Lecture 3 - Mathematical Foundations of Stochastic Processes

Charles R. Doering; notes by Florence Marcotte & Cesar B. Rocha

June 17, 2015

Recalling from lecture 2 that we are studying the general first-order linear stochastic ordinary differential equation

$$\Delta X = f(X(t), t) + g(X(t), t)\xi(t)\Delta t\,, \tag{1}$$

where $\xi(t)$ is a Gaussian white noise, and

$$\Delta X(t) = X(t+\Delta t) - X(t)\,, \tag{2}$$

with

$$X(0) = X_0\,. \tag{3}$$

The first and second moments of the transition density are

$$\int (y-x)\rho(y, t+\Delta t|x, t)\mathrm{d}y = \mathbb{E}(\Delta X(t)|X(t)=x) = f(X,t)\Delta t\,, \tag{4}$$

$$\int (y-x)^2\rho(y, t+\Delta t|x, t)\mathrm{d}y = \mathbb{E}(\Delta X(t)^2|X(t)=x) = f(X,t)^2\Delta t^2 + g(X(t),t)^2\Delta t\,, \tag{5}$$

The higher order moments ($n > 2$) are smaller than $\mathcal{O}(\Delta t)$:

$$\mathbb{E}(\Delta X(t)^n|X(t)=x) = \wr(\Delta t)\,. \tag{6}$$

The Fokker-Planck equation

The Chapman-Kolmogorov relation for the Markovian probability density ρ gives the transition density from state y at time s to state x at time t (where $s < t$) through an intermediate state z at time u, by integrating over all the possible intermediate states z:

$$\rho(x,t|y,s) = \int \rho(x,t|z,u)\rho(z,u|y,s)\mathrm{d}z\,. \tag{7}$$

This relation will be used here to interpret a stochastic differential equation as a continuous limit of the discrete process above. Let $\varphi(X)$ a smooth test-function.

$$\begin{aligned}\int \varphi(x)\rho(x,t+\Delta t|y,s)\mathrm{d}x &= \int \mathrm{d}x\varphi(x)\int \rho(x,t+\Delta t|z,t)\rho(z,t|y,s)\mathrm{d}z\\ &= \int \mathrm{d}z\rho(z,t|y,s)\int \mathrm{d}x\varphi(x)\rho(x,t+\Delta t|z,t)\,.\end{aligned} \tag{8}$$

Expanding $\varphi(x)$ about $\varphi(z)$ for small Δt we obtain

$$\varphi(x) = \varphi(z) + (x-z)\varphi'(z) + \frac{(x-z)^2}{2}\varphi''(z) + \dots\,. \tag{9}$$

Hence, using the moments formulae (4) through (6), we have

$$\int \varphi(x)\rho(x,t+\Delta t|z,t)\mathrm{d}x = \varphi(z)\underbrace{\int \rho(x,t+\Delta t|z,t)\mathrm{d}x}_{=1} + \varphi'(z)\underbrace{\int (x-z)\rho(x,t+\Delta t|z,t)\mathrm{d}x}_{=f(z,t)\Delta t} \tag{10}$$

$$+\frac{1}{2}\varphi''(z)\underbrace{\int (x-z)^2\rho(x,t+\Delta t|z,t)\mathrm{d}x}_{=f(z,t)^2\Delta t^2+g(z,t)^2\Delta t} + \mathcal{O}(\Delta t)\,. \tag{11}$$

Thus

$$\int \varphi(z)\left(\frac{\rho(z,t+\Delta t|y,s)-\rho(z,t|y,s)}{\Delta t}\right)\mathrm{d}z = \tag{12}$$

$$\int \left(\varphi'(z)f(z,t) + \frac{1}{2}\varphi''(z)g(z,t)^2\right)\rho(z,t|y,s)\mathrm{d}z + \mathcal{O}(1)\,. \tag{13}$$

Since $\varphi(z)$ is arbitrary, in the continuous limit we obtain the following partial differential equation (PDE) for the Markovian transition density $\rho(x,t|y,s)$;

$$\frac{\partial}{\partial t}\rho(x,t|y,s) = \frac{\partial}{\partial x}\left[\left(-f(x,t) + \frac{1}{2}\frac{\partial}{\partial x}g(x,t)^2\right)\rho(x,t|y,s)\right]\,, \tag{14}$$

known as the *Fokker-Planck* (or forward Kolmogorov) equation. To solve (14) we need initial data and boundary conditions. Here we will only worry about the former, which is given by

$$\lim_{t\to s}\rho(x,t|y,s) = \delta(x-y)\,, \tag{15}$$

where the limit above is taken from the right (i.e., $t > s$). The derivation of the Fokker-Planck equation can be generalized to higher dimensions. Consider the n-dimensional stochastic process

$$\mathbf{X}(t) = (X_1(t), X_2(t), \dots) = X_i(t)\,. \tag{16}$$

The stochastic differential equation is

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = f_i(\mathbf{x},t) + g_{ij}(\mathbf{x},t)\xi_j(t)\,, \tag{17}$$

where the summation over repeated indices is implicit. The associated Fokker-Planck equation for the Markovian transition density is

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t|\mathbf{y},s) = \frac{\partial}{\partial x_i}\left[\left(-f_i + \frac{1}{2}\frac{\partial}{\partial x_j}g_{ik}g_{jk}\right)\rho(\mathbf{x},t|\mathbf{y},s)\right]\,, \tag{18}$$

where we can introduce the symmetric positive, semi-definite diffusion matrix $D_{ij} \stackrel{\text{def}}{=} g_{ik}g_{jk}$. At this point it is important to remark that different interpretations of the stochastic differential equation lead to different partial differential equations on the transition density. We will take up this important fact later.

Itô's Lemma

Let X_t be a stochastic variable whose evolution is governed by the following SDE, according to Itô's interpretation:

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}W_t\,, \tag{19}$$

and whose transition probability density satisfies the following Fokker-Planck (or Kolmogorov Forward) equation

$$\frac{\partial \rho_X}{\partial t} = \frac{\partial}{\partial X}\left[\left(-f + \frac{1}{2}\frac{\partial}{\partial X}g(X)^2\right)\rho_X\right]. \tag{20}$$

We emphasize that the subscripts above are labels – they do not represent partial differentiation.

Now consider a function of the random process $X(t)$ with a well defined inverse

$$Y = F(X) \Longleftrightarrow X = G(Y)\,. \tag{21}$$

We want to find a stochastic differential equation for Y and thus we write

$$\rho_Y \mathrm{d}Y = \rho_X \mathrm{d}X\,, \tag{22}$$

and

$$\frac{\partial}{\partial X} = F'\frac{\partial}{\partial Y} \qquad \text{or} \qquad G'\frac{\partial}{\partial X} = \frac{\partial}{\partial Y}\,, \tag{23}$$

from which we obtain

$$\frac{\partial \rho_Y}{\partial t} = \frac{\partial}{\partial Y}\left[\left(-f + \frac{1}{2}\frac{\partial}{\partial X}g^2\right)F'\rho_Y\right] = \frac{\partial}{\partial Y}\left[\left(-F'f + \frac{1}{2}\frac{\partial}{\partial X}\frac{1}{F'}(gF')^2\right)\rho_Y\right]. \tag{24}$$

Finally, the equation for the Markovian transition density is then

$$\frac{\partial \rho_Y}{\partial t} = \frac{\partial}{\partial Y}\left[\left(-F'f - \frac{1}{2}F''g^2 + \frac{1}{2}\frac{\partial}{\partial Y}(gF')^2\right)\rho_Y\right]. \tag{25}$$

Using our "recipe", we obtain the associated stochastic differential equation

$$\mathrm{d}Y = \left(F'f + \frac{1}{2}F''g^2\right)\mathrm{d}t + gF'\mathrm{d}W\,, \tag{26}$$

$$\mathrm{d}F(X) = F'(X)\mathrm{d}X + \frac{1}{2}F''(X)(\mathrm{d}X)^2\,, \tag{27}$$

plugging in the equation for $\mathrm{d}X$, using $(\mathrm{d}X)^2 = g^2\mathrm{d}t + \wr(\Delta t)$, we obtain

$$\mathrm{d}F(X) = \left(F'(X)f + \frac{1}{2}F''(X)g^2\right)\mathrm{d}t + gF'\mathrm{d}W\,. \tag{28}$$

Equation (28) is known as Itô's change of variables formula or Itô's lemma (note the extra-term $\frac{1}{2}F''(X)g^2$, due to the presence of the noise).

A linear example: The Ornstein–Uhlenbeck equation

Consider the following model

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\gamma U + \sigma\xi(t)\,, \tag{29}$$

where γ and σ are constants, which is Langevin equation with a linear damping term (first term on the right hand side), the second term being a fluctuation forcing. This could model for example the acceleration of a solid body immersed in a fluid, resulting on the combination of the Stokes drag (if the fluid happens to have a mean flow) and the constant battering of molecules around the body (white noise). Using Itô's interpretation, the associated Fokker-Planck equation is

$$\frac{\partial}{\partial t}\rho(u,t|v,s) = \frac{\partial}{\partial u}\left[\left(\gamma u + \frac{\sigma^2}{2}\frac{\partial}{\partial u}\rho(u,t|v,s)\right)\right]\,, \tag{30}$$

with the initial condition

$$\lim_{t\to s}\rho(u,t|v,s) = \delta(u-v)\,. \tag{31}$$

The solution is then

$$\rho(u,t|v,s) = \frac{1}{\sqrt{2\pi\Sigma(t-s)}}\exp\left[-\frac{1}{2}\frac{(u-v\mathrm{e}^{-\gamma(t-s)})^2}{\Sigma(t-s)}\right]\,, \tag{32}$$

with

$$\Sigma(t-s) \stackrel{\mathrm{def}}{=} \frac{\sigma^2}{2\gamma}\left(1-\mathrm{e}^{-2\gamma(t-s)}\right)\,. \tag{33}$$

To obtain an equation for the expectation of u, we just take the expectation of (29), to obtain

$$\frac{\mathrm{d}\mathbb{E}(U)}{\mathrm{d}t} = -\gamma\mathbb{E}(U)\,, \tag{34}$$

whose solution is

$$\mathbb{E}(U) = v\mathrm{e}^{-\gamma(t-s)}\,. \tag{35}$$

To derive an equation for the second moment, we first obtain an equation for U^2. Using Itô's change of variables (28) we have

$$F = u^2\,, \qquad F' = 2u\,, \qquad \text{and} \qquad F'' = 2\,, \tag{36}$$

we obtain

$$\mathrm{d}U^2 = -2\gamma U^2\mathrm{d}t + 2U\sigma\mathrm{d}W + \sigma^2\mathrm{d}t\,. \tag{37}$$

or

$$\mathrm{d}U^2 = 2U\mathrm{d}U + \sigma^2\mathrm{d}t\,. \tag{38}$$

The last term in (38) arises from the presence of the noise term. Now take the expectation of (38) to obtain

$$\mathrm{d}\mathbb{E}(U^2) = -2\gamma\mathbb{E}(U^2)\mathrm{d}t + \sigma^2\mathrm{d}t\,, \tag{39}$$

or

$$\frac{\mathrm{d}\mathbb{E}(U^2)}{\mathrm{d}t} = -2\gamma\mathbb{E}(U^2) + \sigma^2\,, \tag{40}$$

with the initial condition

$$\mathbb{E}(U^2) = v^2 \,, t \to s \,. \tag{41}$$

The Markovian density equilibrates to a stationary state

$$\lim_{t\to\infty} \rho(u, t|v, s) = \sqrt{\frac{\gamma}{\pi\sigma^2}} \mathrm{e}^{-\frac{\gamma}{\sigma^2}u^2} \,, \tag{42}$$

which is in the form

$$\rho^{stat}(u, t|v, s) = \rho(u, t|v, s)\rho^{stat}(v) \,. \tag{43}$$

For the stationary state, the covariance is

$$\mathbb{E}^{stat}(U(t)U(s)) = \int\int uv\rho(u, t; v, s)\mathrm{d}u\mathrm{d}v = \frac{\sigma^2}{2\gamma}\mathrm{e}^{-\gamma|t-s|} \,. \tag{44}$$

The power spectrum is simply the Fourier transform of the autocovariance $C(t)$

$$S(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} C(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \,. \tag{45}$$

Hence we obtain the Lorenzian spectrum

$$S(\omega) = \int_{-\infty}^{\infty} \frac{\sigma^2}{2\gamma}\mathrm{e}^{-\gamma|t|-\mathrm{i}\omega t}\mathrm{d}t = \frac{\sigma^2}{\gamma^2+\omega^2} \,. \tag{46}$$

Note that with (46) we can recover a flat spectrum in the white noise limit $\gamma \to \infty$ with appropriately rescaled noise amplitude.

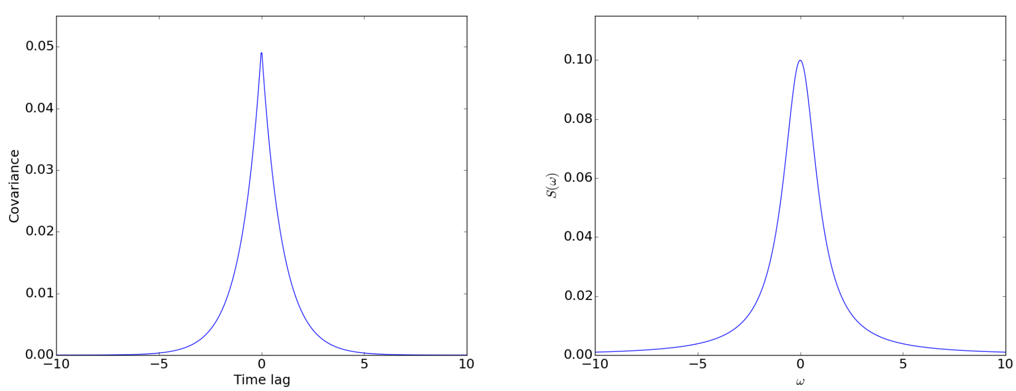

Figure 1: The stationary covariance (left) and spectrum (right) for the Ornstein–Uhlenbeck process with $\gamma = 1$ and $\sigma^2 = 0.1$.

A nonlinear example: The logistic equation

As an example of nonlinear equation, we consider the logistic equation

$$\mathrm{d}X = (\mu X - X^2)\mathrm{d}t\,, \tag{47}$$

where μ represents the ratio of birth rate to death rate, with initial condition $X(0) = X_0 > 0$. This is also known as the *Verhulst model* for population dynamics. The behavior of the deterministic equation (47) is well known; at large t, the solution approaches the fixed point μ (this is sometimes referred to as the carrying capacity of the system).

We now consider a stochastic logistic equation by adding a white noise term to the ratio of birth rate to death rate:

$$\mu(t) = \bar{\mu} + \sigma\xi(t)\,, \tag{48}$$

with $\bar{\mu}$ a constant. The stochastic differential equation is then

$$\mathrm{d}X = (\bar{\mu}X - X^2)\mathrm{d}t + \sigma X\xi(t)\mathrm{d}t\,, \tag{49}$$

with the associated Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial X}\left[\left(X^2 - \bar{\mu}X + \frac{\sigma^2}{2}\frac{\partial}{\partial X}X^2\right)\rho\right]\,. \tag{50}$$

Figure (2) shows numerical simulations for the stochastic logistic equation with two different initial conditions and various levels of noise. As discussed above, the deterministic behavior is well known; the solution initially grows ($X(0) < \mu$) or decays ($X(0) > \mu$) and asymptotically approaches the steady state, or the carrying capacity of the system, μ. For small noise ($\sigma < 0.1$), the solutions oscillate about the deterministic solution. For moderate noise levels ($\sigma^2 \sim \mathcal{O}(1)$), the solutions become very intermittent. For even larger levels of noise, the solutions show a tendency towards extinction.

If there is a stationary state then it must satisfy

$$0 = \frac{\partial}{\partial X}\left[\left(X^2 - \bar{\mu}X + \frac{\sigma^2}{2}\frac{\partial X^2}{\partial X}\right)\rho^{stat}\right]\,. \tag{51}$$

In general we can write

$$0 = \frac{\partial}{\partial X}\left[\left(-f + \frac{1}{2}\frac{\partial}{\partial X}g(X)^2\right)\rho^{stat}\right]\,, \tag{52}$$

from which we find

$$\rho^{stat}(X) = \frac{\mathcal{N}}{g(X)^2}\exp\left[2\int^X \frac{f(\xi)}{g(\xi)^2}\mathrm{d}\xi\right]\,. \tag{53}$$

With the explicit f and g from the stochastic version of the Verhulst model, we obtain

$$\rho^{stat}(X) = \mathcal{N}X^{2(\bar{\mu}/\sigma^2-1)}\mathrm{e}^{-\frac{2}{\sigma^2}X}\,. \tag{54}$$

The normalization constant is defined as

$$\int \rho^{stat}(X)\mathrm{d}X = 1 \Rightarrow \mathcal{N}\left(\frac{\sigma^2}{2}\right)^{(2\bar{\mu}/\sigma^2-1)}\Gamma\left(\frac{2\bar{\mu}}{\sigma^2} - 1\right) = 1\,, \tag{55}$$

where $\Gamma(x)$ is the gamma function. Note that for $\bar{\mu}/\sigma^2 \leq 1/2$ the normalization constant defined above is not bounded. Thus there is no stationary state with support on $x > 0$. For every small $\epsilon > 0$, the probability for X_t to fall above $\epsilon > 0$ as time goes to ∞ is 1 (convergence in probability).

Figure 3 shows a comparison of the stationary probability density (53) with empirical probability densities at $t = 50$ based on 10000 numerical simulations, similar to those presented in figure 2. With small noise, the probability density function almost Gaussian. As the level of noise increases, the probability density becomes highly skewed (the solution becomes highly intermittent). Moreover, there is a qualitative change in the behavior of the stationary density functions for $\sigma^2 > 1.0$ in (53). With a large level of noise, the probability density function is compressed near $x = 0$ (the probability of extinction is very high).

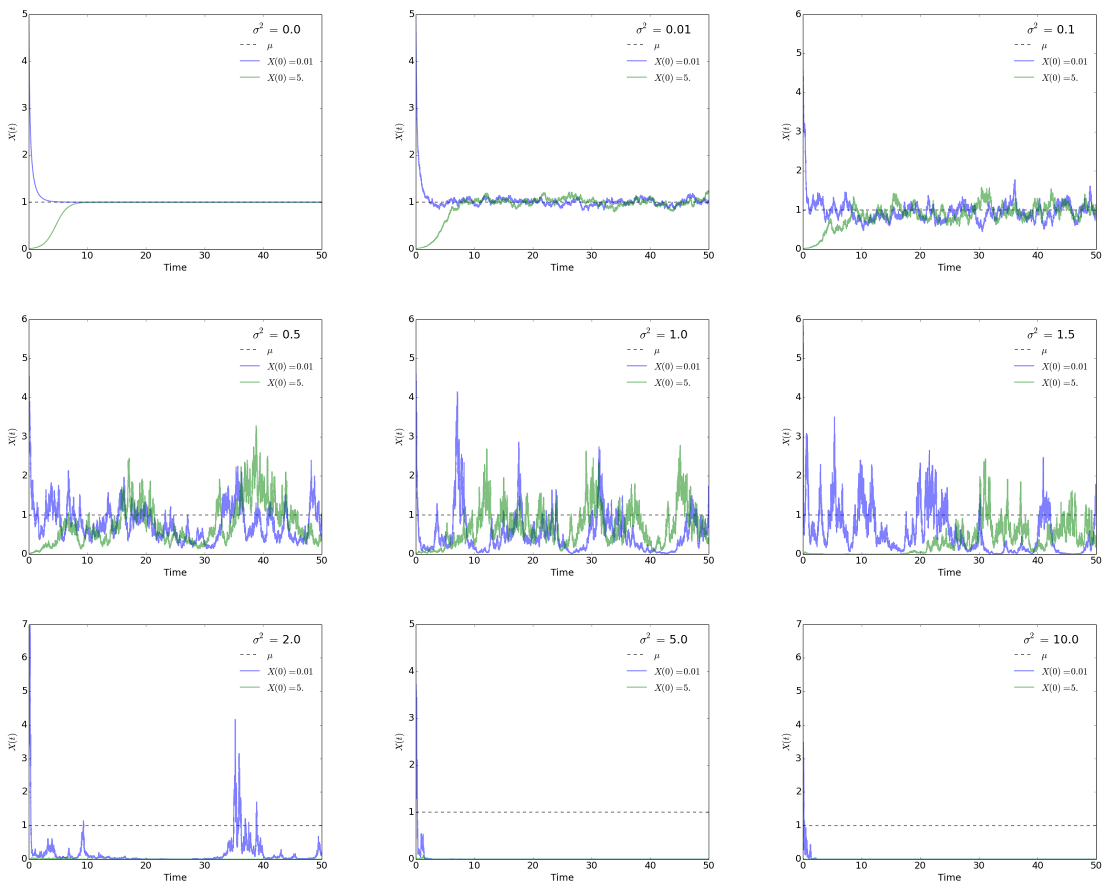

Figure 2: Numerical solutions to the stochastic logistic equation with $\bar{\mu} = 1$ for various levels of noise. Note that $\sigma^2 = 0$ is simply the deterministic solution. For high levels of noise ($\sigma^2 > 1$), one observes a tendency towards extinction.

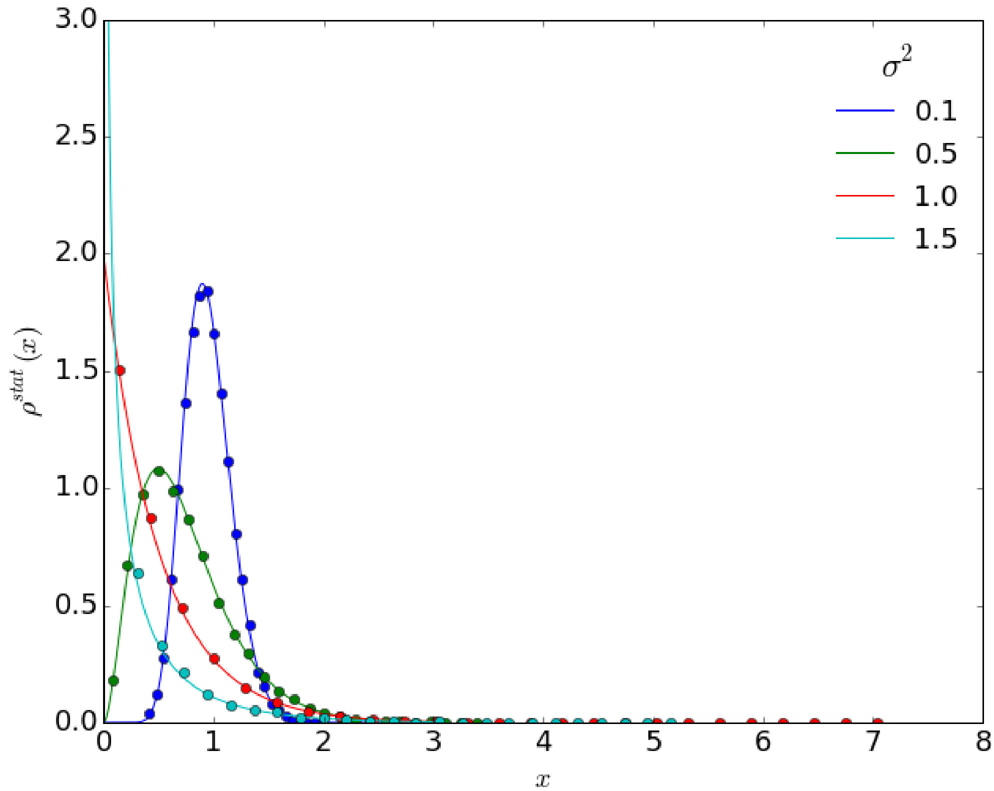

Figure 3: A comparison between the analytical stationary probability density function Eq. (53) (solid lines) against the empirical probability function (dots) at $t = 50$ based on 10000 simulations of the stochastic logistic equation with $\bar{\mu} = 1$. Note the qualitative change at $\sigma^2 \geq 1$. (In these figure labels X from Eq. (53) is written as x.)