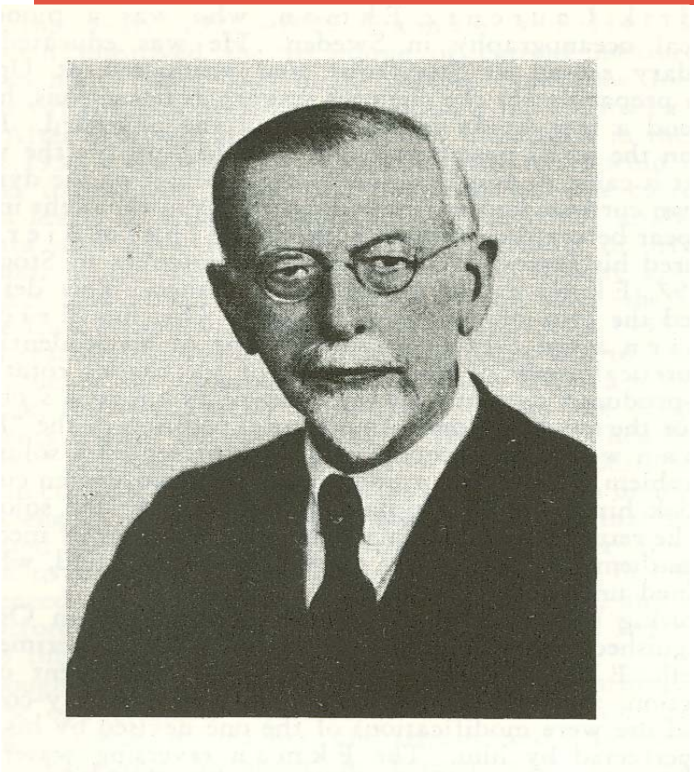
I have set myself the task of investigating systematically the motion of a fluid of which the internal resistance can be assumed very small. In fact, the resistance is supposed to be so small that it can be neglected wherever great velocity differences or cumulative effects of the resistance do not exist. This plan has proved to be very fruitful, for one arrives thereby at mathematical formulations which not only permit problems to be solved but also give promise of providing very satisfactory agreement with observation.

... the investigation of a particular flow phenomenon is thus divided into two interdependent parts: there is on the one hand the *free fluid*, which can be treated as inviscid according to the vorticity principles of Helmholtz, and on the other hand the transition layers at the fixed boundaries, the movement of which is controlled by the free fluid, yet which in turn give the free movement its characteristic stamp by the emission of vortex sheets.



Ludwig Prandtl 1875-1953

Contemporaneously, Ekman…



V.W.Ekman

Young Ekman

Considered the effects of rotation although he did not really think of his solutions in terms of what we would call boundary layer theory.

The principal concept of the boundary originally springs from the particular form of the fluid continuum equations in which the dissipation terms involve higher order derivatives than the inertial, advective terms, e.g. for the Navier Stokes equations for a non rotating fluid:

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\rho F_i+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

For fluids like air or water the coefficient of viscosity μ is often sufficiently small, in a non-dimensional sense to be clarified more formally below, such that the physical effects of friction would seem to be negligible allowing the neglect of the last term on the right hand side of the equation.

This is a singular perturbation.

The order of the equations is reduced and we can no longer satisfy all the boundary conditions if the viscous term is neglected.

The mathematical issue is how to retain the higher order derivatives only where they are needed to help satisfy the boundary conditions and

the physical issue is to understand through the applications of boundary layer theory how (and whether) the action of friction in very localized regions may affect the fluid flow in regions *outside* the domain directly affected by friction. The interplay between the *outer* region, in which friction is not directly important, and the *inner* region in which friction directly acts is a key feature of boundary layer theory (a form of *singular perturbation theory*).

An Oceanic example
Wind-driven ocean circulation model

$$\varepsilon J(\psi, \nabla^2 \psi) + \psi_x = -r\nabla^2 \psi + \nu \nabla^4 \psi + T(x, y)$$

$$J(a,b) \equiv a_x b_y - a_y b_x \qquad \varepsilon = U / \beta L^2$$

$$\nu = \frac{A_H}{\beta L^3}, \quad r = \frac{r_*}{\beta L}$$

If *r* and *v* neglected and the no slip condition is dropped, there will still be a singular perturbation to the equations if the ε term, i.e. the nonlinear advection terms are ignored. This leads to an *inertial* boundary layer.This equation in its entirety will be discussed more fully later.

An outline of where we will be going

1) Linear boundary layer theory

Ekman layers, Boundary layers in density stratified fluids, control of interior, experimental applications.

2) Coastal bottom boundary layer.

Boundary layer on shelf for upwelling and downwelling.

Observations (Lentz)

3) Boundary layers in the General Oceanic Circulation.

Sverdrup theory, Stommel, Munk, inertial boundary layers, inertial runaway,thermocline and its boundary layer structure.

Equations of motion

$$uu_x + vu_y + wu_z - 2\Omega v = -\frac{1}{\rho} p_x + \nu\left[u_{xx} + u_{yy} + u_{zz}\right]$$

$$uv_x + vv_y + wv_z + 2\Omega u = -\frac{1}{\rho} p_y + \nu\left[v_{xx} + v_{yy} + v_{zz}\right]$$

$$uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z - g + \left[w_{xx} + w_{yy} + w_{zz}\right]$$

$$u_x + v_y + w_z = 0$$

Incompressible fluid in a rotating system.

If the density is not constant must add an ernery equation

We are interested in cases where ν is “small”. Must introduce scales.

The Ekman Layer

$$\varepsilon\left[vu_y + wu_z\right] - v = \frac{E}{2}\left[u_{zz} + u_{yy}\right]$$

$$\varepsilon\left[vv_y + wv_z\right] + u = -\frac{\partial p}{\partial y} + \frac{E}{2}\left[v_{zz} + v_{yy}\right]$$

$$\varepsilon\left[vw_y + ww_z\right] = -\frac{\partial p}{\partial z} + \frac{E}{2}\left[w_{zz} + w_{yy}\right]$$

$$v_y + w_z = 0$$

The pressure has been scaled with $\rho 2\Omega L U_0$

Lengths with L and velocity with U_0

$$\varepsilon = \frac{U_o}{2\Omega L},\quad \text{the Rossby number}$$

$$E = \frac{\nu}{\Omega L^2},\quad \text{the Ekman number} \ll 1$$

Far from the boundary the velocity is $U(y)$.

Motion is independent of x (for simplicity)

The solution far from the boundary

$$u_I = U(y),$$

$$p_I = -\int^{y} U(y')dy',$$

$$v_I = 0,$$

$$w_I = 0.$$

This is an exact solution of the equations of motion but does not satisfy the no slip condition on z =0.

We introduce the stretched, boundary layer variable.

$$z = E^{1/2}\zeta$$

Corresponding to using as a vertical scale,

$$\delta_e = \left(\frac{\nu}{\Omega}\right)^{1/2},$$

In the new variable

$$\frac{\partial}{\partial z} = E^{-1/2}\frac{\partial}{\partial \zeta}, \qquad \frac{\partial^2}{\partial z^2} = E^{-1}\frac{\partial^2}{\partial \zeta^2} \qquad\qquad w = E^{1/2}W(y,\zeta)$$

$$\varepsilon\left[vu_y + Wu_\zeta\right] - v = \frac{1}{2}\left[u_{\zeta\zeta} + E\ u_{yy}\right]$$

$$\varepsilon\left[vv_y + Wv_\zeta\right] + u = -\frac{\partial p}{\partial y} + \frac{1}{2}\left[v_{\zeta\zeta} + E\ v_{yy}\right]$$

$$\varepsilon E\left[vW_y + WW_\zeta\right] = -\frac{\partial p}{\partial \zeta} + \frac{E}{2}\left[W_{\zeta\zeta} + E\ W_{yy}\right]$$

$$v_y + W_\zeta = 0$$

Initially consider ε small

Linear Ekman layer problem

$$-v = \frac{1}{2} u_{\zeta\zeta}$$

$$+u = -\frac{\partial p}{\partial y} + \frac{1}{2} v_{\zeta\zeta}$$

$$0 = -\frac{\partial p}{\partial \zeta}$$

The pressure is uniform in the boundary layer and so is equal to its freestream value $p = p_I(y),$

$$v_y + W_\zeta = 0$$

$$\frac{\partial p}{\partial y} = \frac{\partial p_I}{\partial y} = -U(y)$$

thus

$$-v = \frac{1}{2} u_{\zeta\zeta}$$

$$u - U = \frac{1}{2} v_{\zeta\zeta}$$

$$v_y + W_\zeta = 0$$

The solution (1)

let

$$\Lambda = (u - U) + iv \longrightarrow \frac{\partial^2 \Lambda}{\partial \zeta^2} = 2i\Lambda$$

$$\Lambda = \Lambda_o \exp(-[1+i]\zeta)$$

or

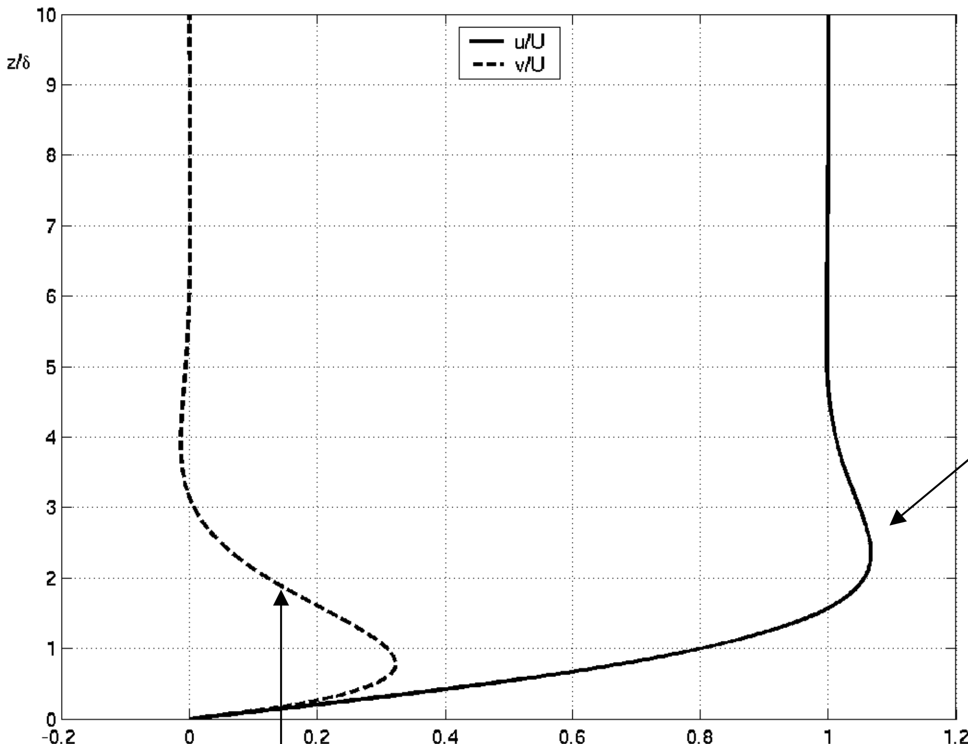
$$u = U + e^{-\zeta}[-A\cos\zeta + B\sin\zeta]$$
$$v = e^{-\zeta}[A\sin\zeta + B\cos\zeta]$$

The conditions that both *u* and *v* vanish on $z=\zeta=0$, yields

$$A = U(y), \quad B = 0$$

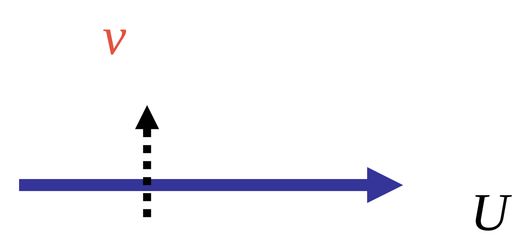
$$u = U(y)\left(1 - e^{-\zeta}\cos\zeta\right)$$
$$v = U(y)e^{-\zeta}\sin\zeta$$

The solution (2)

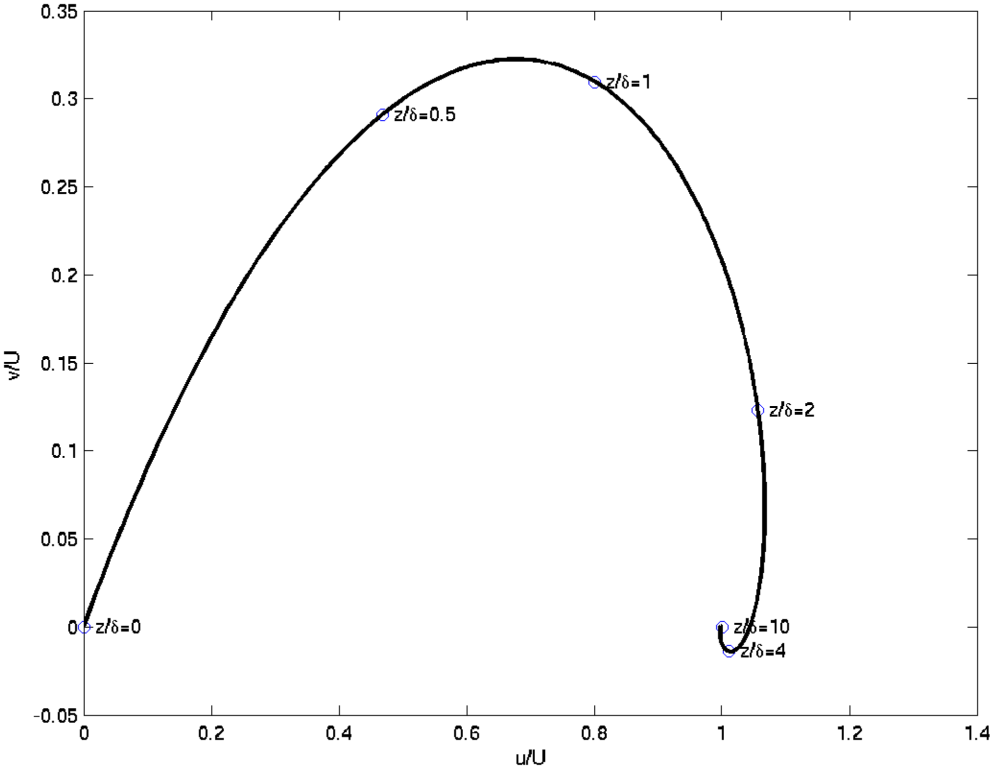


Note the overshoot

cross isobar flow perpendicular to $U(y)$

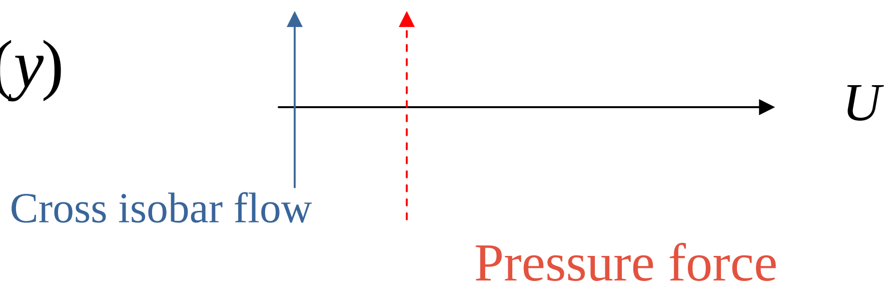


The Ekman spiral



The cross isobar flow (in y direction) and Ekman "pumping"

$$\int_0^\infty v dz = \delta_e \int_0^\infty v d\zeta = \frac{\delta_e}{2} U(y)$$



$$W = -\frac{1}{2}\frac{\partial U}{\partial y}\left[1 - e^{-\zeta}(\cos\zeta + \sin\zeta)\right]$$

and

$$w_I(y,0) = E^{1/2} W(y, \zeta \to \infty) = -E^{1/2} U_y / 2$$

Spin Down (1)

this vertical velocity is small but it can have a substantial effect on the interior flow and is, in many cases the primary mechanism for the destruction of otherwise inviscid motion in the interior. For example, a positive relative vorticity will give rise to a positive vertical velocity out of the boundary layer. If the interior is bounded from above this will usually imply that a column of fluid in the interior is squashed and the result will be to *decrease* the relative vorticity by the inertial effects of vortex squashing. The rate will depend on the Ekman number but the resulting spin-down time will be of the order of $\Omega^{-1} E^{-1/2}$ and so will be long compared with a rotation period of the system but *short* compared to a characteristic diffusion time $\Omega^{-1} E^{-1}$.

Spin down (2)

For small ε

$$\varepsilon \frac{d\omega_z}{dt} = \frac{\partial w}{\partial z}$$

$$\varepsilon \frac{d\omega_z}{dt} = -E_v^{1/2} w(z=0) = -E_v^{1/2}(-U_y/2) = -E_v^{1/2} \frac{\omega_z}{2}$$

$$T_s = \frac{2\varepsilon}{E^{1/2}}$$

Non dimensional decay time

$$T_{*s} = \frac{L}{U_o} \frac{U_o}{\Omega L} \frac{L}{(\nu/\Omega)^{1/2}} = \frac{L}{(\nu\Omega)^{1/2}}$$

In dimensional (non dimensionless) units

Nonlinear modifications of the Ekman Layer

We can expect that nonlinearity will force additional terms in the solution but it will also be the case that it will change the *structure* of the *O(1)* solution. Since the boundary layer, in linear theory depends on the ratio of the viscosity to the rotation Ω, it is often assumed, heuristically, that the first effect of nonlinearity is to change the thickness to something like ,

$$\delta = \left[2\nu / (f + \omega_z) \right]^{1/2}$$

where $f = 2\Omega$ (the Coriolis parameter and the planetary vorticity and where ω is the relative vorticity .

$$\omega_z = v_x - u_y$$

Hence, the expectation is that positive relative vorticity in the interior flow will make the boundary layer thinner. However, positive vorticity produces a vertical velocity that will tend to thicken the layer

A new stretched variable

$$Z = \varepsilon\zeta = \frac{\varepsilon}{E^{1/2}} z$$

And consider all variables to be functions of both ζ and Z

$$\frac{\partial}{\partial z} \to E^{-1/2}\frac{\partial}{\partial \zeta} + \varepsilon E^{-1/2}\frac{\partial}{\partial Z}$$

To order ε

$$\varepsilon\left[vu_y + Wu_\zeta\right] - v = \frac{1}{2}\left[u_{\zeta\zeta} + 2\varepsilon u_{\zeta Z} + E\ u_{yy}\right]$$

$$\varepsilon\left[vv_y + Wv_\zeta\right] + u = -\frac{\partial p}{\partial y} + \frac{1}{2}\left[v_{\zeta\zeta} + 2\varepsilon v_{\zeta Z} + E\ v_{yy}\right]$$

$$\varepsilon E\left[vW_y + WW_\zeta\right] = -\frac{\partial p}{\partial \zeta} + \frac{E}{2}\left[W_{\zeta\zeta} + 2\varepsilon W_{\zeta Z} + E\ W_{yy}\right]$$

$$v_y + W_\zeta + \varepsilon W_Z = 0$$

The ε expansion

$$u = u_o + \varepsilon u_1 + \ldots$$

$$u_o = U - A(y,Z)e^{-\zeta}\cos\zeta + B(y,Z)e^{-\zeta}\sin\zeta,$$

$$v_o = A(y,Z)e^{-\zeta}\sin\zeta + B(y,Z)e^{-\zeta}\cos\zeta$$

$$A(0) = U, \quad B(0) = 0$$

$$W_o = C(Z) + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}\left[\sin\zeta + \cos\zeta\right] + \frac{\partial B}{\partial y}e^{-\zeta}\left[\cos\zeta - \sin\zeta\right] \qquad C(0) = -\frac{1}{2}\frac{\partial U}{\partial y}$$

The solution for W_0

the vertical velocity must be independent of z at least to order ε and E. This implies that C is independent of Z

$$W_o = -\frac{1}{2}\frac{\partial U}{\partial y} + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}\left[\sin\zeta + \cos\zeta\right] + \frac{\partial B}{\partial y}e^{-\zeta}\left[\cos\zeta - \sin\zeta\right]$$

The next order problem

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = Ru + iRv,$$

$$\Lambda_1 = u_1 + iv_1,$$

$$Ru \equiv 2\left[v_o u_{oy} + W_o u_{o\zeta}\right] - 2u_{o\zeta Z},$$

$$Rv \equiv 2\left[v_o v_{oy} + W_o v_{o\zeta}\right] - 2v_{o\zeta Z}$$

Evaluating the terms in *Ru* and *Rv* reveals that a combination of some of the terms will have the form of the homogeneous operator on the left hand side of the eqn. If left unaltered those terms would introduce spatial secular terms, i.e. solutions of the form $\zeta e^{-(1+i)\zeta}$

and would render our expansion invalid for $\varepsilon\zeta = Z = O(1)$. To prevent that, the terms in *Ru* and *Rv* involving the derivatives of the coefficients with *Z* are used to eliminate all secular terms.

Removing secular terms

Eliminating terms of the form $e^{-(1+i)\zeta}$

leaves us with a differential eqn. in *Z* for *A* and *B*

$$\frac{\partial}{\partial Z}\left[A-iB\right]+\left[A-iB\right]\left(\underbrace{C}_{w\ term}+\overbrace{\frac{U_{Iy}}{2i(1+i)}}^{vorticity\ term}\right)=0$$

$$C=-U_{Iy}$$

and so

$$A-iB=U_I e^{-U_{Iy}Z/4}e^{iU_{Iy}Z/4}$$

$$=U_I e^{-U_{Iy}Z/4}\left(\cos(U_{Iy}Z/4)+i\sin(U_{Iy}Z/4)\right)$$

The weakly nonlinear solution

$$A(Z) = Ue^{-U_y Z/4}\cos U_y Z/4,$$

or

$$B(Z) = -Ue^{-U_y Z/4}\sin U_y Z/4$$

$$v_o = Ue^{-\zeta\left[1+\varepsilon U_y/4\right]}\sin(\zeta\left[1-\varepsilon U_y/4\right])$$

$$u_o = U\left[1-e^{-\zeta\left[1+\varepsilon U_y/4\right]}\cos\left(\zeta\left[1-\varepsilon U_y/4\right]\right)\right]$$

The exponential decay decreases when the relative vorticity is positive, i.e. when $U_y < 0$. The *effect of the vertical velocity in thickening the boundary layer dominates the vorticity effect in determining the boundary layer thickness.*

The vertical velocity

At lowest order

$$W_o = -\frac{1}{2}U_y\left[1 - e^{-(\zeta + U_y Z/4)}\left[\sin(\zeta - U_y Z/4) + \cos(\zeta - U_y Z/4)\right]\right]$$

To complete the solution we need to find the next order corrections to *u* and *v*, *i.e.* to solve

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = Ru + iRv,$$

$$\Lambda_1 = u_1 + iv_1,$$

$$Ru \equiv 2\left[v_o u_{oy} + W_o u_{o\zeta}\right] - 2u_{o\zeta Z},$$

$$Rv \equiv 2\left[v_o v_{oy} + W_o v_{o\zeta}\right] - 2v_{o\zeta Z}$$

after the secular terms have been removed.

The non secular problem
(no particular religious meaning implied)

$$Ru_{non\sec} = e^{-2\zeta}\left[AA_y + BB_y + AB_y - BA_y\right] - iU_y e^{-\zeta(1-i)}(A + iB),$$

$$Rv_{non\sec} = e^{-2\zeta}\left[AA_y + BB_y + AB_y - BA_y\right]$$

let

$$\Lambda_1 = u_1 + iv_1$$

$$\Lambda_{1_{\zeta\zeta}} - 2i\Lambda_1 = e^{-2(\zeta+\gamma Z)}(1+i)UU_y - iUU_y e^{-\zeta(1-i)} e^{-\gamma Z(1+i)} \qquad \gamma = U_y / 4$$

$$\Lambda_1 = UU_y\left[\frac{(1+3i)}{10}\left\{e^{-2(\zeta+\gamma Z)} - e^{-\zeta(1+i)}\right\} + 1/4\left\{e^{-\zeta(1-i)-\gamma Z(1+i)} - e^{-\zeta(1+i)}\right\}\right]$$

The cross isobar flux to order ε

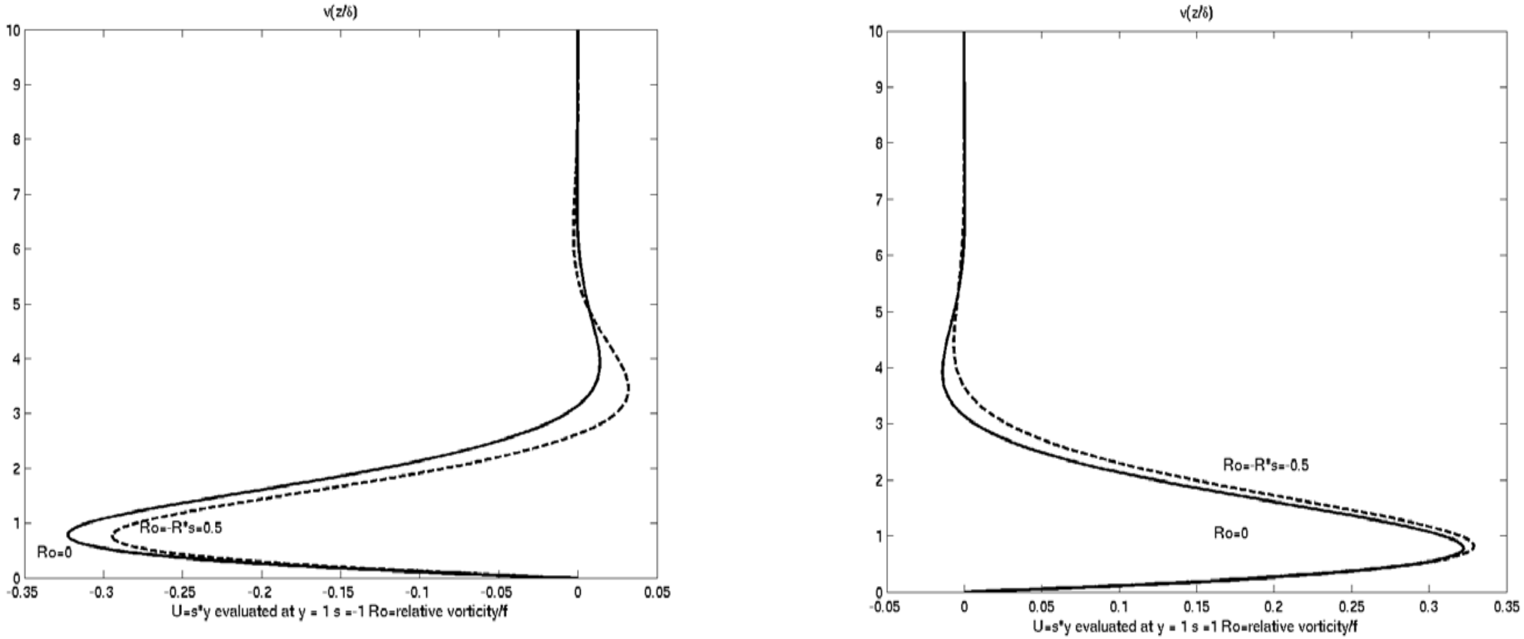
$$E^{1/2}\int_0^\infty (v_o+\varepsilon v_1)\,d\zeta = E^{1/2}\frac{1}{2}\left[U+\varepsilon\frac{7}{20}UU_y\right]$$

linear nonlinear

The divergence of this flow yields the vertical velocity at the edge of the boundary layer

$$w_I(y,0) = -E^{1/2}\left[U_y+\varepsilon\frac{7}{40}\left(U_y^{\,2}+UU_{yy}\right)\right]$$

The form of the cross isobar flow to order to order ε



The panel on the left shows the cross isobar (v) flow for the linear solution (R_0=0, solid line) and the solution corrected for nonlinear effects (R_0= 0.5, dashed line – and clearly "pushing" the validity of the expansion). The panel on the left is for negative , uniform shear (positive relative vorticity) and the panel on the right is for positive shear (negative vorticity).

references

Benton G.S., F.B. Lipps and S.Y. Tuann. 1964 .The structure of the Ekman layer for geostrophic flows with lateral shear. *Tellus* **16**, 186-199

Hart, J.E. 2000 A note on the nonlinear correction to the Ekman layer pumping velocity. *Phys.of Fluids*. **12**, 131-135

Brink, K.H.1997 Time dependent motions and the nonlinear bottom Ekman layer, *J.Marine Res*. **55**, 613-631

Nansen’s problem

sea surface z=0

Scale for velocity

ocean $-\infty \le z \le 0$

$$U_o = \frac{2\tau_o}{\rho f \delta_e}, \qquad f = 2\Omega$$

$$u_\zeta = \tau, \qquad v_\zeta = 0$$

(non dimensional)

$$-2v = u_{\zeta\zeta}$$

$$+2u = v_{\zeta\zeta}$$

$$v_y + W_\zeta = 0$$

linear problem

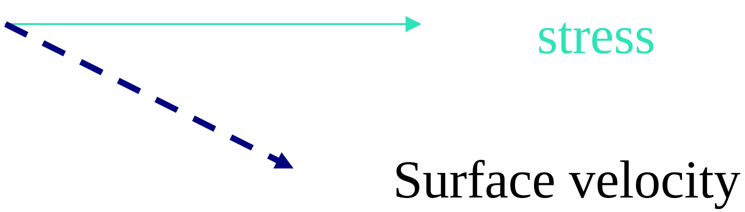
Ekman's solution

$$u = \frac{\tau}{\sqrt{2}} e^{\zeta} \cos(\zeta - \pi/4)$$

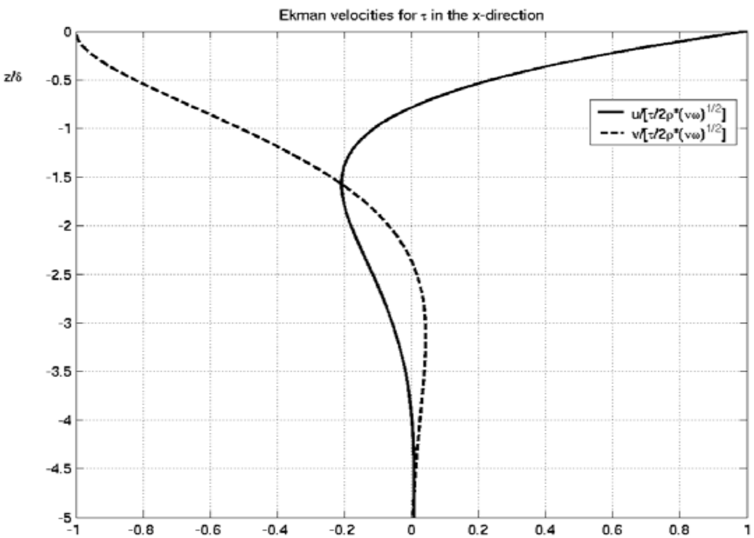
$$v = \frac{\tau}{\sqrt{2}} e^{\zeta} \sin(\zeta - \pi/4)$$

$$u \to \tau/2,\ v \to -\tau/2$$

as ζ goes to zero



The profiles of velocity and the Ekman flux



The Ekman velocities *u* (solid) and *v*(dashed) for a stress in the *x* direction.

$$U_e = \int_0^{\infty} u \, d\zeta = 0,$$

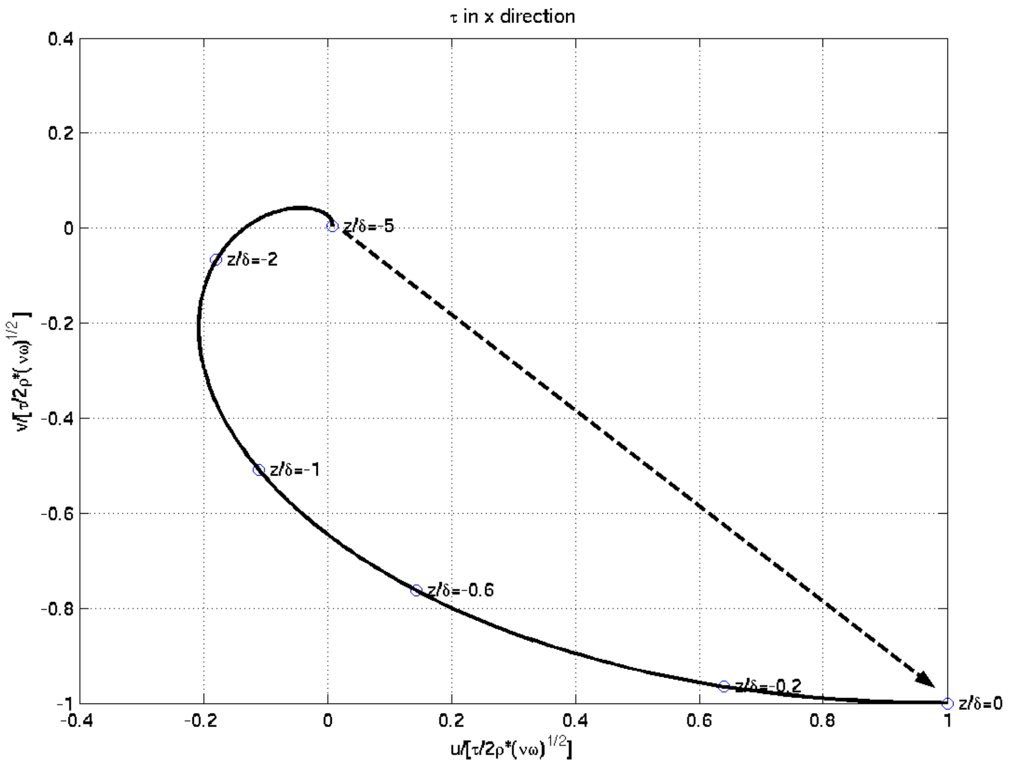
$$V_e = \int_0^{\infty} v \, d\zeta = -\tau / 2$$

total flux is to the right of the stress. In dimensional units:

$$V_e^* = U_o \delta_e \cdot V_e = -\frac{2\tau_o}{\rho f \delta_e}\delta_e \frac{\tau}{2} = -\frac{\tau_*}{\rho f}$$

In general: $\vec{U}_{*e} = -\frac{\hat{k} \times \vec{\tau}_*}{\rho f}$

The Ekman hodograph spiral



The vertical velocity

$$W = -\frac{1}{2}\frac{\partial \tau}{\partial y}\left[1 - e^{\zeta}\cos\zeta\right]$$

$$W(-\infty) = -\frac{1}{2}\frac{\partial \tau}{\partial y}$$

with dimensions restored and in vector form,

$$w_e \equiv \frac{U\delta_e}{L} W(-\infty) = \hat{k}\cdot\nabla\times(\vec{\tau}_* / \rho f)$$

For a wind stress of one dyne/cm^2 the vertical velocity is order of 10^{-4} cm/sec ~ 10 cm/day and is responsible for driving the major part of the ocean circulation.

Nonlinear similar to previous example. *However one simple result follows almost immediately*.

Role of nonlinearity on Ekman transport (and pumping)

Consider stress in x direction but now include a very strong geostrophic current in the x direction whose scale is much greater than the Ekman depth. Again, with solutions independent of x

$$\varepsilon\left[vu_y + Wu_\zeta\right] - v = \frac{1}{2}\left[u_{\zeta\zeta} + E\ u_{yy}\right]$$

$$\varepsilon\left[vv_y + Wv_\zeta\right] + u = -\frac{\partial p}{\partial y} + \frac{1}{2}\left[v_{\zeta\zeta} + E\ v_{yy}\right]$$

$$\varepsilon E\left[vW_y + WW_\zeta\right] = -\frac{\partial p}{\partial \zeta} + \frac{E}{2}\left[W_{\zeta\zeta} + EW_{yy}\right]$$

$$v_y + W_{\zeta z} = 0$$

The linearized equations

$$-\frac{\partial p}{\partial y} = u_g(y) >> u_e \approx \tau / \rho f \delta_e$$

$$\varepsilon v u_{gy} - v = \frac{1}{2} u_{\zeta\zeta}$$

$$u - u_g = \frac{1}{2} v_{\zeta\zeta}$$

Vertically integrating and using

$$u_\zeta = \tau, \quad v_\zeta = 0$$

$$V_e = \frac{-\tau}{2\left[1 - \varepsilon u_{gy}\right]},$$

$$U_e = 0$$

Ekman transport (dimensional)

$$\vec{U}_{*e} = -\frac{\hat{k}\times\vec{\tau}/\rho}{f+\zeta_g}$$

Relative vorticity

This holds only when the stress and geostrophic current are collinear

Refs.
M. E. Stern 1965 Interaction of a uniform wind stress with a geostrophic vortex. *Deep Sea Res.* **12**, 355-367
P.P. Niiler 1969 On the Ekman divergence in an oceanic jet. *J. Geophys. Res.* **74**, 7048-7052

General nonlinear formulation

Again, expand in a series in ε and introduce the "slow" variable

$$Z = \varepsilon\zeta$$

$$\Lambda_0 = A(Z)e^{\zeta(1+i)},\ \Lambda_0 = u_o + iv_o$$

$$u_o = u_g(y) + e^{\zeta}\left(A_r\cos\zeta - A_i\sin\zeta\right),$$
$$v_o = e^{\zeta}\left(A_i\sin\zeta + A_r\cos\zeta\right)$$

$$A_r(0) = \tau/2 \quad A_i(0) = -\tau/2$$

$$W_0 = -\frac{\tau_y}{2} - \frac{e^{\zeta}}{2}\frac{\partial}{\partial y}\left[\left(A_i - A_r\right)\cos\zeta + \left(A_i + A_r\right)\sin\zeta\right]$$

Order ε problem

$$\frac{1}{2}\Lambda_{1\zeta\zeta} - i\Lambda = -e^{\zeta(1+i)}\frac{\partial}{\partial Z}\left[(A_r - A_i) + i(A_r + A_i)\right]$$

$$+e^{\zeta(1+i)}\left[-\frac{\tau_y}{2}\left\{(A_r - A_i) + i(A_r + A_i)\right\} + \frac{u_{gy}}{2i}\left\{A_r + iA_i\right\}\right]$$

$$+e^{\zeta(1-i)}\frac{u_{gy}}{2i}\left[-A_r + iA_i\right] + e^{2\zeta}\frac{(1+i)}{2}\left[A_i(A_{ry} + A_{iy}) + A_r(A_{ry} - A_{iy})\right]$$

Possible resonant terms

Removing the secular terms leads to

$$A_r = \frac{\tau}{2^{1/2}} e^{-Z\left[\tau_y/2 + u_{gy}/4\right]} \cos(Zu_{gy}/4 + \pi/4)$$

$$A_i = -\frac{\tau}{2^{1/2}} e^{-Z\left[\tau_y/2 + u_{gy}/4\right]} \sin(Zu_{gy}/4 + \pi/4)$$

The structure of the O(1) solution

$$u_o = u_g + \frac{\tau}{2^{1/2}} e^{\zeta\left[1-\varepsilon\left(\frac{\tau_y}{2}+\frac{u_{gy}}{4}\right)\right]} \cos\left\{\zeta(1-\varepsilon u_{gy}/4)-\pi/4\right\}$$

$$v_o = \frac{\tau}{2^{1/2}} e^{\zeta\left[1-\varepsilon\left(\frac{\tau_y}{2}+\frac{u_{gy}}{4}\right)\right]} \sin\left\{\zeta(1-\varepsilon u_{gy}/4)-\pi/4\right\}$$

$$\delta_\varepsilon = \frac{\delta}{\left(1-\varepsilon\left\{\tau_y + u_{gy}/2\right\}\right)^{1/2}}$$

Boundary layer thickness

$$\delta_* = \left(\frac{2\nu}{f - \frac{2\tau_{*y_*}}{\rho f \delta} - \frac{u_{*gy_*}}{2}}\right)^{1/2}$$

dimensional

same result if we linearize in transition region around

$u_g(y)$ and $W = -\tau_y/2$

Boundary layer flux to O(ε)

Solution for non secular forcing yields

$$\int_{-\infty}^{0} \varepsilon v_1 \, d\zeta = -\frac{3}{8}\varepsilon \tau u_{gy}$$

And using the O(1) solution's dependence on ε

$$\int_{-\infty}^{0} (v_o + \varepsilon v_1)\, d\zeta = -\frac{\tau}{2} - \varepsilon\tau\left[\frac{\tau_y}{8} + \frac{u_{gy}}{2}\right]$$

Thomas, L.N. and P.B. Rhines, 2002. Nonlinear stratified spinup. *J.Fluid Mech.*, **473**, 211-244

$$W(-\infty) = \frac{\partial}{\partial y}\left(\int_{-\infty}^{0} v_o + \varepsilon v_1\right) d\zeta$$

$$W(-\infty) = -\frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\tau}{1 - \varepsilon\left\{u_{gy} + \tau_y / 4\right\}}\right]$$