Stationary vortices in a Keplerian shear flow

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1 Introduction

1.1 Planetary formation

Science in it's most general definition began as a quest to answer the fundamental questions on the origin of humanity and it's relation to the surrounding Universe. One of the keys to understanding the origin of Life is the mechanism of formation of our own Solar system, and especially the formation of planets. This subject has gained a new interest in the past few years with the discovery of giant planets orbiting some of the nearest neighbouring stars. The generally accepted theory of planet formation consists in the following steps:

- Due to the onset of a large scale gravitational instability, the core of a dense molecular cloud collapses into a protostar; the conservation of its initial angular momentum results in the gradual flattening of the collapsing gas into an accretion disk around the protostar.
- The gas in the accretion disk has two components: a molecular gas, composed mainly of H_2 and other small molecules, and a dust gas, composed of particles of sizes ranging from a few microns to a few centimeters. The interaction between these components takes place mainly via Stokes drag. The vertical stratification in the accretion disk relies on the balance between pressure and the vertical component of the gravitational force. As a result, since the thermal pressure of the dust gas is much smaller than that of the molecular gas, the dust settles into a very thin disk within the accretion disk.
- The dust particles then coalesce into larger and larger grains, up to sizes of a few kilometers; as they grow in mass, the dynamics of these "planetesimals" gradually decouple from that of the molecular gas.
- The planetesimals continue aggregating into planets. Giant planets may accrete some of the molecular gas still left in the accretion disk.

However, although the dust aggregation into larger grains is known to take place, the exact mechanism is poorly understood. The time-scale for this aggregation process has an upper limit of a few Myr (10^6 yr) set by the evolution of the protostar into a T-Tauri star. Indeed, T-Tauri stars are observed to have intense magnetic activity and strong stellar winds which scatter all non-gravitationally bound dust and gas into the interstellar space. Random encounters of the dust particles due to thermal agitation is not sufficient to account for the growth of the dust grains into planetesimals within the T-Tauri evolution time-scale.

In an attempt to remedy this problem, it has been shown in the case of two-dimensional barotropic turbulence in a rotating fluid that dust particles may migrate to the center of anticyclonic vortices [1]. They concentrate there for the lifetime of the vortex. As a result, provided the vortices are long lived, it is possible to greatly increase the aggregation rate, and reach the required sizes of dust grains before the T-Tauri phase. However, it is not yet clear whether a Keplerian flow can undergo self-sustained turbulence. Indeed, from the Rayleigh inflexion theorem, we see that the accretion flow is stable to linear shear instability, and the latest numerical simulations seem to indicate that the primordial solar nebula may be stable to nonlinear hydrodynamic instabilities too [2]. It has been shown that even a very small magnetic field may trigger some linear instability [3], but in this case it is not clear how the magnetic forces would influence the existence or stability of the vortices.

Although the problem of hydrodynamical stability of the accretion flows is not yet fully understood, there has been evidence in two-dimensional decaying turbulence for the spontaneous apparition and the persistence of vortices on time-scales much larger than the turnover time-scale. There is therefore hope in the Keplerian case that even if the turbulence is not self-sustained, the initial anisotropies in the flow are large enough to create these long-lived vortex structures. The work presented in this report is an attempt at finding steady state solutions for vortices in Keplerian accretion flows. If these solutions exist and are found to be stable, they would explain the persistence of the vortices, and therefore solve the remaining dust aggregation problem.

1.2 Mathematical setup

We will always take $\mathbf{u}$ to be the velocity field, ψ the corresponding stream function and ω the potential vorticity. In the work presented here, we have chosen to simplify the problem greatly by considering only 2-dimensional, incompressible fluid motion. As a result of this approximation, we can now write

$$\mathbf{u} = -\nabla\times(\psi\hat{\mathbf{e}}_\mathbf{z}) = \hat{\mathbf{e}}_\mathbf{z} \times \nabla\psi \text{ and } \boldsymbol{\omega} = \omega\hat{\mathbf{e}}_\mathbf{z} = \nabla^2\psi\hat{\mathbf{e}}_\mathbf{z} \ . \tag{1}$$

We will consider the vortex to be a perturbation on the main Keplerian accretion flow. The unperturbed shear flow $\mathbf{u}_\mathrm{K}$ is given by the Keplerian rotation law, which describes the equilibrium between the centrifugal and gravitational forces:

$$\mathbf{u}_\mathrm{K} = v_\mathrm{K}\hat{\mathbf{e}}_\theta = \sqrt{\frac{GM}{R}}\hat{\mathbf{e}}_\theta \tag{2}$$

where R is the distance from the central accreting object of mass M. The corresponding vorticity is $\omega_\mathrm{K} = \frac{1}{2}\sqrt{\frac{GM}{R^3}}$. The vortex studied will be placed at a distance R_0 from the center, at $\theta = 0$. In the following work, we will often have to change from the polar coordinate system around the central mass, (R, θ) to that around the vortex, (r, φ). We chose to take $\varphi = 0$ where $\theta = 0$; this change of coordinate is represented in Fig.1.

The perturbed vorticity and flow are represented by dashed quantities. The equation for the evolution of the vorticity perturbation is

$$\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = \frac{\partial\omega'}{\partial t} + \mathbf{u}_\mathrm{K}\cdot\nabla\omega' + \mathbf{u}'\cdot\nabla\omega_\mathrm{K} + \mathbf{u}'\cdot\nabla\omega' = 0 \ . \tag{3}$$

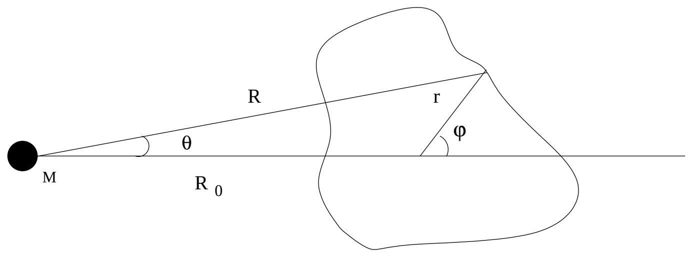

Figure 1: Change of coordinate

This can be rewritten in cylindrical coordinates around the vortex patch as

$$\frac{\partial \omega'}{\partial t} + v_{\rm K}\frac{1}{r}\frac{\partial \omega'}{\partial \varphi} - \frac{1}{r}\frac{\partial \omega_{\rm K}}{\partial r}\frac{\partial \psi'}{\partial \varphi} + \frac{1}{r}\frac{\partial \psi'}{\partial r}\frac{\partial \omega'}{\partial \varphi} - \frac{1}{r}\frac{\partial \omega'}{\partial r}\frac{\partial \psi'}{\partial \varphi} = 0 \ . \tag{4}$$

The vorticity perturbation and the perturbed stream function are related by

$$\omega' = \nabla^2 \psi' \ . \tag{5}$$

1.3 Dimensionless quantities

In order to simplify the expressions, we will now introduce the following new units system:

$$M = 1, \ R_0 = 1, \ \text{and} \ T_0 = 1 \tag{6}$$

where T_0 is the revolution time around the central object at radius R_0, namely $T_0 = 2\pi\sqrt{\frac{R_0^3}{GM}}$. As a result, the Keplerian velocity becomes $v_{\rm K} = 2\pi R^{-1/2}$.

1.4 Change of reference frame

We will be looking for vortex solutions where the vortex is rotating around the central star with a Keplerian velocity. Steady state solutions then only have a meaning when taken in the rotating coordinate frame. We use a frame of reference rotating with velocity which is that of the center of the vortex patch. The relative shear around this point is given by

$$v_{\rm K}(R) = 2\pi(R^{-1/2} - R) \tag{7}$$

The corresponding stream function is

$$\psi_{\rm K}(R) = 2\pi\left(2(R^{1/2} - 1) - \frac{1}{2}(R^2 - 1)\right) \tag{8}$$

Without loss of generality, we have chosen $\psi_{\rm K}(1) = 0$. The Keplerian vorticity and its gradient are given by

$$\omega_{\rm K} = \pi R^{-3/2} - 4\pi, \ \text{and} \ \frac{\partial \omega_{\rm K}}{\partial R} = -\frac{3}{2}\pi R^{-5/2} \tag{9}$$

2 Top hat vortices in a Keplerian shear

In this section we will use an approximation which consists in neglecting the background Keplerian vorticity gradient in the vorticity equation. This approximation is valid primarily for small vortices. In this case, there exists a solution of the steady state problem with ω' piecewise constant. We will therefore try to find solutions of the type

$$\begin{aligned} \omega' = \nabla^2\psi' &= q \text{ inside the vortex patch} \\ &= 0 \text{ outside the vortex patch} \end{aligned} \tag{10}$$

Equation (10) can now be rewritten as

$$\nabla^2\psi' = q\mathcal{H}\left(a + \eta(\varphi) - r\right) \tag{11}$$

where a is the average radius of the patch, and η is the departure from that average. We will linearize this equation by considering $\eta \ll a$, so that

$$\nabla^2\psi' = q\mathcal{H}(a - r) + q\eta(\varphi)\delta(a - r) + O(\eta^2) \tag{12}$$

Replacing ω' in equation (4) by this ansatz, we get the contour dynamics equation (provided we neglect the term involving the vorticity gradient)

$$\frac{\partial\eta}{\partial t}\delta + \frac{v_{\mathrm{K}}(r)}{r}\frac{\partial\eta}{\partial\varphi}\delta + \frac{1}{r}\frac{\partial\psi'}{\partial r}\frac{\partial\eta}{\partial\varphi}\delta + \frac{1}{r}\frac{\partial\psi'}{\partial\varphi}\left(\delta + \eta\delta'\right) + O(\eta^2) = 0 \tag{13}$$

where $\delta \equiv \delta(a - r)$. Taking the steady state part of this equation, we integrate it once across the boundary $r = a$ to get

$$\left(\frac{\partial\psi_{\mathrm{K}}}{\partial r} + \frac{\partial\psi'}{\partial r}\right)_{r=a}\frac{\partial\eta}{\partial\varphi} + \frac{\partial\psi'}{\partial\varphi} + O(\eta^2) = 0 \tag{14}$$

The condition for no fluid to enter or leave the vortex (which defines it as a localized vortex patch) is that the total stream function should be constant along the boundary

$$\left.\frac{\partial}{\partial\varphi}(\psi_{\mathrm{K}} + \psi')\right|_{r=a} + O(\eta^2) = 0 \tag{15}$$

so that if we integrate equation (14) along the boundary, we get

$$\psi_{\mathrm{K}}(a) + \psi'(a) + \left(\frac{\partial\psi_{\mathrm{K}}}{\partial r} + \frac{\partial\psi'}{\partial r}\right)_{r=a}\eta = \psi_\eta \tag{16}$$

where ψ_η is a constant. Since this implies that the velocity field is everywhere parallel to the boundary, there is no net force exerted on the vortex..

2.1 Solution to zeroth order

Let ψ_c be a solution of the zeroth order in η of equation (12):

$$\nabla^2\psi_c = q\mathcal{H}(a-r)\ . \tag{17}$$

We can integrate this on either sides of $r=a$, which yields the solution

$$\begin{aligned}\psi_c(r) &= \frac{qr^2}{4} + c_1\ln r + c_2 \text{ for } r<a\\ &= c_3\ln r + c_4 \text{ for } r>a\end{aligned} \tag{18}$$

Regularity at the origin requires that $c_1 = 0$, and we can choose $c_4 = 0$. Note that the stream function diverges at infinity, but the velocity field is well behaved. Matching the function and its derivative at $r=a$ yields

$$\begin{aligned}\frac{qa^2}{4} + c_2 &= c_3\ln a\\ \frac{qa}{2} &= \frac{c_3}{a}\end{aligned} \tag{19}$$

so that finally, we have

$$\psi_c(r) = \left[\frac{qr^2}{4} + \frac{qa^2}{2}\left(\ln a - \frac{1}{2}\right)\right]\mathcal{H}(a-r) + \frac{qa^2}{2}\mathcal{H}(r-a)\ln r\ . \tag{20}$$

2.2 Solution to first order

If we define $\psi' = \psi_c + \tilde\psi$, subtracting equation (17) from equation (12) yields

$$\nabla^2\tilde\psi = q\eta(\varphi)\delta(a-r)\ . \tag{21}$$

Write that $\tilde\psi = \sum_n \psi_n e^{in\varphi}$, and $\eta = \sum_n \eta_n e^{in\varphi}$, then equation (21) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_n}{\partial r}\right) - \frac{n^2}{r^2}\psi_n = q\eta_n\delta(a-r) \tag{22}$$

so the solution will be of the kind

$$\begin{aligned}\psi_n(r<a) &= a_1(r/a)^{|n|}\\ \psi_n(r>a) &= a_2(r/a)^{-|n|}\end{aligned} \tag{23}$$

provided $n\neq 0$; we have implicitly imposed regularity of the solutions at the origin and at infinity. Matching the solution at $r=a$ requires that $a_1 = a_2$. Finally, integrating equation (21) across the discontinuity, we get

$$\left.\frac{\partial\psi_n}{\partial r}\right|_{a-}^{a+} = q\eta_n \tag{24}$$

For $n\neq 0$, this condition yields $a_1 = a_2 = -\frac{q\eta_n a}{2|n|}$. We therefore get

$$\tilde\psi = -\sum_{n\neq 0}\frac{q\eta_n a}{2|n|}\left((r/a)^{|n|}\mathcal{H}(a-r) + (r/a)^{-|n|}\mathcal{H}(r-a)\right)e^{in\varphi} \tag{25}$$

The case of $n=0$ is discussed in the next session.

2.3 Matching the vortex patch to the Keplerian flow.

The function η is given by equation (16), which corresponds to the requirement that the shape of the vortex patch remains steady. We rewrite it here

$$\psi_{\mathrm{K}}(a) + \eta \frac{\partial \psi_{\mathrm{K}}}{\partial r}(a) + \psi_{\mathrm{c}}(a) + \eta \frac{\partial \psi_{\mathrm{c}}}{\partial r}(a) + \tilde{\psi}(a) = \psi_\eta + O(\eta^2) \tag{26}$$

The $\tilde{\psi}$ term, as we saw, is of first order in η. This equation provides all the η_n's but η_0. A last condition arises from the normalization of the total vorticity of the patch, which is equivalent to fixing the area of the patch. If we require that the area be $A = \pi a^2$ (i.e. the area of a corresponding circular patch), then we have

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^{a+\eta} r \mathrm{d}r \mathrm{d}\varphi = \int_0^{2\pi} \frac{(a+\eta)^2}{2} \mathrm{d}\varphi = \pi (a+\eta_0)^2 + \pi \sum_{n>0} \eta_n^2 \\ &= \pi (a+\eta_0)^2 + O(\eta^2) \end{aligned} \tag{27}$$

The normalization condition on A is therefore $\eta_0 = 0$, and is valid to first order in η. The only terms left to evaluate are $\psi_{\mathrm{K}}(a)$ and $\eta \frac{\partial \psi_{\mathrm{K}}}{\partial r}(a)$.

2.3.1 The linear shear case

Before starting on the Keplerian shear flow case, let's treat the simple linear shear case; in any case, one would expect that the results of the linear shear case are recovered in the limit where the size of the vortex patch a is much smaller than the distance of the patch to the center of the Keplerian shear flow R.

A linear shear is given by $u_{\mathrm{L}}(R) = s(R-1)$, (taking the velocity to be 0 at the position of the center of the vortex patch) so that the corresponding stream function is $\psi_{\mathrm{L}}(R) = \frac{s}{2}(R-1)^2$. Since, to a first approximation $R = 1 + r\cos\varphi + O(r^2)$, we have $\psi_{\mathrm{L}}(r) = \frac{s}{2} r^2 \cos^2\varphi = \frac{sr^2}{8}(e^{2i\varphi} - 2 + e^{-2i\varphi})$. The matching condition then yields

$$\begin{aligned} \frac{sa^2}{8}(e^{2i\varphi} - 2 + e^{-2i\varphi}) \; &+ \; \left(\sum_{n \neq 0} \eta_n e^{in\varphi} \right) \frac{sa}{4}(e^{2i\varphi} - 2 + e^{-2i\varphi}) \\ &+ \; \frac{qa^2}{2} \ln a + \left(\sum_{n \neq 0} \eta_n e^{in\varphi} \right) \frac{qa}{2} - \sum_{n \neq 0} \frac{q \eta_n a}{2|n|} e^{in\varphi} = \psi_\eta \end{aligned} \tag{28}$$

An important point is that the first term in that expression is potentially much larger than the other ones. In order for this term to be balanced, one requires that $\eta q \approx sa$. Since $\eta \ll a$, this condition is equivalent to $q \gg s$. Hence this work is only valid for vortex patches with vorticity much larger than the local Keplerian vorticity. We also see that the second term in that expression is of order of η/a compared to the other ones, and will be neglected in the coming analysis. As a result, if we take $\int_0^{2\pi}$ (28) $e^{-im\varphi}\mathrm{d}\varphi/2\pi$, we get

$$\psi_\eta = \frac{qa^2}{2} \ln a - \frac{sa^2}{4} \text{ and } \eta_{\pm 2} = -\frac{sa}{2q} \tag{29}$$

and all other η_n are zero. We see again that the condition $\eta \ll a$ is equivalent to the condition $s \ll q$. We therefore have

$$\eta(\varphi) = -\frac{sa}{q} \cos(2\varphi) \tag{30}$$

To first order in η, this corresponds to an elliptical shape[1].

2.3.2 The Keplerian shear case

As a first approximation, let's take $R = 1 + r \cos\varphi$. This approximation will be discussed later. In this case, if we define

$$\psi_{\mathrm{K}}(a) \equiv \psi_{\mathrm{K}}(1 + a \cos\varphi) \equiv \sum_n I_n^{\mathrm{K}} e^{in\varphi} \tag{31}$$

we have, from the matching condition given by equation (26)

$$\psi_\eta = I_0^{\mathrm{K}} + \frac{qa^2}{2} \ln a \tag{32}$$

$$\eta_n = \frac{2I_n^{\mathrm{K}}|n|}{qa(1-|n|)} \text{ for } |n| > 1 \tag{33}$$

Because of the symmetry in $\varphi \to -\varphi$ of the Keplerian shear flow, we know that $I_n^{\mathrm{K}} = I_{-n}^{\mathrm{K}}$, which is confirmed by expression (33). The case $n = 1$ corresponds to a translation of the vortex along the $\theta-$direction (azimuthally around the central mass), so that η_1 can always be taken to be 0 by an appropriate change of referential[2]. The I_n^{K} are given by

$$\begin{aligned} I_n^{\mathrm{K}} &= \int_0^{2\pi} \psi_K(\sqrt{1 + a \cos\varphi}) e^{-in\varphi} \frac{\mathrm{d}\varphi}{2\pi} \\ &= 2\pi \int_0^{2\pi} \left[2(\sqrt{1 + a \cos\varphi} - 1) - \frac{1}{2}(a^2 \cos^2\varphi + 2a \cos\varphi) \right] e^{-in\varphi} \frac{\mathrm{d}\varphi}{2\pi} \end{aligned} \tag{34}$$

In order to solve this integral, we need to expand it as a Taylor series (which is necessarily convergent since we have $a < 1$). So

$$I_n^{\mathrm{K}} = -\pi a^2 \int_0^{2\pi} \cos^2\theta e^{-in\varphi} \frac{\mathrm{d}\varphi}{2\pi} + 4\pi \int_0^{2\pi} \sum_{k>1} \frac{(1/2)_k}{k!} a^k \cos^k\varphi e^{-in\varphi} \frac{\mathrm{d}\varphi}{2\pi} \tag{35}$$

where we define $(\nu)_k = \nu(\nu-1)(...)(\nu-k+1)$, and $(\nu)_0 = 1$. Then

$$I_n^{\mathrm{K}} = -\pi a^2 J_{2,n} + 4\pi \sum_{k=|n|}^{\infty} \frac{(1/2)_k}{k!} a^k \frac{1}{2^k} C^k_{\frac{k-n}{2}} \tag{36}$$

[1]Indeed, the equation for an ellipse being $r = b\sqrt{1 - a \cos^2\varphi/e}^{-1}$, if the eccentricity e is very small, then $r \approx b(1 + e \cos^2\theta/2a) = r_0 + \frac{be}{2a} \cos(2\varphi)$.

[2]Indeed, let's take the example of the displacement of a circular patch: the equation for a circle centered on $x = \eta$ (instead of $x = 0$) is $(x-\eta)^2 + y^2 = a^2$. Expanding this to first order in η and changing coordinates from (x, y) to (r, φ), we get $r = a + \eta \cos\varphi$, which corresponds to an $n = 1$ deformation mode.

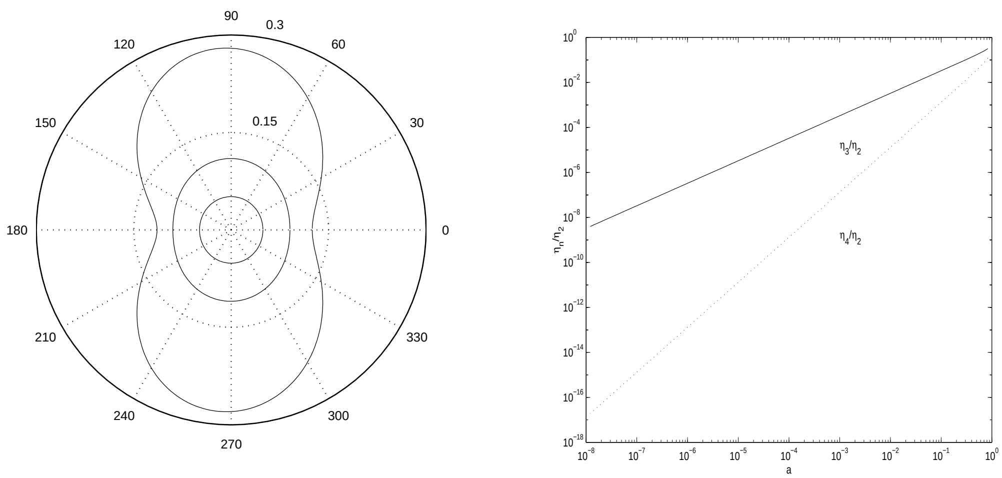

Figure 2: On the left: steady-state shape of the vortex patch, for $a = 0.05, 0.1$ and 0.2 in the Keplerian shear case, with the approximation $R = 1 + r\,\cos\varphi$. In all 3 plots $q = 1$. On the right: Ratio of the 3rd and 4th order of deformation to the 2nd as a function of a.

where the C_n^k are the binomial coefficients. Finally, we compute the deformation by adding the Fourier coefficients as

$$\eta(\varphi) = \sum_{n>0} 2\eta_n \cos(n\varphi) \tag{37}$$

Fig. 2 present the results for some values of a and q. One can however guess (and check) that:

- The larger q, the smaller the deformation from a circular patch. Since q only appears in the "normalization" of η rather than in the relative amplitude of the modes η_n of deformation, changing q only amounts to changing the total amplitude of the deformation. In the following plots, a small value of q was chosen on purpose to let the vortex deformation be more easily identifiable. In reality, we should take $q \gg 1$ to have the required $\eta \ll a$.

- On the other hand, the value of a will influence the relative importance of the η_n, and will dictate the shape of the vortex. The larger a, the larger the higher order modes of deformation, and the more difficult the convergence.

We can see in Fig. 2 that for a (a/R_0 in real units) small, the dominant term in the deformation η is η_2, the ellipsoidal term. The ratio of the 3rd and 4rd order deformation modes to the second is also shown. As we can see from this plot, for $a/R_0 < 0.1$, the shape of the vortex patch is very well approximated by an ellipse.

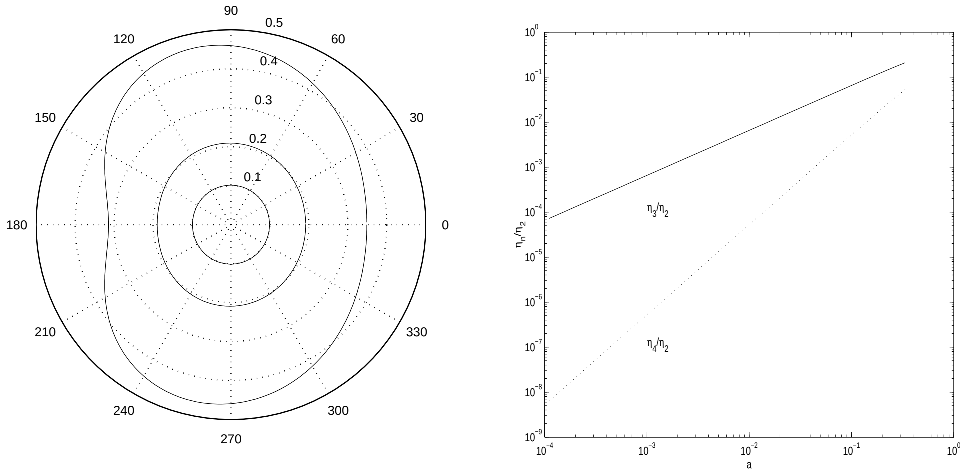

Figure 3: On the left: steady-state shape of the vortex patch, for $a = 0.1, 0.2$ and 0.4 in the linear shear case. In all 3 plots $q = 1$. On the right: relative amplitude of the 3rd and 4th moment of deformation compared to the second, for varying values of a.

2.3.3 Validity of the approximation $R = 1 + r\ \cos\varphi$.

The full expression for R is

$$R = \sqrt{1 + r^2 + 2r\ \cos\varphi} \approx 1 + r\ \cos\varphi + O(r^2) \tag{38}$$

therefore the approximation is only valid for $r \ll 1$. Let's study again the example of the linear shear case: this time, we have

$$\psi_{\rm L}(R) = \frac{s}{2}(R-1)^2 = \frac{s}{2}\left(2 + r^2 + 2r\ \cos\varphi - 2\sqrt{1 + r^2 + 2r\ \cos\varphi}\right) \tag{39}$$

Following the same procedure as before (Taylor expansion + Fourier decomposition) we can obtain the Fourier coefficients of $\psi_{\rm L}(a)$: successively

$$\begin{aligned}\psi_{\rm L}(a) &= -s\sum_{k>1}(a^2 + 2a\ \cos\varphi)^k\frac{(1/2)_k}{k!} \\ &= -s\sum_{k>1}\frac{(1/2)_k}{k!}\sum_{p=0}^{k} C_p^k a^{k+p} 2^{k-p}\ \cos^{k-p}\varphi \end{aligned} \tag{40}$$

so that

$$I_n^{\rm L} = -s\sum_{k>1}\frac{(1/2)_k}{k!}\sum_{p=0}^{k} 2^{k-p} C_p^k a^{k+p}\frac{1}{2^{k-p}} C^{k-p}_{\frac{k-p-n}{2}} \tag{41}$$

The η_n are given by equation (33). The resulting steady-state shape of the vortex patch is shown in Fig.3 This deformation is due to the fact that the flow is not a plane parallel flow,

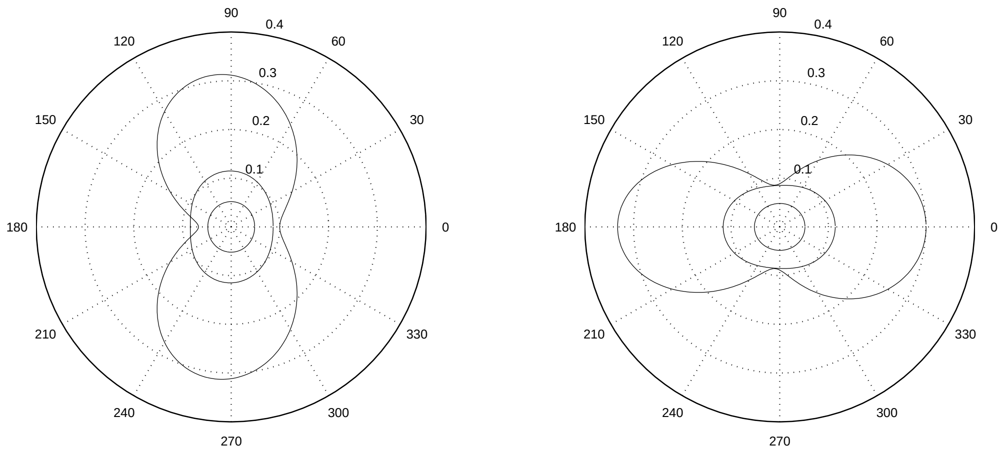

Figure 4: Deformation of a vortex patch in a Keplerian shear for $a = 0.05, 0.1, 0.2$, and $q = 1$.

but rather curves around the central accreting object. When the size of the patch is large, this curvature acts to deform it.

To conclude, if we interpolate the linear shear results to the Keplerian case, it is likely that the approximation will fail for $a > 0.1$. The full expression for R should therefore be kept. The results for the Keplerian shear, using equation (38) in expression (35), are the following:

$$I_n^{\mathrm{K}} = -4\pi \sum_{k>1} \frac{(1/4)_k}{k!} \sum_{p=0}^{k} a^{k+p} 2^{k-p} C_p^k C_{\frac{k-p-n}{2}}^{k-p} \tag{42}$$

The corresponding vortex patches are presented in Fig. 4, for both a cyclonic and an anticyclonic vortex.

2.4 Discussion.

Assuming that the background vorticity is constant, it has been possible to calculate the steady state shape of top-hat (i.e. constant piece-wise) vortices. In a linear shear, it is a well known result that the shape of the vortices should be elliptical [4]. For very small vortices, for which the variation of the background vorticity is negligible, we could expect, and saw that the vortices were mainly elliptical in shape. However, for larger vortices[3], there is a systematic variation from elliptical, and this has two main causes: firstly, the curvature of the Keplerian flow around the central star, and secondly (and this is the dominant effect), the background variation in the velocity field. The next step in this analysis would be to consider the stability of these vortices, in a similar way as has been done by Meacham *et al.*[5]. This is not in the scope of this project.

[3]In fact the approximation of constant vorticity around the vortex is no more valid for the larger vortices anyway.

3 Including the main flow vorticity gradient

In the previous section, the vorticity gradient term in the vorticity equation (4) has been neglected in order to allow solutions with piece-wise constant vorticity. However, including the vorticity gradient term forbids this solution. In particular, when the nonlinear terms can be neglected, we will see that there exist stationary wave-like solutions: the lee waves. These have to be taken into account in their interaction with the vortex patch. In the following work, we will therefore study two main regimes:

- Far from the vortex, the perturbation induced by the vortex is small; the vorticity equation becomes a linear equation for stationary lee waves. The far field of the vortex can then be obtained by studying the lee waves which are created around a point vortex.
- Near the vortex, the perturbation is much stronger than the Keplerian flow. By rescaling the coordinate system to emphasize the region near the vortex, we will see that to a first approximation, the gradient of the vorticity can be neglected. This zeroth order solution resembles closely that presented in Section 1 for the linear shear. The deformation of the vortex patch is then given by the next order in the approximation.

In the steady state, equation (4) becomes

$$J(\psi_{\mathrm{K}} + \psi', \omega_{\mathrm{K}} + \omega') = 0 \tag{43}$$

where the Jacobian J is, in the cylindrical coordinate system

$$J(A, B) = \frac{1}{R}\frac{\partial A}{\partial R}\frac{\partial B}{\partial \theta} - \frac{1}{R}\frac{\partial A}{\partial \theta}\frac{\partial B}{\partial R} \tag{44}$$

Let's now use the new coordinate system

$$\begin{aligned} \theta &= -X \\ R &= \exp(Y) \end{aligned} \tag{45}$$

The Keplerian velocity and stream function in this new ordinate system is then (using equations (7), (8), (9) and $\Gamma = 3\pi$)

$$v_{\mathrm{K}} = \frac{2\Gamma}{3}\left(e^{-\frac{1}{2}Y} - e^{Y}\right) \tag{46}$$

$$\psi_{\mathrm{K}} = \frac{2\Gamma}{3}\left(2e^{\frac{1}{2}Y} - \frac{1}{2}e^{2Y} - \frac{3}{2}\right) \tag{47}$$

$$\omega_{\mathrm{K}} = \frac{\Gamma}{3}e^{-\frac{3}{2}Y} - \frac{4\Gamma}{3} \tag{48}$$

The Jacobian equation then becomes (dropping the primes on the perturbed quantities)

$$J\left(\frac{2\Gamma}{3}\left(2e^{\frac{1}{2}Y} - \frac{1}{2}e^{2Y} - \frac{3}{2}\right) + \psi, \frac{\Gamma}{3}e^{-\frac{3}{2}Y} - \frac{4\Gamma}{3} + e^{-2Y}\nabla^2\psi\right) = 0 \tag{49}$$

where we now have $J(A, B) = \partial_X A \partial_Y B - \partial_Y A \partial_X B$, and the Laplacian operator is $\nabla^2 = \partial_{XX} + \partial_{YY}$.

3.1 The far field: a linear approach

3.1.1 Assumptions and equations

We will assume that the perturbation is much smaller than the Keplerian stream function. This should be valid everywhere but around $Y = 0$, where $\psi_{\rm K}$ vanishes. Equation (49) implies that

$$\frac{\Gamma}{3}e^{-\frac{3}{2}Y} - \frac{4\Gamma}{3} + e^{-2Y}\nabla^2\psi = F\left[\frac{2\Gamma}{3}\left(2e^{\frac{1}{2}Y} - \frac{1}{2}e^{2Y} - \frac{3}{2}\right) + \psi\right] \tag{50}$$

The function F is not unique. This means that there are many steady state solutions to the problem we are considering, each of them depending on the type of forcing, the symmetries required, the behaviour far from the vortex. We must choose the function F carefully to represent the physics of the system considered. We want to represent the presence of a small vortex patch, and its influence on the Keplerian accretion flow. Far from the vortex (the region considered here), we hope that there exist solutions in which the disturbance caused by the vortex is very small, so that the Keplerian stream lines are merely displaced by a small amount. Taking these two ideas in consideration, we see that a possible prescription for F is

$$F(\psi_{\rm K} + \psi) = F_{\rm K}(\psi_{\rm K} + \psi) + Q\delta(X)\delta(Y) \tag{51}$$

where Q is the total vorticity of the patch, and the function $F_{\rm K}$ is defined as

$$\omega_{\rm K} = F_{\rm K}(\psi_{\rm K}), \;\Leftrightarrow\; \frac{\Gamma}{3}e^{-\frac{3}{2}Y} - \frac{4\Gamma}{3} = F_{\rm K}\left[\frac{2\Gamma}{3}\left(2e^{\frac{1}{2}Y} - \frac{1}{2}e^{2Y} - \frac{3}{2}\right)\right] \tag{52}$$

Putting this ansatz back into equation (50) we get

$$\begin{aligned} e^{-2Y}\nabla^2\psi &= F_{\rm K}(\psi_{\rm K} + \psi) - F_{\rm K}(\psi_{\rm K}) + Q\delta(X)\delta(Y) \\ &\approx \psi F'_{\rm K}(\psi_{\rm K}) + Q\delta(X)\delta(Y) \end{aligned} \tag{53}$$

since we assumed that $\psi \ll \psi_{\rm K}$. The function $F'_{\rm K}$ can easily be obtained by taking the Y-derivative of equation (52), and is

$$F'_{\rm K} = \frac{\frac{\partial\omega_{\rm K}}{\partial Y}}{\frac{\partial\psi_{\rm K}}{\partial Y}} = -\frac{3}{4}e^{-\frac{3}{2}Y}\left(e^{\frac{1}{2}Y} - e^{2Y}\right)^{-1} \tag{54}$$

We finally get

$$\left(e^{\frac{3}{2}Y} - 1\right)\nabla^2\psi = \frac{3}{4}\psi + Q\delta(X)\delta(Y)\left(e^{\frac{7}{2}Y} - e^{2Y}\right) \tag{55}$$

Note that equation (53) is an equation for stationary waves, the lee waves.

3.1.2 Localized solutions to the point vortex problem

The periodicity in X suggests the expansion $\psi = \sum_m \psi_m e^{imX}$. The symmetry of the system as $X \to -X$ limits the sum to $m \geq 0$. In this case, we will have to solve

$$(e^{\frac{3}{2}Y} - 1)\left(\frac{\partial^2\psi_m}{\partial Y^2} - m^2\psi_m\right) = \frac{3}{4}\psi_m + \frac{Q}{2\pi}\delta(Y)\left(e^{\frac{7}{2}Y} - e^{2Y}\right) \tag{56}$$

Asymptotically, we see that there exists solutions to this equation which are localized in the radial direction:

- for $Y \gg 1$, we get
$$e^{\frac{3}{2}Y}\left(\frac{\partial^2\psi_m}{\partial Y^2} - m^2\psi_m\right) \approx \frac{3}{4}\psi_m \tag{57}$$
This equation can be solved exactly by using the change of variable $t = e^{-\frac{3}{4}Y}$, which leads to the solutions
$$\psi_m = I_{\pm\frac{4}{3}m}(\sqrt{\frac{4}{3}}t) \tag{58}$$
For $Y \gg 1$, $t \to 0$ so we must keep the $I_{+\frac{4}{3}m}$ solution to ensure the decay of the solutions. Note that when $m = 0$, there is no decaying solution. This will be discussed later.

- for $Y \ll -1$, which corresponds to the center of the accretion disk, the equation becomes
$$\left(\frac{\partial^2\psi_m}{\partial Y^2} - m^2\psi_m\right) \approx -\frac{3}{4}\psi_m \tag{59}$$
which has the decaying solutions
$$\psi_m \propto \exp(\sqrt{m^2 - \frac{3\Gamma}{4}}Y) \tag{60}$$
when $m \neq 0$. When $m = 0$, we get oscillatory solutions.

Near the point vortex (X, Y small) the equation becomes
$$Y\left(\frac{\partial^2\psi_m}{\partial Y^2} - m^2\psi_m\right) \approx \frac{1}{2}\psi_m + Y\frac{Q}{2\pi}\delta(Y) \tag{61}$$
By changing the variable to $t = \alpha Y$, we get, for the homogeneous part
$$\frac{\partial^2\psi_m}{\partial t^2} - \frac{m^2}{\alpha^2}\psi_m - \frac{1}{2\alpha t}\psi_m = 0 \tag{62}$$
which corresponds to a Whitaker equation with coefficients $\kappa = -1/2\alpha$, and $\mu^2 = 1/4$ (cf. Abramovitz & Stegun [6]) provided $\frac{m^2}{\alpha^2} = 1/4$. The solutions are the Whitaker functions $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ such that
$$\begin{aligned} M_{\kappa,\mu} &= e^{-\frac{1}{2}t}t^{\frac{1}{2}+\mu}M(\frac{1}{2}+\mu-\kappa, 1+2\mu, t) \\ W_{\kappa,\mu} &= e^{-\frac{1}{2}t}t^{\frac{1}{2}+\mu}U(\frac{1}{2}+\mu-\kappa, 1+2\mu, t) \end{aligned} \tag{63}$$
where M and U are the regular and singular confluent hypergeometric functions. Since we are actually solving the problem of stationary waves in a shear flow, we know that the point at which the velocity vanishes is a critical layer for the waves. The singularity of the equations near $Y = 0$ reflects the presence of this critical layer. As a result we expect solutions of the kind [7]
$$\begin{aligned} \psi_r &\propto tP_1(t) \\ \psi_s &\propto t\ln|t|P_1(t) + P_2(t) \end{aligned} \tag{64}$$

with $P_1(t) = a_0 + a_1 t + \cdots$ and $P_2(t) = b_0 + b_1 t + b_2 t^2 + \cdots$ near the origin, we know that we should take $\mu = 1/2$ [6]. The expansion of the functions near the origin is then

$$M_{\kappa,1/2} = e^{-\frac{1}{2}t} t M(1-\kappa, 2, t) = e^{-\frac{1}{2}t} t \sum_{n=0}^{\infty} \frac{(1-\kappa)_n t^n}{(n!2)_n} \tag{65}$$

$$\begin{aligned} W_{\kappa,1/2} &= e^{-\frac{1}{2}t} t U(1-\kappa, 2, t) \\ &= \frac{e^{-\frac{1}{2}t} t}{\Gamma(-\kappa)} \left[M(1-\kappa, 2, t) \ln t + \sum_{n=0}^{\infty} \frac{(1-\kappa)_n t^n}{(2)_n n!} (\psi_\Gamma(1-\kappa+n) \right. \\ &- \left. \psi_\Gamma(1+n) - \psi_\Gamma(2+n)) + \frac{\Gamma(-\kappa)}{\Gamma(1-\kappa)} \frac{1}{t} \right] \end{aligned} \tag{66}$$

where $\psi_\Gamma(a) = \Gamma'(a)/\Gamma(a)$. We see that in order for $W_{\kappa,\mu}$ to be well defined, we need to choose t positive everywhere, which means taking $\alpha_+ = 2m$ for $Y > 0$ and $\alpha_- = -2m$ for $Y < 0$. This also means that the $Y > 0$ and $Y < 0$ branches will have different values of κ: $\kappa_+ = -1/4m$ and $\kappa_- = 1/4m$.

Let's now write the full solutions:

$$\psi_m(Y > 0) = A M_{\kappa_+, \frac{1}{2}}(2mY) + B W_{\kappa_+, \frac{1}{2}}(2mY) \tag{67}$$

$$\psi_m(Y < 0) = C M_{\kappa_-, \frac{1}{2}}(-2mY) + D W_{\kappa_-, \frac{1}{2}}(-2mY) \tag{68}$$

Since $M_{\kappa,\frac{1}{2}}(0) = 0$, and $W_{\kappa,\frac{1}{2}}(0) = \frac{1}{\Gamma(1-\kappa)}$, the continuity of ψ_m across the origin implies

$$\frac{B}{\Gamma(1+1/4m)} = \frac{D}{\Gamma(1-1/4m)} \tag{69}$$

Integrating equation (56) across $Y = 0$, we get the jump condition

$$\left. \frac{\partial \psi_m}{\partial Y} \right|_{0-}^{0+} = \int_{0-}^{0+} \frac{\psi_m}{2Y} \mathrm{d}Y + \frac{Q}{2\pi} \tag{70}$$

The integral term on the RHS is mathematically ill-defined. However, if we assume that it should really be the principal value of this integral, then we can show that this term is 0 using the expansion in Y of ψ near $Y = 0$, and we are left with the simple jump condition

$$\left. \frac{\partial \psi_m}{\partial Y} \right|_{0-}^{0+} = \frac{Q}{2\pi} \tag{71}$$

Since we have $M'_{\kappa,\frac{1}{2}}(0) = 1$ and $W'_{\kappa,\frac{1}{2}}(0) = \frac{1}{\Gamma(-\kappa)} \ln t + c(\kappa)$ where $c(\kappa)$ is a constant term, we see that the singular part of the derivative is continuous across the origin when the function ψ_m itself is: indeed, the derivative is

$$\psi'_m(0_+) = 2m \left(A + B \frac{\ln(2mY)}{\Gamma(-\kappa_+)} + Bc(\kappa_+) \right) \tag{72}$$

$$\psi'_m(0_-) = -2m \left(C + D \frac{\ln(-2mY)}{\Gamma(-\kappa_-)} + Dc(\kappa_-) \right) \tag{73}$$

so that the continuity of the singular part of the derivative implies

$$\frac{B}{\Gamma(1/4m)} = -\frac{D}{\Gamma(-1/4m)} \Leftrightarrow \frac{B}{\Gamma(1+1/4m)} = \frac{D}{\Gamma(1-1/4m)} \tag{74}$$

using the property $\Gamma(1+x) = x\Gamma(x)$.

This comment implies that although the derivative of the function ψ_m becomes singular near the origin, it is still possible to have a finite jump of the derivatives across the origin. The asymptotic behaviour and the jump condition define uniquely the four coefficients A, B, C and D for each value of m but 0, to yield a unique solution for the far field depending only on the vortex strength Q.

3.1.3 The axisymmetric (m=0) case

In this case, the solutions do not decay at infinity. In fact, we see that for $Y \gg 1$, the equation becomes $\psi_0'' = 0$, which has the general solutions $\psi_0 = aY + b$, and for $Y \ll -1$ there is an oscillatory solution $\psi = c \cos\left(\sqrt{\frac{3}{4}Y}\right) + d\left(\sin(\sqrt{\frac{3}{4}Y}\right)$. There is here an arbitrariness in the choice of the boundary conditions, which is solved by the matching with the inner solution. For the purpose of plotting the results only, we chose to take the following boundary conditions $\psi_0(Y_c) = \psi_0(-Y_c) = 0$. Again, there exists a unique solution fulfilling these 2 boundary conditions and the jump condition at the origin.

3.1.4 Numerical procedure and results

Having established that there exists a unique solution to the problem, it is now easy to find it numerically. We start by integrating equation (56) from $+\infty$ and $-\infty$ towards the origin using the asymptotic behaviour as a first boundary condition. We define a free parameter h as $\psi_m(0) = h$, and use this as a second boundary condition for both branches of the solution. We then calibrate this parameter h so that the jump across the origin is indeed $Q/2\pi$. Since both parts of the solution are linear, increasing h by a factor of 2 amounts to increasing Q by a factor of 2: there is a linear relation between h and Q, namely

$$\frac{Q}{2\pi} = s(m)h \tag{75}$$

The coefficients $s(m)$ can be found numerically by fixing $h = 1$. The result is shown in Fig.5, on the left. On the right, the resulting solutions for $Q = -50$ are shown. Note the slow convergence of the modes for large m. This is due to the fact that the point vortex is a logarithmic singularity, and the amplitude of the modes vary as $1/m$.

Finally, we are left to sum the Fourier coefficients to reconstruct the function: we have

$$\psi = \psi_0 + 2\sum_{m=1}^{\infty} \psi_m(Y)\cos(mX) \tag{76}$$

The contour lines of the total stream function (the perturbation and the Keplerian shear) have been plotted for 4 values of Q, and are represented in Fig.6. In all cases, the summation over m has been truncated at $m = 20$.

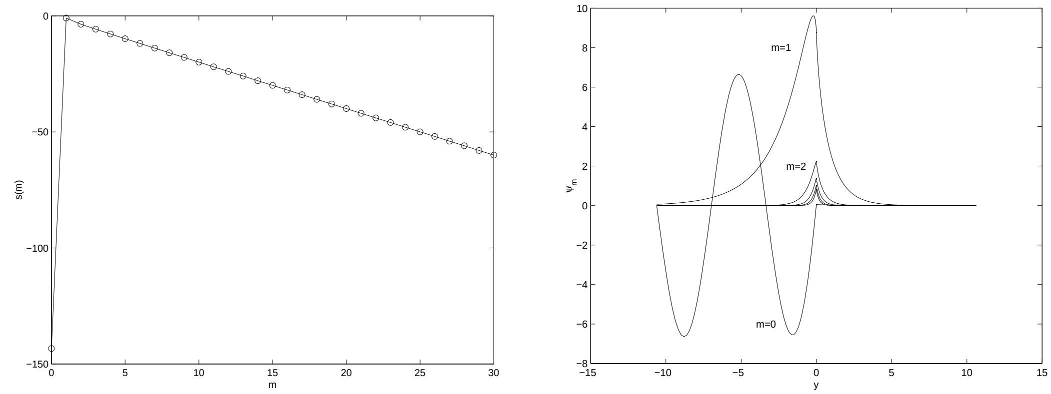

Figure 5: On the left, the coefficients $s(m)$ have been calculated and are represented as a function of m. On the right, this calibration has been used to calculate the functions ψ_m for $m = 0, 1, 2, 3, 4, 5$ for $Q = -50$. Only the first few have been labelled. The cutoff for ψ_0 has been chosen at $Y_{\rm c} = 10$.

3.1.5 Discussion

The solutions obtained correspond well to what might have been expected. There are here two main features to the result. Firstly, the presence of a point vortex in any shear flow induces the deformation of stream lines seen in Fig.7. This type of deformation is also seen in the results presented here. The second feature corresponds to the presence of the critical layer at the radius $R = 1$, and is characterized by the discontinuity in the velocities at that radius. This is qualitatively similar to the case of the Cat's Eyes patterns seen in the plane parallel shear flows [7]. The linear approximation theoretically fails as $Y \to 0$, and a full non-linear theory would normally be necessary; however, it was shown that the nonlinear boudary layer simply connects to the linear branches of the solution far from the critical layer, without change in the phase of the logarithm, so that the solution found here is a good approximation to the nonlinear solution provided $Y \gg \epsilon$

3.2 Close to the vortex

In this case, we want to chose a new scaling to represent the region near the vortex. Let's chose to take $Y = \epsilon y$ and $X = \epsilon x$, and expand the equations in ϵ, assuming that $\epsilon \ll 1$. We also assume the following form for the stream function ψ and the vorticity:

$$\psi = \epsilon^2(\psi_0 + \epsilon\psi_1) \tag{77}$$

$$\omega = \omega_0 + \epsilon\omega_1 \tag{78}$$

The Jacobian equation becomes

$$J\left(-\frac{\Gamma}{2}y^2 - \frac{5}{12}\Gamma\epsilon y^3 + \psi_0 + \epsilon\psi_1 + O(\epsilon^2)\,,\right.$$
$$\left.\frac{\Gamma}{3}(1 - \frac{3}{2}\epsilon y) + (1 - 2\epsilon y)\nabla^2\psi_0 + \epsilon\nabla^2\psi_1 + O(\epsilon^2)\right) = 0 \tag{79}$$

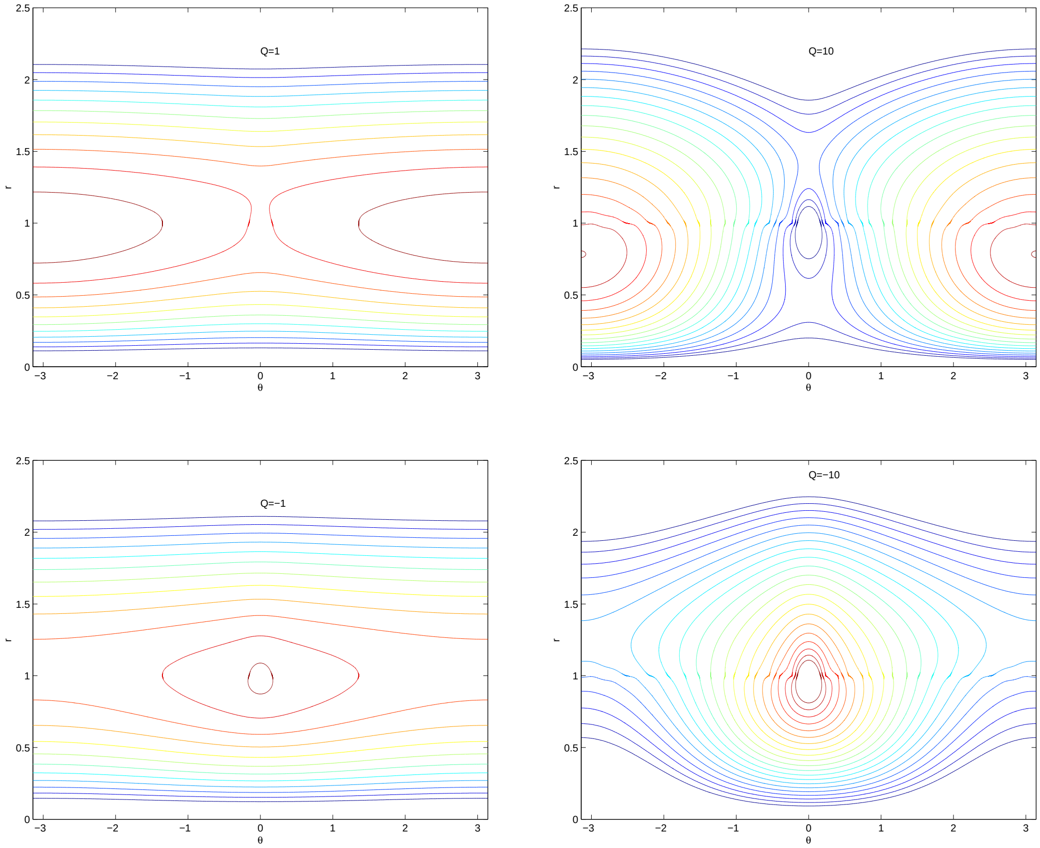

Figure 6: Stream line contours of around an anticyclonic point vortex in Keplerian shear flow, Q=1, (upper left), Q=10 (upper right), Q=-1 (lower left) and Q=-10 (lower right)

and we also have

$$\omega_0 = \nabla^2\psi_0 \tag{80}$$

$$\omega_1 = \nabla^2\psi_1 - 2y\nabla^2\psi_0 \tag{81}$$

The successive orders in ϵ from the Jacobian yield

$$J\left(\psi_0 - \frac{\Gamma}{2}y^2, \nabla^2\psi_0\right) = 0 \tag{82}$$

$$J\left(\psi_0 - \frac{\Gamma}{2}y^2, -\frac{\Gamma}{2}y + \omega_1\right) + J\left(\psi_1 - \frac{5}{12}\Gamma y^3, \nabla^2\psi_0\right) = 0 \tag{83}$$

3.2.1 Zeroth order solutions

The solutions to

$$J\left(\psi_0 - \frac{\Gamma}{2}y^2, \nabla^2\psi_0\right) = 0 \tag{84}$$

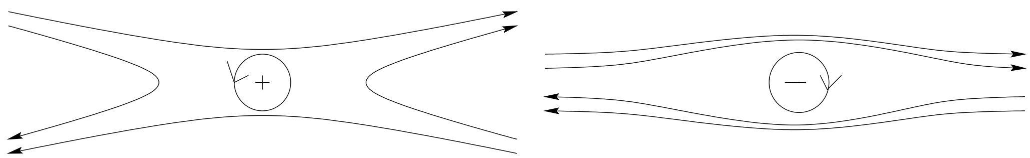

Figure 7: Stream lines around a cyclonic (left) and an anticyclonic (right) vortex positioned at a zero-velocity point

are well known, and have been calculated in the previous section: taking piece-wise constant solutions, we get an elliptical vortex patch of constant vorticity. This suggests the use of the elliptical orthogonal coordinate system (χ, ζ), such that

$$\begin{aligned} x &= f\chi\zeta \\ y &= f\sqrt{\chi^2 - 1}\sqrt{1 - \zeta^2} \end{aligned} \tag{85}$$

with $1 < \chi$ and $-1 < \zeta < 1$. The boundary of the vortex is given by $\chi = a$ so that the solution can be written as

$$\omega_0 = q\mathcal{H}(\chi - a) \tag{86}$$

The stream function ψ_0 is then given by equation (82). The Laplacian in the elliptical coordinate system is

$$\begin{aligned} \nabla^2\psi_0 &= \frac{1}{f^2(\chi^2 - \zeta^2)}\left[\sqrt{\chi^2 - 1}\frac{\partial}{\partial\chi}\left(\sqrt{\chi^2 - 1}\frac{\partial\psi_0}{\partial\chi}\right)\right. \\ &+ \left.\sqrt{1 - \zeta^2}\frac{\partial}{\partial\zeta}\left(\sqrt{1 - \zeta^2}\frac{\partial\psi_0}{\partial\zeta}\right)\right] = q\mathcal{H}(a - \chi) \end{aligned} \tag{87}$$

Looking for separable solutions, such that $\psi_0(\chi, \zeta) = \sum_n G_n(\chi)H_n(\zeta)$, we must solve

$$\sqrt{\chi^2 - 1}\frac{\partial}{\partial\chi}\left(\sqrt{\chi^2 - 1}\frac{\partial G_n}{\partial\chi}\right) = \lambda_n^2 G_n + qf^2\chi^2 \tag{88}$$

$$\sqrt{1 - \zeta^2}\frac{\partial}{\partial\zeta}\left(\sqrt{1 - \zeta^2}\frac{\partial H_n}{\partial\zeta}\right) = -\lambda_n^2 H_n - qf^2\zeta^2 \tag{89}$$

The homogeneous part generates the Chebyshev polynomials for both χ and ζ with $\lambda = n$. The polynomials form the basis for the regular solution. The H_n solutions must always be regular, so that we simply have

$$H(\zeta) = \sum_{n=0}^{\infty} b_n T_n(\zeta) \tag{90}$$

where the T_n are the Chebyshev polynomials of the first kind. However, because of the matching of Ψ, we also need to find the singular solution for G outside the vortex. In order to do this, let's perform the change of variables $\chi = \cosh\alpha$

$$\frac{\partial^2 G_n}{\partial\alpha^2} = n^2 G_n \tag{91}$$

We have the solutions, for $n > 0$

$$\begin{aligned} G_n = a_n e^{n\alpha} + \tilde{a_n} e^{-n\alpha} &= a_n \left(\chi + \sqrt{\chi^2 - 1}\right)^n + \tilde{a_n} \left(\chi + \sqrt{\chi^2 - 1}\right)^{-n} \\ &\equiv a_n R_n + \tilde{a}_n S_n \end{aligned} \tag{92}$$

which defines the functions R_n and S_n, and and for $n = 0$

$$G_0 = a_0 \ln(\chi + \sqrt{\chi^2 - 1}) + \tilde{a_0} = a_0 S_0 + \tilde{a_0} \tag{93}$$

which defines S_0, and $R_0 \equiv 1$. The special solution, necessary inside the vortex, is a second order polynomial in the variables χ or ζ respectively, so that the final solution is

$$\psi_0^{\text{in}} = \sum_{n=0}^{\infty} \left(A_n T_n(\chi) + (\chi^2 - 2)\frac{qf^2}{2}\right)\left(B_n T_n(\zeta) + (\zeta^2 - 2)\frac{qf^2}{2}\right) \tag{94}$$

$$\psi_0^{\text{out}} = \sum_{n=0}^{\infty} C_n S_n(\chi) T_n(\zeta) \tag{95}$$

where we have used the fact that the solutions must be regular inside the vortex, and that they should decay outside the vortex. Note that we can rewrite $x^2 = (T_0(x) + T_2(x))/2 = (1 + T_2(x))/2$. The matching condition at the boundary $\chi = a$ yields the following relation between the coefficients [4]:

$$\begin{aligned} \left(A_n T_n(a) + (a^2 - 2)\frac{qf^2}{2}\right)\left(B_n + \frac{qf^2}{4}\delta_{n,2}\right) &= C_n S_n(a) \text{ for } n > 0 \\ \left(A_0 + (a^2 - 2)\frac{qf^2}{2}\right)\left(B_0 - 3\frac{qf^2}{4}\right) &= C_0 S_0(a) \text{ for } n = 0 \end{aligned}$$

The matching of the derivatives yields a similar system,

$$\begin{aligned} \left(A_n T_n'(a) + qf^2(a-1)\right)\left(B_n + \frac{qf^2}{4}\delta_{n,2}\right) &= C_n S_n'(a) \text{ for } n > 0 \\ qf^2(a-1)\left(B_0 - 3\frac{qf^2}{4}\right) &= C_0 S_0'(a) \text{ for } n = 0 \end{aligned}$$

Using equation (82) with the fact that $\frac{\partial \omega_0}{\partial \zeta} = 0$ yields

$$\frac{\partial}{\partial \zeta}\left(\psi_0 - \frac{1}{2}\Gamma y^2\right)\bigg|_{\chi = a} = 0 \tag{98}$$

[4] In order to derive these conditions, we use the orthogonality relation between the Chebyshev polynomials

$$\begin{aligned} \int_{-1}^{1} T_n(\zeta) T_m(\zeta) \frac{d\zeta}{\sqrt{1 - \zeta^2}} &= \frac{\pi}{2}\delta_{m,n} \text{ if } n \neq 0 \qquad (96) \\ &= \pi \delta_{m,n} \text{ if } n = 0 \qquad (97) \end{aligned}$$

or rather,

$$\psi_0 - \frac{1}{2}\Gamma y^2\Big|_{\chi=a} = \psi_0(a,\zeta) - \frac{1}{2}\Gamma f^2(a^2-1)(1-\zeta^2) = c \tag{99}$$

where c is a constant. This implies that we must take

$$C_n S_n(a) = \frac{\Gamma}{2} f^2(a^2-1)(\delta_{n,0} - \frac{1}{2}\delta_{n,2}) \tag{100}$$

so we see that only 2 coefficients are non-zero, namely C_0 and C_2.

If we were to match this with the Keplerian shear flow, and ignore the far field solution, we would then obtain a unique relation between the size of the vortex a and it's vorticity q.

3.2.2 First order solutions

We now have to solve equation (83). The Jacobians in this equation can directly be transformed into Jacobians for the new coordinate system: with

$$\mathcal{J}(A,B) = \frac{\partial A}{\partial \chi}\frac{\partial B}{\partial \zeta} - \frac{\partial A}{\partial \zeta}\frac{\partial B}{\partial \chi} \tag{101}$$

we get

$$\mathcal{J}\left(\psi_0 - \frac{\Gamma}{2}y^2, -\frac{\Gamma}{2}y + \omega_1\right) + \mathcal{J}\left(\psi_1 - \frac{5}{12}\Gamma y^3, \nabla^2\psi_0\right) = 0 \tag{102}$$

Since $\nabla^2\psi_0 = q\mathcal{H}(a-\chi)$, we see that

$$\mathcal{J}\left(\psi_0 - \frac{\Gamma}{2}y^2, -\frac{\Gamma}{2}y + \omega_1\right) = -q\delta(a-\chi)\frac{\partial}{\partial\zeta}\left(\psi_1 - \frac{5}{12}\Gamma y^3\right)_{\chi=a} \tag{103}$$

This equation suggests the ansatz $\omega_1 = q\eta(\zeta)\delta(a-\chi) + \omega_2$, so that we have the condition

$$\frac{\partial\eta}{\partial\zeta}\frac{\partial}{\partial\chi}\left(\psi_0 - \frac{\Gamma}{2}y^2\right)_{\chi=a} + \frac{\partial}{\partial\zeta}\left(\psi_1 - \frac{5}{12}\Gamma y^3\right)_{\chi=a} = 0 \tag{104}$$

which can be integrated along the boundary to yield

$$\eta(\zeta)\frac{\partial}{\partial\chi}\left(\psi_0 - \frac{\Gamma}{2}y^2\right)_{\chi=a} + \psi_1(a,\zeta) - \frac{5}{12}\Gamma f^3(a^2-1)^{3/2}(1-\zeta^2)^{3/2} = \psi_\eta \tag{105}$$

The equation for ω_2 is

$$\mathcal{J}\left(\psi_0 - \frac{\Gamma}{2}y^2, -\frac{\Gamma}{2}y + \omega_2\right) = 0 \tag{106}$$

which implies

$$\omega_2 = G\left(\psi_0 - \frac{\Gamma}{2}y^2\right) + \frac{\Gamma}{2}y \tag{107}$$

As in the far field solution, we must chose the function G to represent the presence of a vortex. Ideally, the functions F should be the linear continuation of G when $\psi \ll \psi_{\mathrm{K}}$. As a first guess, we chose to take simply $G \equiv 0$, so that

$$\omega_2 = \frac{\Gamma}{2}y \tag{108}$$

We now have to express the stream function ψ_1 as a function of y. This can be done by solving the equation

$$\nabla^2\psi_1 = 2yq\mathcal{H}(a-\chi) + q\eta(\zeta)\delta(a-\chi) + \frac{\Gamma}{2}y \tag{109}$$

Set $\psi_1 = \phi_1 + \phi_2 + \phi_3$, where the ϕ_i satisfy respectively

$$\begin{aligned} \nabla^2\phi_1 &= q\eta(\zeta)\delta(a-\chi) && (110)\\ \nabla^2\phi_2 &= \frac{\Gamma}{2}y && (111)\\ \nabla^2\phi_3 &= 2qy\mathcal{H}(a-\chi) && (112) \end{aligned}$$

In all three cases we will have to solve the homogeneous equation $\nabla^2\phi_h = 0$. This has already been done in the zeroth order case, and the result is

$$\phi_h^{\text{out}} = \sum_{n=0}^{\infty}(a_n^{\text{out}}R_n(\chi) + b_n^{\text{out}}S_n(\chi))T_n(\zeta) \tag{113}$$

outside the vortex, where the singular solution must be kept, and

$$\phi_h^{\text{in}} = \sum_{n=0}^{\infty} a_n^{\text{in}}T_n(\chi)T_n(\zeta) \tag{114}$$

inside the vortex. For ϕ_2 and ϕ_3, the special solutions are easy to find and we get

$$\begin{aligned} \phi_2 &= \phi_h + \phi_{2,s} = \phi_h + \frac{\Gamma}{12}y^3 + c_1 y + c_0 && (115)\\ \phi_3 &= \phi_h + \phi_{3,s} = \phi_h + \left(\frac{q}{3}y^3 + d_1 y + d_0\right)\mathcal{H}(a-\chi) && (116) \end{aligned}$$

Note that the solutions are divergent for large y, and that the true solution is obtained by matching the near-vortex solution to a far field, wave-like solution. Inside the vortex, however, the solutions must be regular.

To summarize, renormalizing the coefficients a_n and b_n, we have

$$\begin{aligned} \psi_1^{\text{in}} &= \sum_{n=0}^{\infty} a_n^{\text{in}}T_n(\chi)T_n(\zeta) + \phi_{2,s} + \phi_{3,s}\\ \psi_1^{\text{out}} &= \sum_{n=0}^{\infty}\left(a_n^{\text{out}}R_n(\chi) + b_n^{\text{out}}S_n(\chi)\right)T_n(\zeta) + \phi_{2,s} && (117) \end{aligned}$$

where by definition, $R_0(\chi) = 1$. The continuity of the function across the boundary of the vortex implies that

$$\frac{\pi}{2}a_n^{\text{in}}T_n(a) + \int_{-1}^{1}\frac{\phi_{3,s}(a,\zeta)}{\sqrt{1-\zeta^2}}T_n(\zeta)\mathrm{d}\zeta = \frac{\pi}{2}\left(a_n^{\text{out}}R_n(a) + b_n^{\text{out}}S_n(a)\right) \tag{118}$$

for $n > 0$ and

$$\pi a_n^{\text{in}} + \int_{-1}^{1} \frac{\phi_{3,s}(a,\zeta)}{\sqrt{1-\zeta^2}} \mathrm{d}\zeta = \pi \left(a_0^{\text{out}} + b_0^{\text{out}} S_0(a) \right) \tag{119}$$

for $n = 0$. For ϕ_1, the function must be continuous across the boundary, but the derivative has a jump given by equation (112). Integrating (112) across the boundary, we get

$$\frac{1}{f^2} \frac{a^2-1}{a^2-\zeta^2} \left[\frac{\partial \phi_1}{\partial \chi} \right]_{a-}^{a+} = q\eta(\zeta) \tag{120}$$

Let's write $\eta(\zeta) = \sum_{n=0}^{\infty} \eta_n T_n(\zeta)$. The matching condition of the derivatives therefore implies that, for $n > 0$

$$\left(a_n^{\text{out}} R_n'(a) + b_n^{\text{out}} S_n'(a) \right) - a_n^{\text{in}} T_n'(a) = q\eta_n \tag{121}$$

and for $n = 0$,

$$b_0^{\text{out}} S_0'(a) = q\eta_0 \tag{122}$$

The coefficients η_n can actually be determined from self-consistently using equation (105), provided we know a_n and b_n for all n. If we truncate the system at the order $N-1$, there are in total $3N+3$ coefficients to solve for, and $2N$ matching conditions. The remaining coefficients are given by the matching of this solution to a far field.

3.3 Matching of the far field to the vortex solution.

The behaviour of the far-field is mostly determined by the total vorticity Q of the vortex patch (with the exception of the axisymmetric component). In order to be consistent between the far-field and the close-field, we require that $Q = \epsilon^2 q$, since we assumed the size of the vortex patch to be of order of ϵ. The aim of this section is more to assess whether such a matching is possible rather than to perform it. The actual matching, as we shall see, can only be done numerically, and will be the aim of future work.

In order to do this matching, it is necessary to study the behaviour of the inner solution for $\chi \to \infty$ and the outer solution as $\chi \to 1$. The elliptical coordinate system asymptotically tends to the polar coordinate system as $\chi \gg 1$. Indeed, we then have $r^2 = x^2 + y^2 \approx f^2\chi^2$ and $\zeta \approx \cos\varphi = x/\sqrt{x^2+y^2}$. Also, we use the property that $T_n(\cos\varphi) = \cos(n\varphi)$, and that $\chi + \sqrt{\chi^2 - 1} \approx 2\chi$. As a result, for $\chi \gg 1$, the inner solution tends to

$$\begin{aligned} \psi^{\text{inner}} &= \epsilon^2 \sum_{n=0}^{\infty} C_n \left(\frac{2r}{f} \right)^{-n} \cos(n\varphi) + \epsilon^3 \left[a_0^{\text{out}} + b_0^{\text{out}} \ln \left(\chi + \sqrt{(}\chi^2 - 1) \right) \right. \\ &+ \left. \sum_{n=1}^{\infty} \left(a_n^{\text{out}} \left(\frac{2r}{f} \right)^{n} + b_n^{\text{out}} \left(\frac{2r}{f} \right)^{-n} \right) \cos(n\varphi) + \phi_{2,s} \right] \end{aligned} \tag{123}$$

On the other hand, as we saw, the outer solution tends to

$$\begin{aligned} \psi^{\text{outer}} = \alpha_{\pm} \psi_0^{\text{outer}}(|Y|) \quad &+ \quad \sum_{n=1}^{\infty} \left(A_{m,\pm} M_{\pm 1/4m, 1/2}(|Y|) \right. \\ &+ \quad \left. B_{m,\pm} W_{\pm 1/4m,1/2}(|Y|) \right) \cos(nX) \end{aligned} \tag{124}$$

where the $\pm$ sign refers to the difference in the $Y > 0$ and $Y < 0$ branches. All the coefficients $A_{m,\pm}$ and $B_{m,\pm}$ are uniquely defined, with the exception of the $m = 0$ mode where we imposed some additional boundary conditions to determine them. These may be taken as free parameters if necessary to perform the matching on to the inner solution.

Let's study the various terms that appear in the inner and the outer, and that may cause problems in the matching. The most obvious term is the axisymmetric term, which has the main component as $(\Gamma/12)\epsilon^3 y^3$ in the inner, and that can be shown to behave as $c_0 + c_1 Y + c_2 Y^2 + c_3 Y^3 + ...+$ logarithmic terms in the outer. The Y^3 terms can be matched, since we can to choose the coefficients c_3 on either sides of $Y = 0$ to be $\Gamma/12$. This is possible since we had the freedom of varying the boundary conditions on the axisymmetric mode to fit this requirement. Next, we must fit the logarithmic terms. The main logarithmic dependence in the inner comes from the $O(\epsilon^2)$ term. As we take $y \to \infty$, this term can be assimilated to the contribution from a point vortex only in the outer region. We expect this term to match exactly onto the outer solution for a point vortex only, which has been studied in Section 3.2.5. Finally, it can be shown that the remaining difference between the point vortex case and the vortex+waves case is non-singular, so that this could possibly be matched onto the $O(\epsilon^3)$ term in the inner. This last matching would yield the coefficients a_n and b_n, and therefore determine the shape of the boundary by determining η.

4 Conclusion

In an attempt to understand the dynamics of vortices in accretion flows, we have been looking for steady state solutions of such a system, since the existence of stable steady states might be reason for the observed longevity of the vortices. The first part of the project was a simple attempt at finding such solutions using the rather crude assumption of a constant vorticity field, which is only truly justified in the case small vortex patches. This assumption allowed us to consider top-hat vortex solutions, and study their steady state shapes. The second part of the project was an attempt at dropping this assumption. In that case, it has been shown that a general stationary lee wave solution must be added to the vortex solution in order to satisfy the vorticity equation. This problem can only be solved asymptotically in two limits: far from the vortex, it is possible to find linearized solutions. Closer to the vortex, an expansion in the small parameter ϵ which is really the ratio of the size of the vortex to the distance to the center of the shear flow, yields results very similar to the first section: to zeroth order, we recover the elliptical vortex solution, and to first order, the deformation of the vortex matches onto the "background flow", which consists of the Keplerian flow and the lee waves. The possibility of the matching between the two solution has been considered, and will be the purpose of future work.

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