Towards a Non-Linear Model for Spicules

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1 Introduction

Spicules are approximately $7,000$ km high jets of hot plasma seen on the sun extending from the chromosphere into the lower corona. Each jet lasts roughly ten minutes and then disappears. These jets have been previously discussed in the literature (for example, see [Hasan and Venkatakrishnan, 1981], [Athay, 1984], and [Umurhan et al., 1998]); there is no consensus on the driving mechanism. This paper follows the approach of Umurhan, Tao, and Spiegel ([Umurhan et al., 1998]) who suggested that spicules are a non-linear effect of overstable acoustic modes.

There are two main parts to this work. The first is to write down an eigenvalue equation that can be numerically solved to obtain frequencies and growth rates for the various oscillation modes, considered as infinitesimal perturbations to a background state, of a chromosphere-like layer. The second part is to use the results of the linear stability problem to derive a non-linear equation to describe the time evolution of the layer.

The main outline of the paper is as follows: Section 2 contains a brief outline of the basic structure of the sun and spicules. Section 3 describes the time scales involved in the problem. The basic equations that will be used to describe the plasma are introduced in Section 4 and linearized in Section 5. Section 6 contains a discussion of the energy equation and Newton's law of cooling. Section 7 discusses the various background atmospheres around which the linearization is done. The boundary conditions for the problem are described in Section 8. Results for the case of the isothermal and polytrope background are shown in Sections 9 and 10. Sections 11 and 12 develop the amplitude equation. The remaining sections discuss the conclusions and possibilities for future work.

2 A Brief Review of Spicules and The Sun

The sun has radius of 7×10^5 km. The inner 20% by radius is where the nuclear burning takes place and is responsible for energy generation. The region below 70% of the solar radius is called the radiative zone, in this region energy is carried outwards by radiation. Above the radiative zone and extending to the surface is the convection zone, where as the name implies energy is carried outwards mainly by convection. The top of the convection zone, known as the photosphere, produces most of the visible light we see. At the top of the photosphere the temperature, which decreases steadily outwards from the core, reaches its minimum of about $6,000$ K. Above the photosphere is the chromosphere, a layer roughly $4,000$ km thick, in which the temperature increases with height from the photospheric value to approximately $10,000$ K while the density and pressure are steadily decreasing. At the top of the chromosphere the temperature starts increasing very rapidly with height, up to the two million degrees of the corona.

The surface of the photosphere shows two main spatial scales of convection. The smaller of these scales is the granule scale, which is 1000 km. Granules are up-drafts of hot plasma, with thin boundaries of down-going cooler plasma. The larger length scale is due to supergranules, each of which contains approximately $1,000$ granules. Supergranules, like granules, consist of a widespread warm up-welling surrounded by thin lanes of cooler down-drafts. The flow due to the supergranular structure sweeps granules, as well as magnetic flux, towards the boundaries between supergranules. As a result the magnetic field above the supergranule lanes can be up to 2000 G, compared to the usual background of a few gauss. Spicules tend to occur above these supergranule lanes. As a result, it is suggested that the formation of spicules is related to the presence of the magnetic field.

Spicules can be seen in $H\alpha$, the 660 nm spectral line of hydrogen, images of the solar limb as well as of the disk. In limb images they appear as extended, much taller than they are wide, bright regions. They are seen to extend to heights of $7,000$ to $10,000$ km above the photosphere and last for roughly ten minutes (e.g. [Suematsu et al., 1995]).

3 Time Scales

The time scales involved in the problem determine what physics needs to be included when describing the chromosphere. As only the fundamental modes of waves in a chromosphere-like layer are considered the length scale is taken to be the height of the chromosphere, $4,000$ km. The sound speed is approximately 10 $\frac{km}{s}$. 200 G is taken as the magnetic field strength. The important times are the sound speed transit time, the Alfvén wave transit time, and resistive diffusion time, the viscous damping time, and the rotation period. In the expressions for these times L is the length scale, c_s is the sound speed, c_a is the Alfvén speed, and η is the resistivity. The expressions and values for these times are then:

$$T_{sound} = \frac{L}{c_s} = 400s$$

$$T_{Alfvén} = \frac{L}{c_a} = 200s$$

$$T_{resist} = \frac{L^2}{\eta} = 10^7 years$$

T_{sound} is the time for a vertically pressure perturbation to cross the layer, $T_{Alfvén}$ is the time for a vertically propagating magnetic field disturbance to cross the layer. The resistive diffusion time T_{resist} is the time scale for the decay of magnetic field due to finite conductivity. Because the resistivity of the chromosphere is so small compared to the length scale in the problem, the resistive diffusion term will be neglected in the MHD equations in the next section. The viscous diffusion time is the time for a velocity perturbation to decay in the absence of other effects. It is not clear what value to use for the viscosity, perhaps turbulent viscosity is important. The rotation period is a approximately a month, so rotation is not a large effect on waves with periods of minutes, of order the sound or Alfvén wave transit times.

4 MHD equations

The standard ideal MHD equations are used to describe the plasma. ρ is the density, P is the pressure, $\vec{v}$ is the velocity, $\vec{J}$ is the current, $\vec{B}$ is the magnetic field, $\vec{E}$ is the electric field, T is the temperature. These quantities are functions of time and space. C_v is the heat capacity at constant volume and is considered fixed. R is the gas constant. The equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla P + \rho g \hat{z} + \frac{1}{c} \vec{J} \times \vec{B} \tag{2}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{3}$$

$$C_v \rho \frac{dT}{dt} + P \nabla \cdot \vec{v} = Q(T, \rho) \tag{4}$$

$$P = R \rho T \tag{5}$$

$$\vec{J} = \frac{c}{4\pi} \nabla \times \vec{B} \tag{6}$$

$$\vec{E} = -\vec{v} \times \vec{B} \tag{7}$$

Equation 1 is the usual continuity equation, expressing the fact that there are no sources or sinks of mass in the problem. Conservation of momentum is expressed by Equation 2. The forces on the right hand side are the pressure gradient, gravity, and the Lorentz force. Gravity is assumed to act only the in the $+\hat{z}$ direction. Equation 3 is the usual Faraday's Law. Conservation of energy is given by Equation 4, the function Q gives the rate at which energy is injected into (or removed from) the fluid at a given point and is in principle a function of temperature and density. The equation of state is assumed to be that for an ideal gas and is given in Equation 5. Equation 6 is the pre-Maxwell prescription for the current. The displacement current is assumed to be negligible, so this set of equation is not expected to be valid at high frequencies. The final equation gives the electric field in terms of the magnetic field. It arises from Ohm's law and the infinite conductivity assumption of Ideal MHD. If the conductivity σ is infinite then in the frame of reference moving with the fluid the current $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ must be zero, from which a relation between $\vec{E}$ and $\vec{B}$ is obtained. Once Q is specified this set of equations, along with boundary conditions and initial conditions, gives a complete description of the time evolution of the system.

5 Linearization

In order to study the linear stability problem the MHD equations must be linearized around a background state. In this work only the two dimensional plane-parallel problem is studied, x describes the horizontal direction, z describes the vertical. The background state is assumed static. Gravity and the background magnetic field are assumed to be constant and in the $\hat{z}$ direction. The background pressures is given by P_0, the density by ρ_0, and field by $B_0\hat{z}$. The variables P, ρ, θ, $\vec{v} = (U, W)$, $\vec{B} = (B_x, B_z)$ describe the perturbations to the pressure, density, temperature, velocity, and magnetic field respectively. The perturbations are assumed to have the form $f(z) \exp i(kx - \omega t)$. The linearized equations, given below, along with a prescription for boundary conditions and a background state then constitute a linear eigenvalue problem for the complex frequency ω.

$$-i\omega\rho + W\rho' + \rho_0(ikU + W') = 0$$

$$-i\omega U = -\frac{ikP}{\rho_0} + \frac{i{B_0}^2}{4\pi\rho_0\omega}(-k^2U + U'')$$

$$-i\omega W = -\frac{P'}{\rho_0} + g\frac{\rho}{\rho_0}$$

$$C_v\rho_0(-i\omega\theta + W{T_0}') + P_0(ikU + W') = Q(\theta, \rho, T_0, \rho_0)$$

$$\frac{P}{P_0} = \frac{\rho}{\rho_0} + \frac{\theta}{T_0}$$

The real part of ω gives the frequency of the mode and the imaginary part is the growth rate. The linearization makes it clear the the magnetic field causes only a force in the horizontal direction and is only due to motions in the horizontal direction. Motion along the background field is not affected at all. Note now that the function Q, which originally was a function of the total T and ρ must now be considered a function of both the background and perturbation values of these quantities. In order to further simplify these equations a form of Q must be chosen.

6 Heating Function

As mentioned earlier the function Q describes that rate at which heat is added or removed from the plasma as a function of its temperature and density. The chromosphere, to a good approximation, is optically thin (e.g. [Syrovat-skii and Zhugzhda, 1968]). Optically thin perturbations can be described by Newton's law of cooling: $Q(\theta, \rho) = -q\rho_0 C_v\theta$ ([Spiegel, 1957]). In general q is a function of the background state. For this work only the case in which q is a constant is considered. For q constant and no flow the non-linearized heat equation is:

$$\frac{\partial T}{\partial t} = -q(T - T_0)$$

For the case $q > 0$ deviations in the temperature T from the background value T_0 decay with a time scale of $\frac{1}{q}$. If $q < 0$ then any temperature perturbation is unstable, growing exponentially with time scale $\frac{-1}{q}$. In general, though not always, $q > 0$ has the effect of damping waves that produce temperature perturbations, and $q < 0$ increases the growth rate of waves that produce temperature perturbations.

7 Background State

In addition to the function Q the background state also must be specified. The full problem of heating in the chromosphere is unsolved. As a result it is not clear what the appropriate background model is. For lack of a better model, the three simplest temperature profiles are examined: a constant, a linear function of depth, and a quadratic function of depth. All three of these background temperatures can be obtained from the diffusion equation, with constant thermal conductivity κ and heating rate h,

$$-\kappa \frac{\partial^2 T}{\partial z^2} = h$$

For the case $h = 0$ a linear temperature profile is obtained, of which the constant profile is a special case. For nonzero h a parabolic profile is obtained. In all cases gravity is assumed to be constant and the equations of state is that of an ideal gas. A further assumption, for the sake of simplicity, is that the background state is in hydrostatic equilibrium. This assumption is probably not good for the chromosphere.

7.1 Isothermal Atmosphere

For the case of T constant,

$$\frac{dP}{dz} = g\rho = \frac{gP}{RT}$$

With the definition of the scale height $H = \frac{RT}{g}$, $P(z)$ has the form $\tilde{P} \exp \frac{z}{H}$, where $\tilde{P}$ is the pressure at $z = 0$. The scale height H is the distance it takes for the density and pressure to increase by a factor e. It is reasonable the this distance increases with temperature and decreases with increasing strength of gravity, g. It can be shown that this atmosphere is always convectively stable.

7.2 Polytrope Atmosphere

Suppose $T = \beta z + \alpha$. For $\beta \neq 0$, α can be made zero by choice of coordinates. Then:

$$\frac{dP}{dz} = g\rho = \frac{gP}{R\beta z}$$

The solution of which is $P(z) = \tilde{P}(\frac{z}{z_s})^{(\frac{g}{R\beta})}$. Here $\tilde{P}$ is the value of P at $z = z_s$. z_s has the units of length and is put in only to make the units look correct. The definition of $m = \frac{g}{R\beta} - 1$ is commonly used. With this definition $P(z) = \tilde{P}(\frac{z}{z_s})^{m+1}$ and $\rho(z) = \tilde{\rho}(\frac{z}{z_s})^m$. This atmosphere is known as a polytrope and can be either convectively stable or unstable depending on the relationship between m and $\gamma = \frac{C_p}{C_v}$. For $m < \frac{1}{\gamma-1}$ the atmosphere is unstable, while for $m > \frac{1}{\gamma-1}$ it is not. The instability arises from the combination of the stratification with gravity.

7.3 Constant Heating

For the case of $h \neq 0$ the temperature has the form $\frac{-h}{2\kappa}z^2 + \beta z + \alpha$. Again by change of coordinates α can be made zero. By the same method as in the previous two sections the pressure profile of this atmosphere is found to be $P(z) = \tilde{P}(\frac{z}{\beta - \frac{H}{2\kappa}z})^{(\frac{g}{R\beta})}$. This atmosphere is more interesting then the previous two. In the other cases the temperature and pressure both increase with depth. With constant non-zero heating there is the possibility of having the temperature decrease with depth. This feature makes this atmosphere more like the chromosphere than the previous two.

8 Boundary Conditions

Boundary conditions must be specified in order to complete the formulation of the linear stability problem. The linearized version of the equations are fourth order in space. As a result four boundary conditions are required. The traditional boundary condition that the vertical velocity, W, vanish on the top and bottom of the layer is used. This condition ensures that the mechanical part of the energy flux across the boundary is zero.

Two more boundary conditions are required. Limits on the possible boundary conditions on the magnetic field can be found be requiring that the Poynting vector $\vec{S} = \vec{E} \times \vec{B}$ at the boundary be parallel to the boundary. The result of the calculation is that for $k \neq 0$ either B_x or B_z must be zero on the boundary. The physical meaning of these boundary conditions requires some discussion.

The basic problem of selecting boundary conditions for this problem is that the boundaries do not correspond to clear physical boundaries, but rather have been selected arbitrarily. In order to deduce the physical meaning of the various boundary conditions on $\vec{B}$ it is necessary to make assumptions about what is outside the layer in which the eigenvalue problem is to be solved.

8.1 $B_x = 0$

For the case $k \neq 0$ the boundary condition $B_x = 0$ is equivalent to $U' = 0$ which is a statement of no stress on the boundary. In particular U is allowed to be nonzero on the boundary. Another point of view is that if the magnetic perturbation to the field outside the layer is required to be vertical, then $B_x = 0$ on the boundary implies that no surface currents are allowed, as can be seen by the standard Ampere's law argument. In order to maintain $\nabla \cdot \vec{B} = 0$ there would have to be a vertical perturbation to the field outside the layer to match the perturbation inside.

8.2 $B_z = 0$

The boundary condition $B_z = 0$ is equivalent to considering the outside of layer to be a perfect conductor with fixed field $B_0\hat{z}$ in it. As a result of $\nabla \cdot \vec{B} = 0$, B_z must vanish at the boundary. A consequence of this condition is that for $k \neq 0$ the horizontal velocity must vanish at the boundary. The can be understood in terms of line pinning. Any horizontal motion along the boundary would drag field lines with it, this motion is therefore not allowed as the field lines are stuck into the boundary and cannot move.

9 Polytrope Atmosphere

The problem of the stability of adiabatic motions ($q = 0$) in a polytrope atmosphere without magnetic field is well understood (e.g. [Lamb, 1925]). The adiabatic problem with magnetic field has been studied in the case of a complete polytrope, i.e. a polytrope that extends from $z = 0$ to $z = \infty$ (e.g. [Bogdan and Cally, 1997]) as well as for a generic polytrope layer (e.g. [Kaplan and Petrukhin, 1965]). The case without magnetic field but including heat transfer by conduction rather than Newton's law of cooling has been previously studied (e.g. [Lou, 1990]), as has the case of particular spatially dependent cooling times $\frac{1}{q}$ (e.g. [Macdonald and Mullen, 1997], [Spiegel, 1964]). There are few generalizations than can be obtained from these studies besides that the existence of unstable modes depends sensitively on the exact form of heat transfer that is used. In this study only the case where the cooling time is independent of depth is considered.

9.1 Non-Dimensionalization

For the polytrope background atmosphere it is convenient to non-dimensionalize the MHD equations by scaling the pressure, density and velocity perturbations by the background pressure, density, and adiabatic sound speed at the bottom of the layer. The length scale, z_s, is the distance from the height where the density is zero to the bottom of the layer. In all of the calculations for polytrope atmospheres shown here the layer is chosen to extend from $z = 0.1$ to $z = 1$. A result of this scaling is that the dimensionful frequency is given by $\frac{\omega c}{z_s}$ where c is the adiabatic sound speed. For the values assumed for the chromosphere this implies that the dimensionful frequency is $\frac{1}{400}$ Hz times the dimensionless frequency. The dimensionful horizontal wavenumber is given by $\frac{k}{z_s}$ where k is the dimensionless horizontal wave number.

9.2 Results

The results of numerical calculations, done in terms of the dimensionless variables, of the frequencies and growth rates for the fundamental acoustic and gravity modes are shown for a sample stable polytrope in Figure 1 and for an unstable polytrope in Figure 2, in both cases without magnetic field. The first figure shows that both the acoustic and gravity wave branches have zero growth rate as one would expect for a stable atmosphere. The gravity wave frequency goes to zero with decreasing wavenumber. The acoustic wave has a minimum frequency, this would be this case even in an infinitely thick layer though the actual minimum would be different. The second figure shows that the acoustic mode has zero growth rate and has a dispersion relation much like that for the stable atmosphere. The gravity mode is completely different, however. The gravity mode is not oscillatory in time at all, rather it grows exponentially.

If magnetic field is added to the problem, but the adiabaticity of the oscillations is maintained, a new set of features arises. The most striking feature of the magnetized polytrope, with boundary conditions $B_x = 0$ on the top and bottom of the layer, is that the magnetic field can make unstable polytropes behave in a stable way. Kaplan and Petrukhin ([Kaplan and Petrukhin, 1965]) argued that stabilization of the transverse mode by the magnetic field only occurs, in the adiabatic case, for $\frac{m+1}{m} < \gamma + \frac{B^2}{4\pi}$. This claim has not been investigated here. One example where the magnetic fields makes the layer stable is shown in Figure 3. The dotted line represents the more longitudinal mode and the solid the more transverse mode. Growth rates are not shown because they are all zero. The modes are identified by their $k = 0$ behavior. At $k = 0$ the equations separate into two decoupled eigenvalue problems, one for $U(z), \vec{B(z)}$ and one for $W(z), P(z)$. The first eigenvalue problem describes the vertically propagating Alfvén mode, which is transverse and causes no density or pressure perturbations. The second eigenvalue problem describes the vertically propagating acoustic mode, which is unaffected by the magnetic field. By starting at $k = 0$ with one of these solutions and moving up in k the behavior of each branch can be calculated.

The ability of the magnetic field to completely suppress convective instability is likely due to the reversibility of fluid parcel motions. For nonzero q as well as B_0 the unstable polytrope regains its instability. This is shown in Figure 4, again using the boundary conditions $B_x = 0$. This figure has a number of important features. As in previous plots the dashed line is for the longitudinal mode and the solid line for the transverse mode. The transverse mode is unstable for large k while the longitudinal mode is unstable for small k. this is a general feature that appears in many of the calculations shown in this paper. Another feature is that the modes come very close to crossing. As a result, resonance between the modes, despite their different physical characteristics, might be an important effect.

10 Isothermal

Adiabatic oscillations in an isothermal atmosphere in the absence of magnetic field have been studied and well understood (e.g. [Lamb, 1925]). The isothermal atmosphere is always convectively stable and supports three types of modes: the acoustic, Lamb, and the gravity waves. With the addition of the magnetic field there are only two fundamental modes, a mostly transverse and a mostly longitudinal mode.

The problem of the isothermal atmosphere with a constant magnetic field and Newton's law of cooling has been previously studied (e.g. [Babaev et al., 1995]). Babaev and coworkers considered the case of an infinite isothermal atmosphere and found analytic solutions to the linear stability problem. They found that all the modes were stable and discussed the damping rates in the limits of strong and weak magnetic field.

10.1 Non-Dimensionalisation

For the numerical calculations with isothermal atmosphere background, pressure and density perturbations are scaled by the values of background pressure and density at the middle of the layer. The velocity perturbations are scaled by the adiabatic sound speed. The scale height is used as the length scale. The calculations were done on a layer that extends from $z = -0.5$ to $z = 0.5$.

10.2 Results

The results of the calculations for the isothermal atmosphere are qualitatively similar to those of [Babaev et al., 1995] in that for $q > 0$ there are no unstable modes, as long as the boundary conditions on $\vec{B}$ are either $B_x = 0$ or $B_z = 0$. For $q < 0$, all modes are overstable. For the case $q < 0$ the fundamental longitudinal mode is most unstable for $k = 0$ while the transverse mode is most unstable for some finite k which depends on the parameters of the problem. The transverse mode is always marginally stable for $k = 0$ as the vertically propagating transverse mode does not produce temperature perturbations.

11 General Form of The Amplitude Equation

As a result of the unrealistic nature of the background atmospheres considered in this work no attempt was made at the derivation of a detailed amplitude equation. Instead the procedure of Fauve ([Fauve, 1991]) is followed in order to derive a schematic amplitude equation.

Consider the isothermal atmosphere with a background magnetic field. For $q = 0$ the fundamental longitudinal mode is marginally stable. For $q > 0$ the fundamental longitudinal mode is overstable with maximum growth at $k = 0$. Define $\epsilon = -q$ as the control parameter, which will be useful later. The amplitude $A(k, t)$ is then introduced through the equation:

$$\psi(x, z, t) = \int A(k, t)e^{ikx}\phi_k(z)dk \tag{8}$$

Where $\phi_k(z) = (\vec{v_k}(z), \vec{B_k}(z), P_k(z))$ describes the velocity, magnetic field, and density perturbation eigenfunctions associated with the fundamental longitudinal mode with horizontal wavenumber k. $\psi(x, z, t)$ then describes the behavior of these fields in time as well as space. It is important to note that the effect of modes other than the fundamental longitudinal mode is ignored. The other modes could be easily included. With the simplifying, but incorrect, assumption that $\phi_k(z) = \phi_0(z)$ Equation 8 becomes:

$$\psi(x, z, t) = \phi_0(z) \int A(k, t)e^{ikx}dk = \phi_0(z)A(x, t)$$

Thus if the amplitude $A(x,t)$ is known, the behavior of our physical fields can be found by multiplying by the eigenfunction for $k=0$.

Fauve explains that for the case of overstable modes with most unstable wave number k the lowest order non-linear equation for the time evolution of $A(k,t)$ is

$$\frac{\partial A(k,t)}{\partial t} = (i\omega(k) + \eta(k))A(k,t) + \alpha|A(k,t)|^2 A(k,t)$$

Here $\omega(k)$ is the mode frequency as a function of k and $\eta(k)$ is the growth rate. The Fourier transform of this equation gives the time dependence of the amplitude $A(x,t)$. The Fourier transform of the linear part is straightforward and gives a linear operator involving even power of ∂_x, as only even powers of k can appear by symmetry. The non-linear part is quite complicated. In order to proceed without working out the details, the Swift-Hohenberg argument ([Manneville, 1990]) is used to show that:

$$\int |A(k,t)|^2 A(k,t) e^{ikx} dk \approx |A(x,t)|^2 A(x,t)$$

With this assumption:

$$\frac{\partial A(x,t)}{\partial t} = L({\partial_x}^2)A(x,t) + \alpha|A(x,t)|^2 A(x,t)$$

12 Particular Form of the Amplitude Equation

In the previous section the general form of the amplitude equation was derived. In this section the linear operator $L({\partial_x}^2)$ is obtained from the Fourier transform of $i\omega(k) + \eta(k)$. In order facilitate this computation a function of the form $\epsilon\delta + \frac{\epsilon\kappa}{1+\beta k^2}$ is fit to the numerical results for the growth rate. In addition the frequency is assumed constant, equal to ω_0. This approximation could easily be relaxed. The calculation that needs to be done then is:

$$\int (i\omega_0 + \epsilon\delta + \frac{\epsilon\kappa}{1+\beta k^2})A(k,t) \exp ikxdk$$

which gives:

$$(i\omega_0 + \epsilon\delta)A(x,t) + \epsilon\kappa C(x,t)$$

with

$$(1-\beta\Delta)C(x,t) = A(x,t)$$

The complete form of the amplitude equation is then, (explicitly noting that α is order ϵ) :

$$\frac{\partial A(x,t)}{\partial t} = (i\omega_0 + \epsilon\delta)A(x,t) + \epsilon\kappa C(x,t) + \epsilon\alpha|A(x,t)|^2 A(x,t)$$

with

$$(1-\beta\Delta)C(x,t) = A(x,t)$$

13 Conclusions

The general eigenvalue problem for the growth rates of normal modes of 2D MHD with Newton's law of cooling in a layer with a variety of boundary conditions has been set up and solved for both isothermal and polytrope backgrounds with constant vertical magnetic field. The linear stability of the different modes depends strongly on the boundary conditions as well as the background atmosphere.

For the particular case of the boundary condition $B_z = 0$ and an isothermal background atmosphere it was found the modes are always stable for $q > 0$ and unstable for $q < 0$. An amplitude equation was developed for the overstable fundamental acoustic modes for the case $q < 0$. The amplitude equation was derived with the assumption that the only important branch is the fundamental acoustic branch and by making a Swift-Hohenberg type simplification of the Fourier Transform of the nonlinear term.

14 Future Work

This work is far from complete. Viscosity of any sort has not been included. It is likely that introducing a finite viscosity would reduce the growth rates or perhaps even damp the large horizontal wave number modes. This would introduce a new horizontal length scale into the problem and would alter the amplitude equation significantly. The expansions in the eigenfunctions may have to be done around finite k instead of $k = 0$.

The issue of resonance between modes was not treated in the work, and may be an important effect. For many choices of parameters there appears to be nearly a mode crossing between the longitudinal and transverse waves.

A vast improvement in the realism of the problem could be made through the use of a more accurate background atmosphere. For both the polytrope and the isothermal atmosphere the temperature either increases with pressure and density or remains constant. The temperature of the chromosphere decreases with increasing density and pressure.

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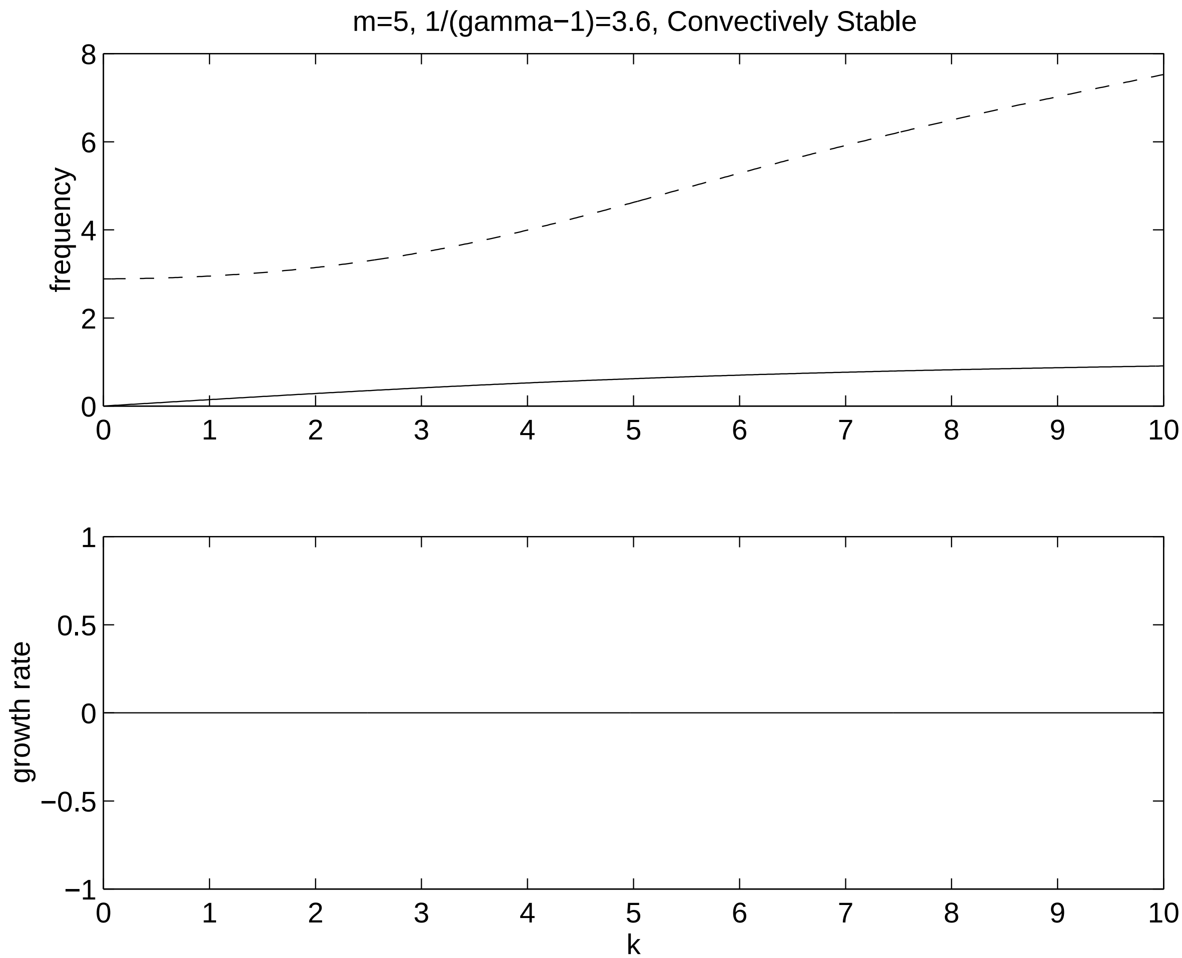

Figure 1: The dispersion relation for the fundamental acoustic, in the dashed line, and gravity mode, in the solid line, are shown for the $m = 5$, $\gamma = 1.28$, non-magnetic polytrope atmosphere. The layer extends from $z = .1$ to $z = 1$. The atmosphere is stable and the oscillations are adiabatic so the growth rate is zero for both branches.

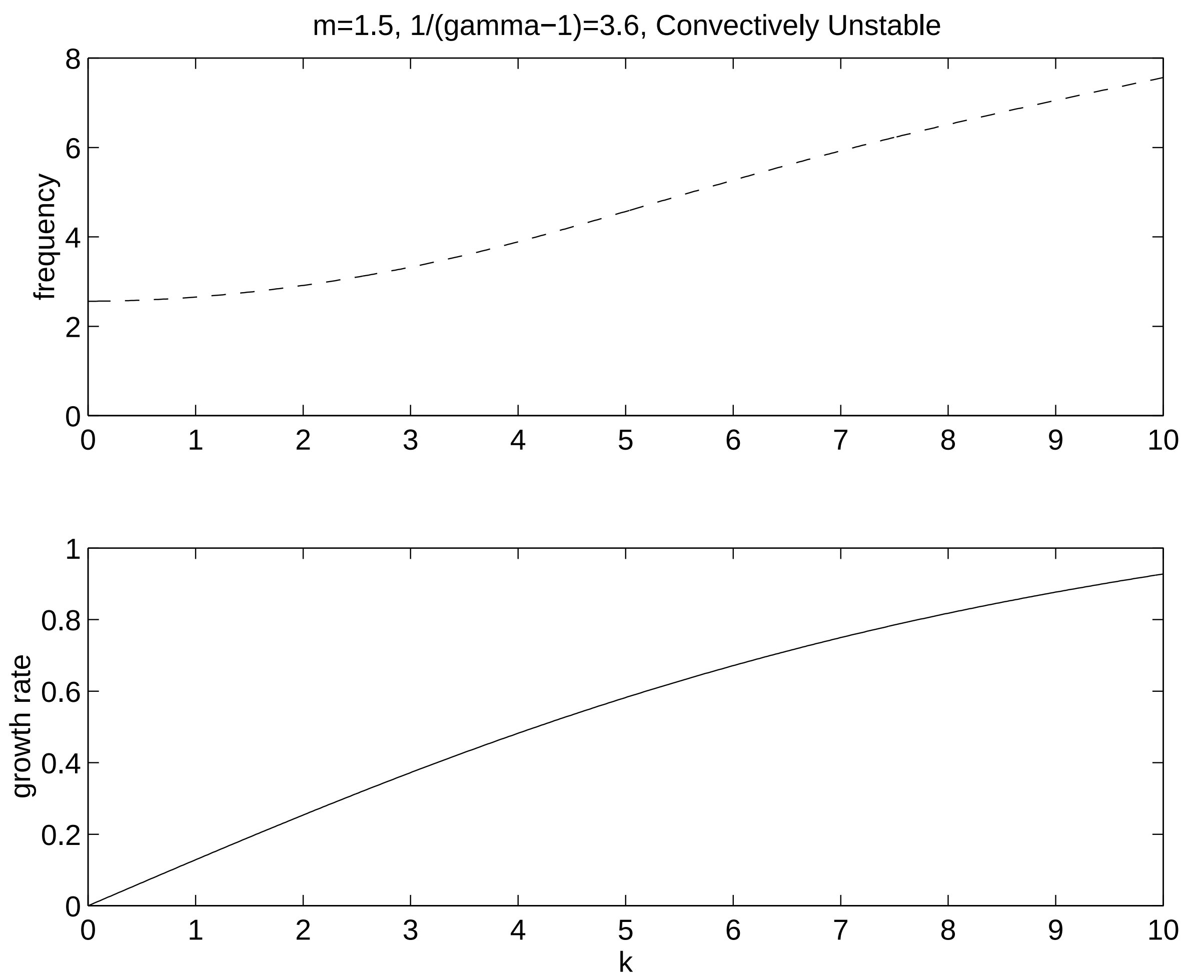

Figure 2: The dispersion relation for the fundamental acoustic, in the dashed line, and gravity mode, in the solid line, are shown for the $m = 1.5$, $\gamma = 1.28$, non-magnetic polytrope atmosphere. The layer extends from $z = .1$ to $z = 1$. The atmosphere is unstable and the oscillations are adiabatic so the growth rate is zero for the acoustic mode and positive for the gravity modes. The gravity modes have zero frequency and thus are not oscillatory in time

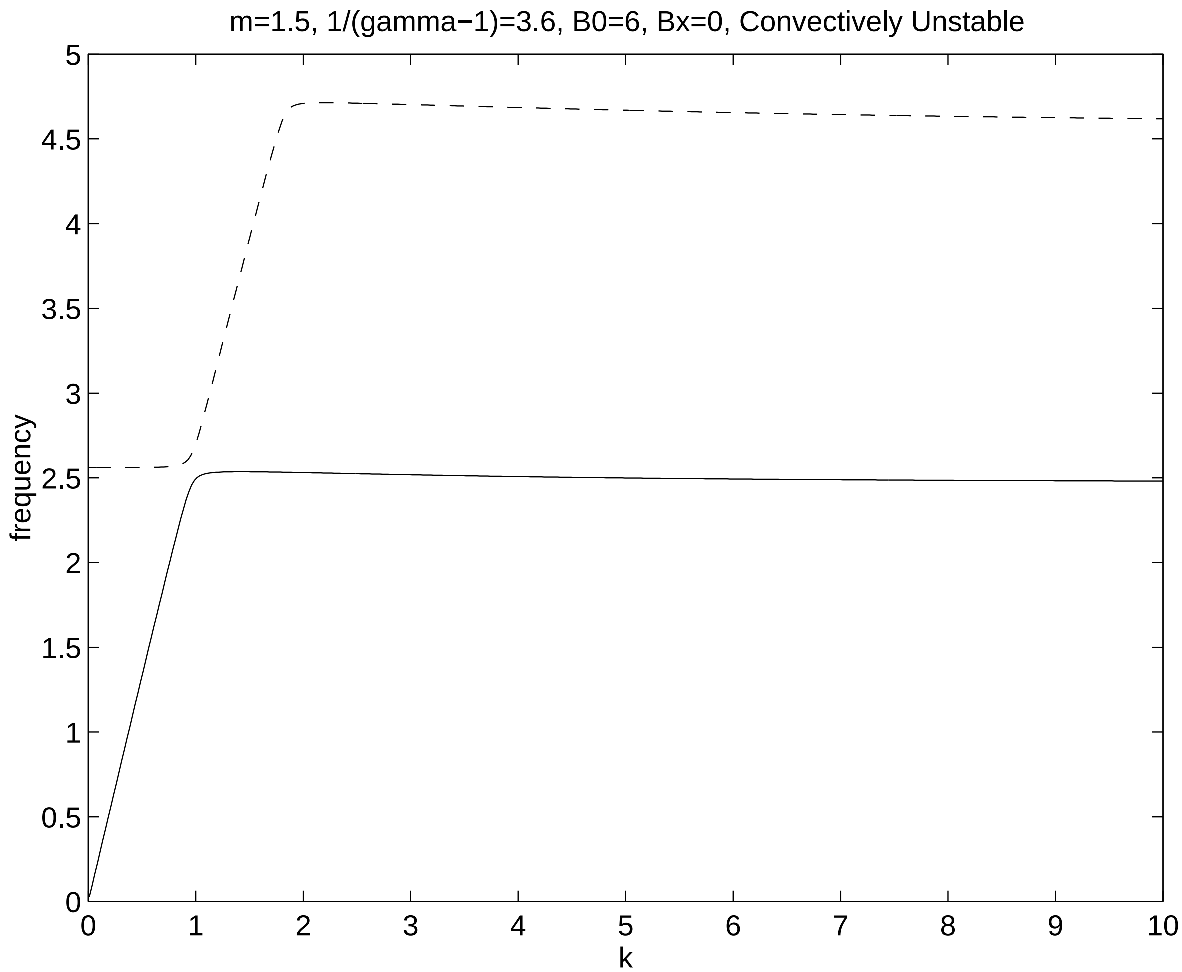

Figure 3: The dispersion relation for the fundamental longitudinal, in the dashed line, and transverse mode, in the solid line, are shown for the $m = 1.5$, $\gamma = 1.28$ magnetic polytrope. The layer extends from $z = .1$ to $z = 1$ and the magnetic boundary conditions are $B_x = 0$ on top and bottom. For this plot $q = 0$ so the oscillations are adiabatic.The growth rates are not shown because they are zero. Without magnetic field this atmosphere would be unstable, but the transverse modes are stabilized by the magnetic field.

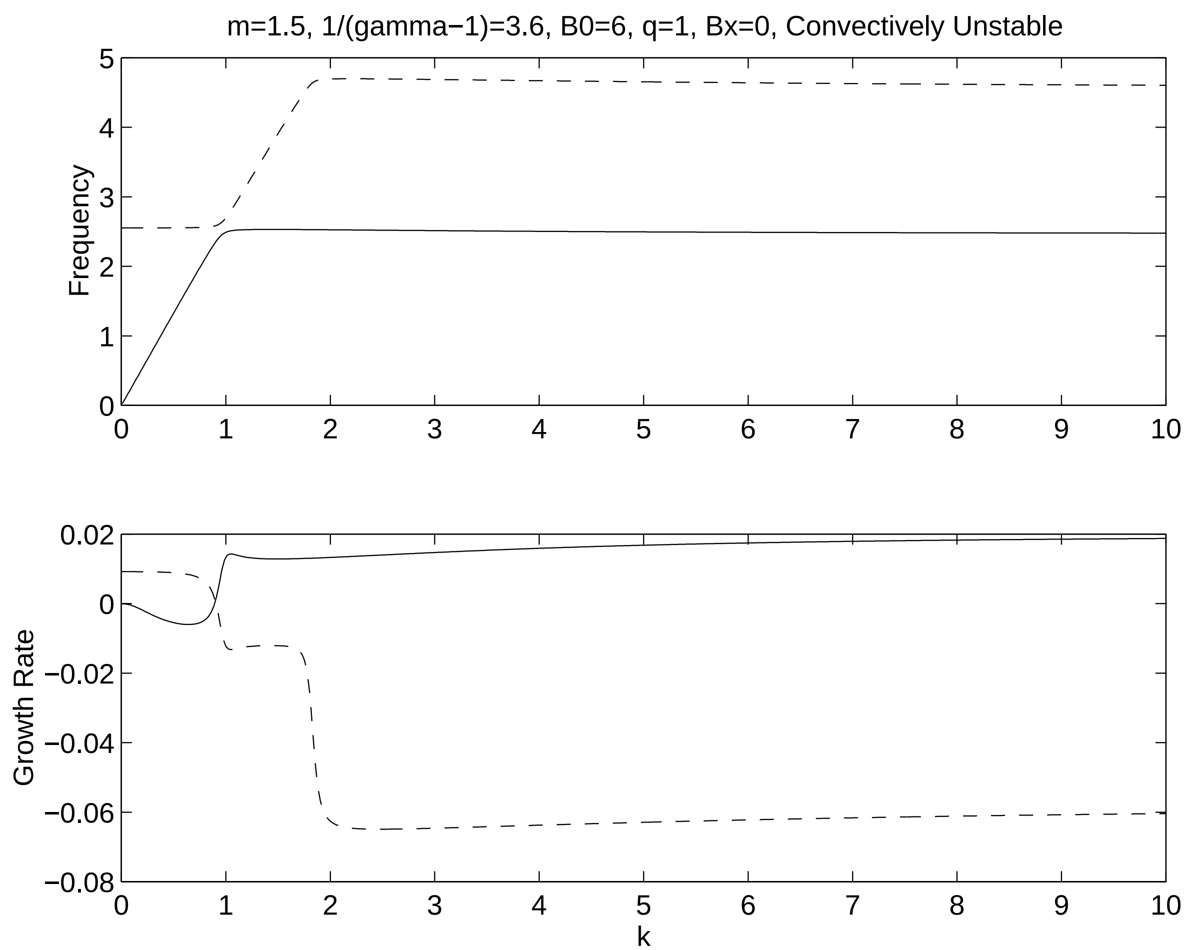

Figure 4: The dispersion relation for the fundamental longitudinal, in the dashed line, and transverse mode, in the solid line, are shown for the $m = 1.5$, $\gamma = 1.28$ magnetic polytrope. The layer extends from $z = .1$ to $z = 1$ and the magnetic boundary conditions are $B_x = 0$ on top and bottom. For this plot $q = 1$ so the oscillations are non-adiabatic. The finite cooling time makes fluid motions irreversible so the transverse modes are unstable. Even though this atmosphere is convectively unstable the transverse modes have an oscillatory component. In addition for the given B_0 and q the longitudinal mode is unstable for $k < 1$.

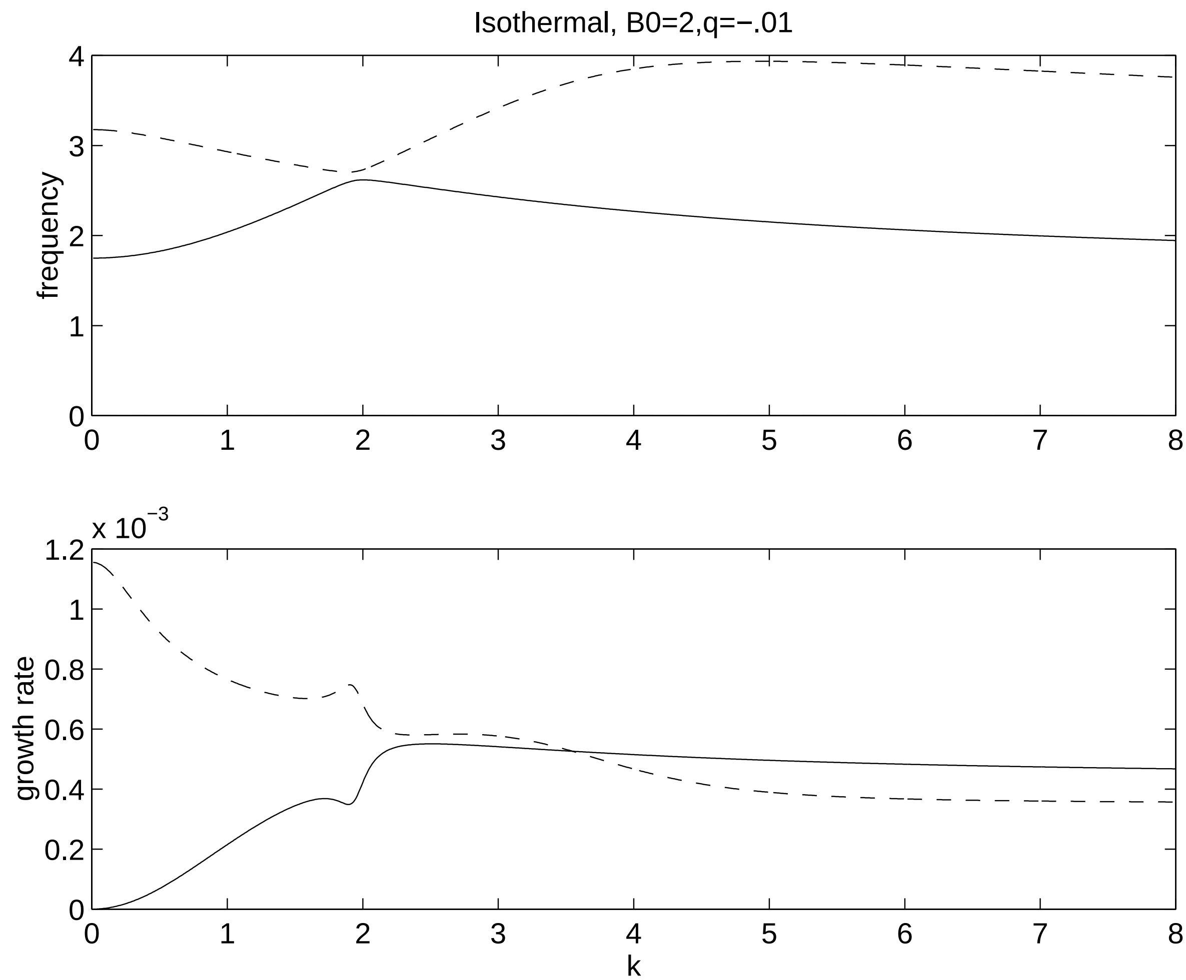

Figure 5: The dispersion relation for the fundamental longitudinal, in the dashed line, and transverse mode, in the solid line, are shown for the isothermal atmosphere. The layer extends from $z = -0.5$ to $z = 0.5$ and the magnetic boundary conditions are $B_z = 0$ on top and bottom. For this plot q is small and negative. The background magnetic field is 2 which corresponds roughly to real magnetic field of 60 gauss. The modes in this diagram almost cross, which results in small features in the growth rate near the close approach of the frequencies.

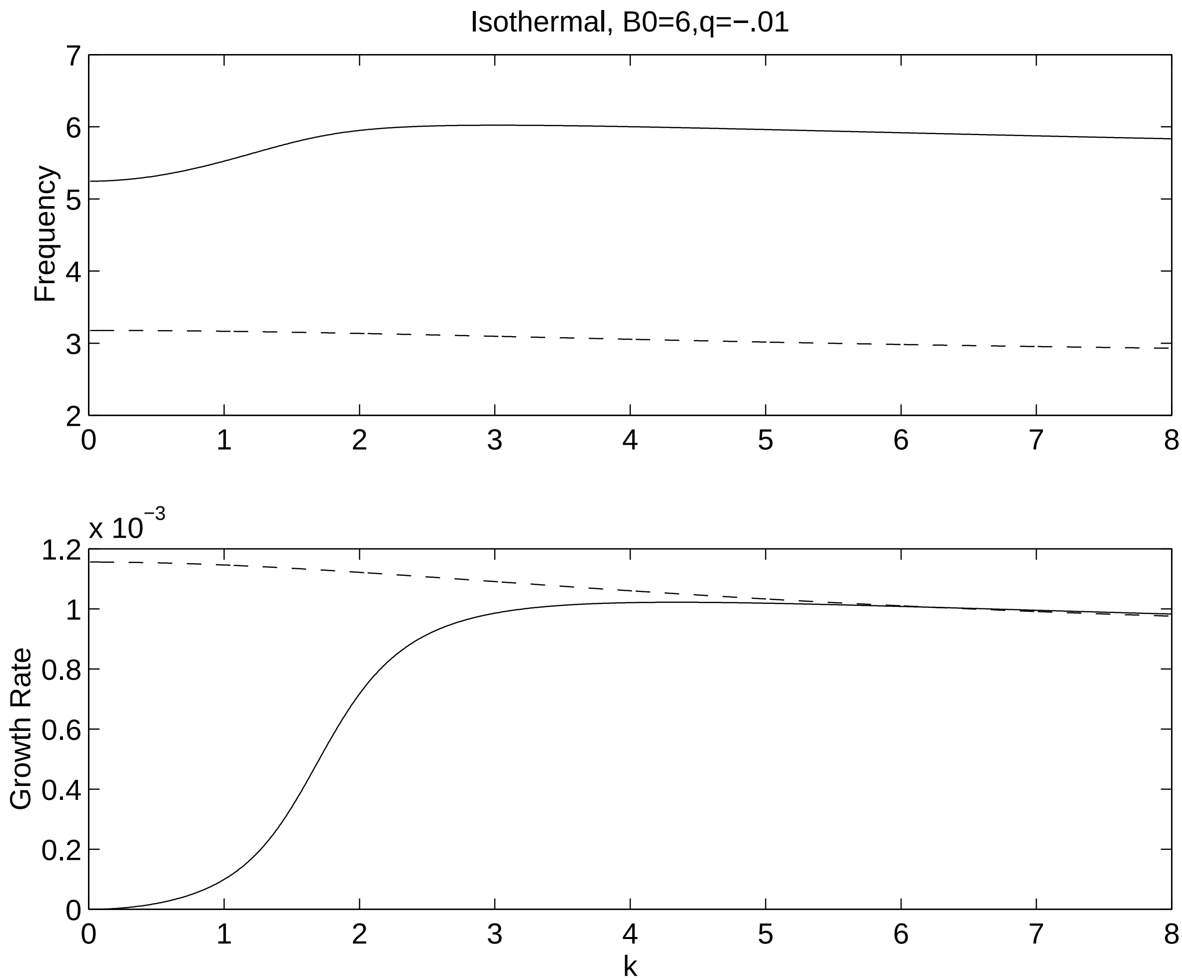

Figure 6: The dispersion relation for the fundamental longitudinal, in the dashed line, and transverse mode, in the solid line, are shown for the isothermal atmosphere. The layer extends from $z = -0.5$ to $z = 0.5$ and the magnetic boundary conditions are $B_z = 0$ on top and bottom. For this plot q is small and negative. The background magnetic field is 6 which corresponds roughly to real magnetic field of 180 gauss. In contrast with the previous diagram, for a weaker field, the modes don't come close to crossing. This is a result of the higher frequency for the $k = 0$ Alfvén mode.