Lecture 6

Energy Balance Models

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How I learned to stop worrying, and taught myself radiative transfer...

1 Simple energy balance models

The Earth and other planets in the solar system are heated by radiation from the sun. In turn, the planets reprocess the radiation and emit energy into space, leading to a global radiative balance which plays a key role in determining the planetary climate. As a result, a detailed treatment of radiative transfer is a necessary ingredient in models of climate dynamics.

For terrestrial planets (those with a solid crust), the influx of solar radiation must balance the outflow from the surface and atmosphere. It was known to Aristotle that the source of energy on earth is the sun, but it took 20th century quantum mechanics (specifically Planck and his understanding of black body radiation) to understand how the earth loses energy back to space. Based on the notion that radiation comes in discrete bundles of energy, quanta, $\Delta E = h\nu$, where h is Planck's constant ($6.6262 \times 10^{-34} Js$) and ν is the frequency of the radiation in Hertz, Planck explained the Stefan-Boltzmann law, which states that $E = \sigma T^4$, where E is the energy output of a black body, $\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$ is the Stefan-Boltzmann constant and T is the absolute temperature of the body. He expressed his result in form of the spectral energy density, $B_\nu(T)$, at frequency ν as

$$B_\nu(T) = \frac{2\pi h}{c^2} \frac{\nu^3}{\exp(h\nu/kT) - 1} [J m^{-2}], \tag{1}$$

where $c = 2.998 \times 10^8 m s^{-1}$ is the speed of light and $k = 1.37 \times 10^{-23} J K^{-1}$ is Boltzman's thermodynamic constant. (B_ν gives the energy emitted outward per unit area and time over the frequency interval $[\nu, \nu + d\nu]$; an integration over frequency gives the black-body radiation law.) Planck's theory also explained Wien's Law which states that the frequency at which the radiation from the black body is maximal is proportional to the absolute temperature of the body:

$$\nu_{max} = \left(5.879 \times 10^{10} \frac{Hz}{K}\right) T \tag{2}$$

The simplest radiative-convective model is zero-dimensional in space: the entire planet is given one temperature, T. Such simple models are the first line of defense against the onslaught of complexity present in climate problems. We consider terrestrial planets, like the Earth and Mars, that have solid surfaces, as opposed to gaseous planets like Jupiter.

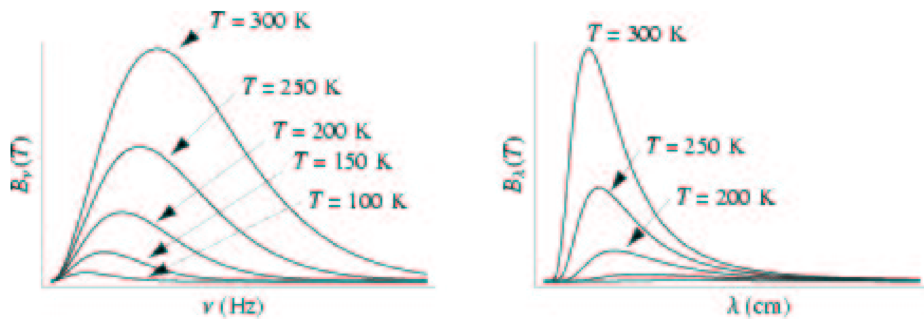

Figure 1: Emission spectra of black bodies at selected temperatures.

Standing on the shoulders of our scientific forefathers, we write a simple energy balance equation,

$$H_{sun}(1-\alpha) = \sigma T^4 \tag{3}$$

where H_{sun} is the radiation flux incident from the sun, averaged over time and over the planet's surface, α is the albedo, the fraction of the incident radiation that is reflected back into space, and hence never absorbed, and σ is the Stefan-Boltzmann constant. Thus we equate the net energy absorbed by the earth with the energy it loses to space as a black body. Given that H_{sun} is approximately 340 W/m^2, and taking $\alpha \approx 0.3$ (a crude estimate of the combined effect of sea, land, ice, clouds and so on), we find that $T = 255K$, much colder than the global average temperature we experience. Of course, we have here the grossest of models; the earth is basically treated as a metallic sphere. The more complicated models described next build on this model by incorporating the atmosphere. However, a key idea is clearly expressed in this model: the incoming radiation from the sun must be balanced by the outgoing radiation from earth.

2 Atmospheric structure

According to Wien's Law, a black body at 6000K, the temperature of the solar surface, emits the most radiation in the visible spectrum. We will assume that the atmosphere absorbs none of this incident radiation. This approximation is not too bad, the atmosphere actually absorbs less than 20% of incoming radiation. A fraction of the radiation incident on the earth, (1-α), is absorbed on the surface, causing the surface to warm. The warmed surface radiates energy back to space, primarily in the infra-red (IR) region of the spectrum, in accord with Wien's law for a black body near 300 K.

The atmosphere is, however, not transparent to IR radiation, and part of the outgoing radiation is absorbed; this upsets the energy balance, thereby increasing the surface temperature. How do we build such features into a radiative balance model? We start with the empirical data. The atmospheric radiation spectrum can be observed by looking directly upwards on a clear day with an infra-red interferometer. Spectra can also be taken from satellites, looking down, but one must then cancel out the radiation from the earth. Such spectra reveal a rough continuum interrupted by a immense number of molecular absorption lines. Fig. 2 shows the absorption spectrum for CO_2 in the IR region. Note that the

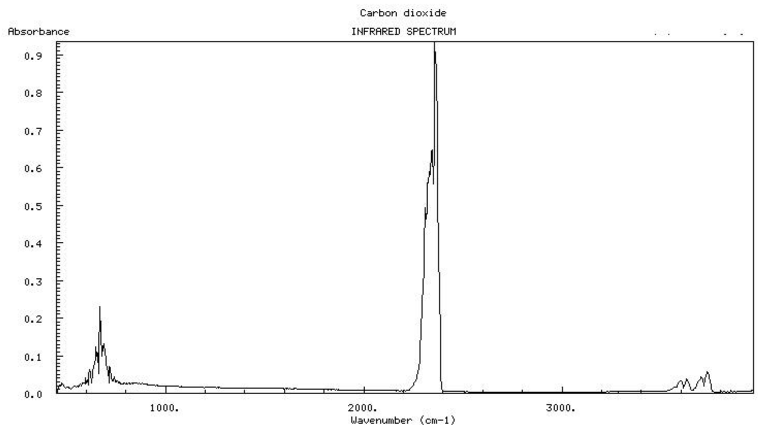

Figure 2: Absorption Spectrum of CO_2 (www.webbook.nist.gov).

wavenumber is the inverse of the wavelength measured in cm, and hence proportional to energy. It is through these spectral lines that certain molecules, such as CO_2, affect the atmospheric heat balance, and is how the greenhouse effect comes into play.

Now, according to quantum mechanics, molecules absorb light at discrete wavelengths. The so-called greenhouse gases are those that absorb in the IR, where photons are of the same energy as the translational, rotational, and bending modes of the molecules. A crude requisite to be a greenhouse molecules is to be polar and/or to support rotational or bending modes that create an oscillating dipole moment. Water and CFC's fall into the former category, while CO_2 and CH_4 satisfy the latter. An oscillating dipole moment is necessary to interact with the incoming electromagnetic radiation. Nonpolar molecules like N_2 and O_2 are transparent in the IR, but do play an indirect role in the greenhouse effect, as indicated below.

Under ideal conditions, the restriction of the absorption to narrow lines places severe limitations on the greenhouse effect: the absorption lines saturate quickly, so that the addition of more greenhouse gas does not result in a proportionate increase in absorption. In the atmosphere, however, conditions are far from ideal, and absorption lines can be broadened by molecular motion. As a result, the greenhouse effect is considerably extended by mechanisms that broaden the absorption lines. These mechanisms are:

1. Collisions with other molecules, which allow the absorber to take in a photon of smaller/larger energy, and transfer the energy difference to another molecule during the collision. This is how the greenhouse-neutral gases like N_2 and O_2 come into play. Because the collision frequency is proportional to the pressure of the gas, this broadening depends on the atmospheric pressure.

2. Doppler shifting of the absorbing molecule. If the molecule is moving towards/away from the source of radiation, it experiences a different frequency. Doppler broadening is a function of temperature (as temperature dictates the speed of the gas particles); the higher the temperature the broader the windows.

3. Ultimately, Heisenberg uncertainty puts a lower bound on the peak breadth.

The collisional effect dominates on earth. In fact, even though Mars has a pure CO_2 atmosphere, the warming effect is rather less than that of the CO_2 on earth due to our N_2 and O_2, even though the content of CO_2 on earth is far less. Collisional broadening decreases with air density, causing the absorption lines to narrow with height. At the top of our atmosphere the Doppler effect starts to dominate. However, the bulk of absorption takes place in the lower atmosphere, where the atmosphere is thickest, so that Doppler broadening can be neglected. In any case, the importance of line spectra in determining atmospheric absorption has the unappealing consequence that one needs a sophisticated treatment of radiative transfer in order to construct properly a model of the climatic energy balance.

To understand radiative transfer, we need more information about the atmosphere's vertical temperature structure. Roughly speaking the atmosphere is composed of the "troposphere" and "stratosphere." There is also a relatively shallow boundary layer just above the ground, which we will ignore. Inside the troposphere, the temperature decreases with height. The decline of temperature halts at a level referred to as the "tropopause," where the temperature is about $200K$, and then in the stratosphere above it, the temperature begins to increase. Some observed vertical temperature profiles for a location in the tropics are shown in figure 3. Crudely speaking, the reason why the globally averaged temperature is higher than the $255K$ expected from the simple energy balance argument above is that the effective "photosphere" of the Earth's emission into space is higher in the atmosphere than ground level. The earth must emit energy as a black body at $255K$ to maintain radiative balance with the sun. The surface, however, can be warmer as long as the radiation it loses is trapped by the colder atmosphere, which radiates at $255K$.

But why does temperature decline with height? The simplest argument, ignoring details such as the effect of water vapour, leads to what is called the "dry adiabat:" As a parcel of air rises off the ground, it expands as the pressure decreases. The gas does work as it expands, loses energy, and hence cools. We invoke the ideal gas approximation, which is quite accurate for the earth's atmosphere. On Venus, or in the protoclimate of Mars, however, increased pressures cause significant deviations from the ideal gas law. The potential temperature θ, a measure of the entropy of a gas, is defined by

$$\theta = T \left(\frac{p}{p_*} \right)^{-R/C_p}, \tag{4}$$

where p is the pressure, p_* some reference pressure (say 1 atmosphere), R the ideal gas constant, and C_p, the heat capacity of the gas at constant pressure. Quantum theory tells us that R/C_p is approximately 2/7 for a diatomic gas, and this approximation works well for our atmosphere.

If we assume constant entropy (constant θ), a "dry" atmosphere's temperature should be a function of pressure according to

$$T = \theta \left(\frac{p}{p_*} \right)^{R/C_p}. \tag{5}$$

Constant entropy is a good assumption, as the timescale on which fluid motions mix up the atmosphere, homogenizing scalar invariants such as entropy, is shorter than the timescale on which radiation warms the atmosphere.

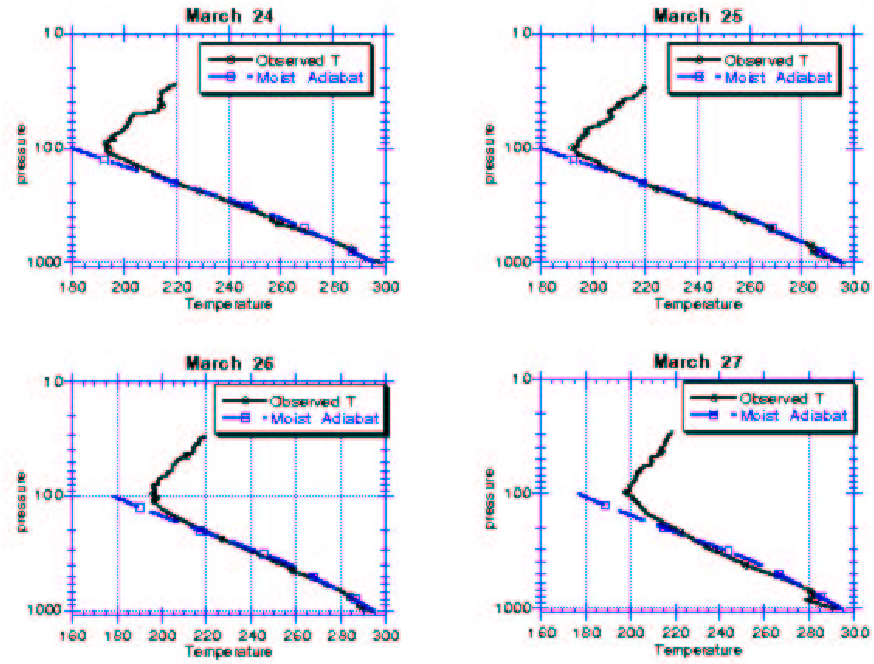

Figure 3: Air temperatures as function of altitude.

Equation (5) is the dry adiabat. The temperature of our atmosphere, however, does not fall as quickly as this relation predicts. The error stems from the fact that we have neglected the effect of water vapor, which can have a significant effect as a result of the release of latent heat. As the temperature cools with height, H_2O evaporated on the surface condenses, heating up the air and reducing the temperature gradient. Just 1 kg of water vapor releases 2.5 megaJoules when it condenses in the upper atmosphere. The moist adiabat is calculated by assuming that the air remains saturated with water vapour all the way up, that is, that there is no entrainment of dry air and thus the relative humidity is held constant at 100% once condensation starts. This gives a remarkably good fit for air in the tropics. This is shown in Fig. 3. Note we are only fitting the temperature in the troposphere. In the stratosphere absorption of solar radiation dominates, and the temperature deviates strongly from the moist adiabat.

The fit is quite remarkable in light of the fact that, outside the inter-tropical convergence zone (ITCZ) near the equator, the air in the tropics above the surface boundary layer is quite dry. The relative humidity is just 5-10%, "as dry as a desert." (See Fig. 4.) The Hadley circulation in the tropics explains why this dry air fits the moist adiabat, but we leave this until lecture 7. The mid latitudes do not follow the moist adiabat, but the temperature still falls with height up until the stratosphere.

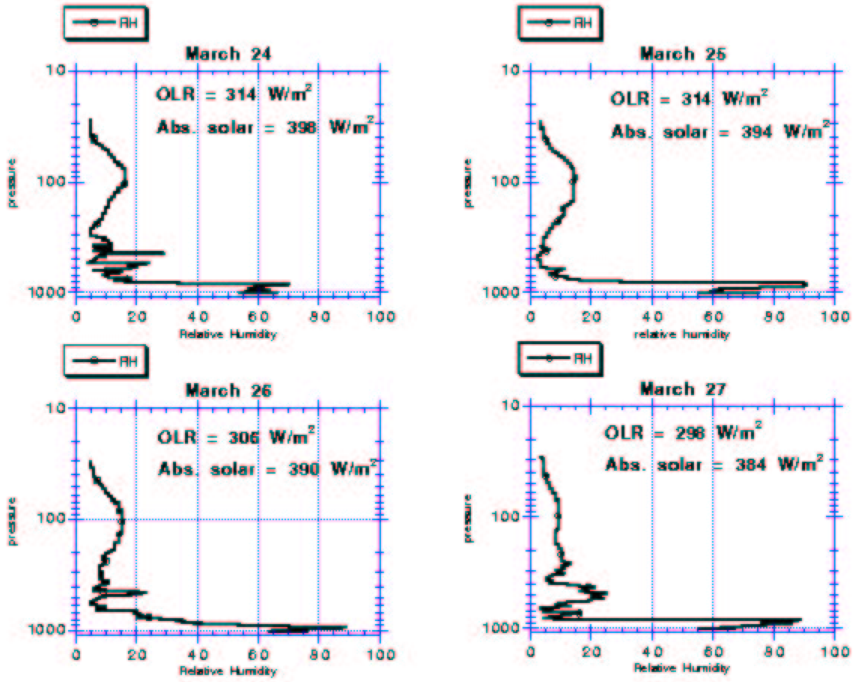

Figure 4: Relative humidity as a function of altitude.

3 The OLR curve

We now have the machinery needed to explain the greenhouse effect, which is most succinctly described in terms of the "OLR curve" – the dependence of the Outgoing Long-wave Radiation on surface temperature. This curve is also the key ingredient in a variety of toy climate models that will be described later.

In most places in the world the surface temperature is approxmately the same as the surface air temperature. Exceptions are deserts, where the surface can be 10 to 15 degrees warmer,and ice where the surface can be tens of degrees colder than the overlying air a few meters up. In simple models it is usually acceptable to equate surface temperature with surface air temperature.

Given the surface temperature, the thermal structure of the atmosphere above roughly follows the moist adiabat up to the tropopause. The stratosphere is ignored, as its overall effect is unimportant. The crucial step in constructing the climatic energy balance is then to determine the radiative transfer through the troposphere of the infra-red radiation leaving the surface. That transfer ultimately determines the total outgoing long-wave radiation (OLR), which must balance the incident solar energy flux. All told, this amounts to a complicated radiative transfer computation that approximates the collective effect of all the absorption and emission lines of every important molecule in the atmosphere. The computation involves thousands of lines of coding and a multitude of clever approximations to meet the computational efficiency requirements dictated by climate modeling.

The result of the calculations is the total OLR emitted by the earth as a function of the surface temperature; sample computations of this function are shown in figure 5. The key to understanding global warming is predicting how the addition of CO_2 and other greenhouse gases change the OLR, which in turn force a change in surface temperature in order to bring the outgoing energy into balance with the solar heating. The trickiest part is predicting how the relative humidity, RH, changes as the temperature increases. Unlike CO_2, the concentration of water vapor is highly dependent on temperature. Manabe proposed that the relative humidity remains constant as the temperature increases. This assumption is widely employed in conceptual climate models, but has never really been justified on the basis of first-principals physical arguments.

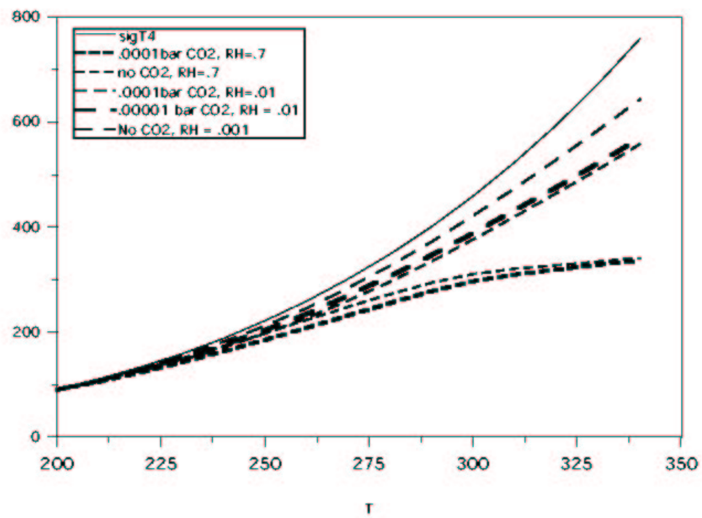

Figure 5: OLR as a function of surface temperature.

Fig. 5 shows the OLR as a function of surface temperature, for calculations based on different compositions of greenhouse gases and relative humidity. Recall that the OLR must be $340W/m^2$ to maintain radiative balance with the sun. To find the steady state surface temperature from the diagram, we draw a horizontal line at $340W/m^2$, and read off the surface temperature from where it intersects the OLR curve. The curves that do not reach $340W/m^2$ have no steady state and reveal a runaway greenhouse effect. Given an initially cool surface temperature, the OLR is below the incoming flux value, so the planet warms. The OLR thereby increases, and provided the OLR continues to increase with surface temperature, an equilibrium arrangement will eventually be struck. However, for the curves that flatten out, the OLR can never reach the input, so a runaway greenhouse ensues, at least if the physical input to the computation remains the same.

The specific mechanism for the runaway can be attributed largely to the effect of water vapour. Water vapour is a powerful greenhouse gas, there is plenty of it on earth, and the saturation pressure of water in air increases exponentially with temperature. The positive feedback of the runaway follows the route:

higher temperature $\Rightarrow$ more water vapor $\Rightarrow$ more global warming $\Rightarrow$ higher temperature

Once the feedback starts, the temperature keeps increasing without bound, and the OLR can never rise high enough to balance the incoming radiation. In such a situation, the energy continues to build up, leading to the melting of the ice caps, the evaporation of the oceans, and the eventual dissociation of H_2O. At this juncture, the hydrogen would escape into space, leaving an atmosphere much like that of Venus. Fortunately, the earth's atmosphere is much drier than that necessary for this doomsday scenario, at least for now.

4 A simple model of the greenhouse effect

According to the preceding arguments, the following phenomena are crucial to greenhouse effect on earth:

1. Decreasing atmospheric temperature with altitude.
2. Decreasing atmospheric pressure with altitude. This affects both the amount of greenhouse gasses and the peak broadening, and has its basis in the hydrostatic balance of the atmosphere, $dp/dz \approx -\rho g < 0$.
3. Presence of greenhouse gases in atmosphere, where the greenhouse gases are defined as the components of the atmosphere having absorption bands in the infra-red.

In this section, we build a simple model that illustrates the construction of the OLR, together with an implicit greenhouse effect.

The model consists of a plane-parallel atmosphere in which the pressure, $p(z)$, and temperature, $T(z)$, fall with height, z, from their values at ground level, p_0 and T_0 respectively. Since we are interested in only demonstrating how the ingredients add together to keep the earth surface warm, we will make some pretty crude idealizations. For one, the temperature and pressure fields will be specified by the piece-wise linear functions:

$$T(z) = T_0 \left(1 - \frac{z}{H}\right) \vartheta(H - z) \qquad p(z) = p_0 \left(1 - \frac{z}{H}\right) \vartheta(H - z), \tag{6}$$

where $\vartheta(x)$ is the step function; see figure 6.

The atmosphere is assumed to consist mainly of an inert gas except for small fraction of a greenhouse gas with a single absorption line at the frequency corresponding to maximum emission for a black body at temperature T_0 (see Wien's law in (2)).

$$\nu^* = \left(5.879 \times 10^{10} \frac{Hz}{K}\right) T_0. \tag{7}$$

The line is, however, broadened to a degree determined by the pressure. Let the width of the line be given by

$$\Delta\nu(z) = 2\delta_{\nu^*} p(z). \tag{8}$$

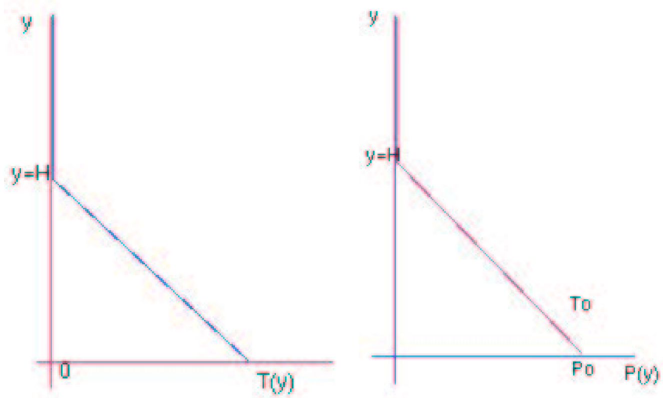

Figure 6: Temperature and pressure of a piece-wise linear toy model.

Thus the absorption depends on both frequency and height. We choose the simple model absorption coefficient shown in figure 7, in which constant absorption occurs within the broadened spectral line:

$$\begin{aligned} \epsilon_\nu(z) &= \epsilon_0 \qquad \text{if } \nu^* - \delta_{\nu^*} p(z) < \nu < \nu^* + \delta_{\nu^*} p(z) && (9) \\ &= 0 \qquad \text{otherwise.} && (10) \end{aligned}$$

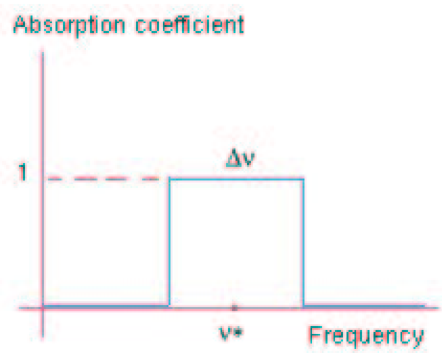

Figure 7: Model absorption coefficient for greenhouse gas.

The transfer of radiation is described by the intensity, $I_\nu(z)$, the upward energy flux density in the frequency interval $[\nu, \nu + d\nu]$, by a simple energy balance between layers, this satisfies

$$\frac{dI_\nu}{dz} = \frac{\epsilon_\nu(z)}{2} B_\nu\left(T(z)\right) - \epsilon_\nu(z) I_\nu(z). \tag{11}$$

That is, the change of intensity equals emission minus absorption, with the boundary condition,

$$I_\nu(0) = B_\nu(T_0), \tag{12}$$

which assures that the intensity at the base of the atmosphere is given by the emission from the ground.

Outside the greenhouse window on the ground, $[\nu^* - \Delta\nu(0), \nu^* + \Delta\nu(0)]$, the radiation is never affected by the greenhouse gas:

$$\frac{dI_\nu}{dz} = 0 \qquad \text{for } |\nu - \nu^*| > \Delta\nu, \tag{13}$$

and so $I_\nu(H) = B_\nu(T_0)$.

If the frequency lies inside the ground-level greenhouse window, radiation can be absorbed at certain heights. The decline in pressure with height narrows the window of absorbtion. For radiation of frequency ν we denote the height H_ν to be the point at which absorbtion of this frequency ceases. For our simplified model,

$$H_\nu = \left[1 - \frac{|\nu - \nu^*|}{\delta_{\nu^*} p_0}\right] H. \tag{14}$$

The governing equations for radiation flux then become

$$\begin{aligned} \frac{dI_\nu}{dz} &= -\epsilon_0 I_\nu + \frac{h\epsilon_0}{c^2}\frac{\nu^3}{\exp[h\nu/kT(z)] - 1}, \qquad \text{for } 0 < z < H_\nu \\ \frac{dI_\nu}{dz} &= 0 \qquad \text{for } H_\nu < z < H, \end{aligned} \tag{15}$$

with $I_\nu(z)$ continuous at $z = H_\nu$. Since $T(z)$ is an decreasing function of temperature, and $\exp[h\nu/kT(z)] > \exp[h\nu/kT(0)] \approx 16 \gg 1$, we may simplify still further:

$$\begin{aligned} \frac{dI_\nu}{dz} &= -\epsilon_0 I_\nu + \frac{h\epsilon_0}{c^2}\nu^3 \exp\left(-\frac{\chi_\nu}{1 - z/H}\right) \qquad \text{for } 0 < z < H_\nu \\ \frac{dI_\nu}{dz} &= 0 \qquad \text{for } H_\nu < z < H, \end{aligned} \tag{16}$$

where

$$\chi_\nu = \frac{h\nu}{kT_0}. \tag{17}$$

The radiation leaving the atmosphere, $I_\nu(H) = I_\nu(H_\nu)$, and is therefore given by the integral,

$$I_\nu(H) = e^{-\epsilon_0 H_\nu} B_\nu(T_0) + \frac{\epsilon_0 h\nu^3}{c^2}\int_0^{H_\nu} \exp\left(z - \epsilon_0 H_\nu - \frac{H\chi_\nu}{H - z}\right) dz). \tag{18}$$

The first term in this expression is the residual attenuated radiation from the earth's surface; the second term is the net radiation from the atmosphere, also suitably attenuated. Fig. 8 shows a representative spectrum as given by (18).

The expression (18) can be integrated over all frequencies to yield an OLR curve as a function of surface temperature; see figure 9. It can be seen that the OLR for the greenhouse system is always below the OLR curve for the black body, and so the surface temperature is always higher. The parameter values used are merely representative, and chosen chiefly to bring out the difference between the black-body law and our toy model. Real models of greenhouse effect incorporate absorption spectrum of all the greenhouse gases present in the atmosphere and use realistic stratifications for the temperature and pressure.

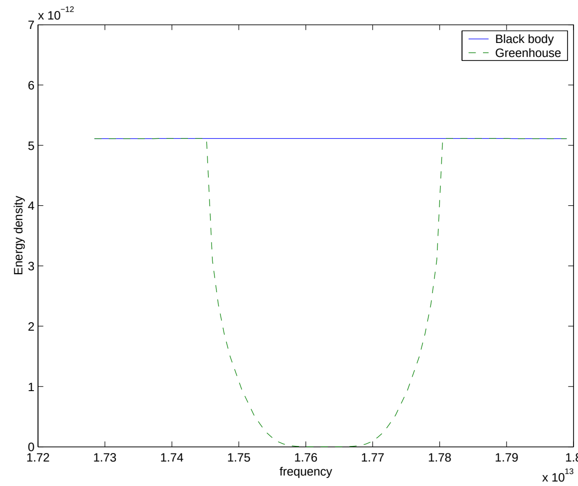

Figure 8: Intensity at the top of the atmosphere for the toy model.

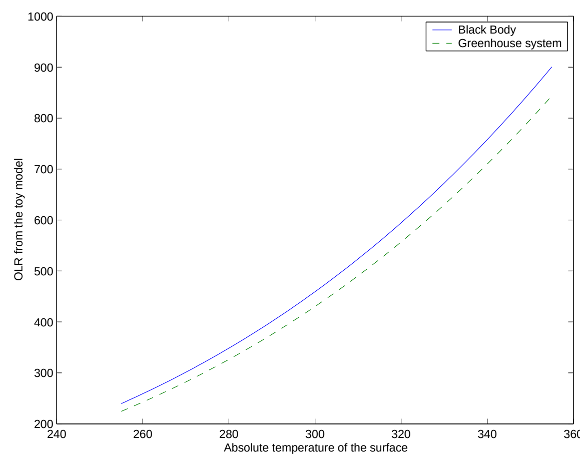

Figure 9: OLR curve for the toy model.

5 Radiative balance models

In general, the OLR curve as a function of surface temperature is the main ingredient in a radiative balance model. To compute this curve we follow the recipe outlined above, which requires as input the structure and composition of the atmosphere. With the balance of incoming and outgoing radiation, we then may infer surface temperatures. Often the procedure can be simplified by tabulating the OLR curve and fitting both its shape, the dependence on surface temperature, together with the dependence upon other significant variables, such as CO_2 concentration. One can then make relatively fast global warming calculations with the OLR curve and build conceptual climate models.

For example, consider CO_2 on earth. Once all the calculations are done, it turns out

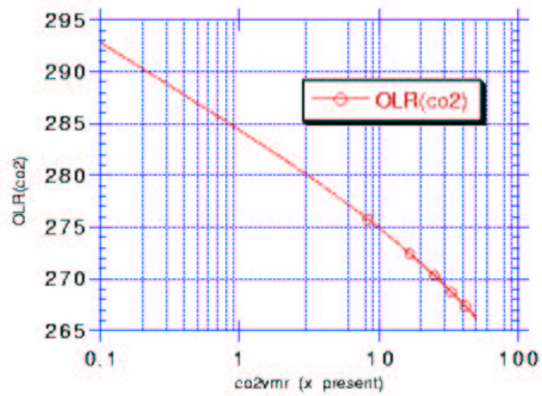

Figure 10: Effect of carbon dioxide concentration on OLR.

that the OLR response to CO_2 is roughly logarithmic, as shown in Fig. 10. Hence,

$$OLR(T, \ln CO_2) \approx OLR(T, \ln CO_{2*}) + D(T) \ln\left(\frac{CO_2}{CO_{2*}}\right), \tag{19}$$

where CO_{2*} is some reference value and the linear coefficient $D(T)$ may be a function of T. Here we see how important it is that absorption is limited to narrow bands – without saturation, the absorption would grow linearly with concentration, generating a much more pronounced sensitivity of the OLR to CO_2. The OLR exhibits the same kind of logarithmic dependence on concentration for most greenhouse gases.

From figure 10, we see that doubling the CO_2 concentration lowers the OLR by 4 W/m^2, assuming that the total water vapour content stays constant. In order to balance the incoming solar radiation, the surface temperature must then increase in order to raise the OLR. Based on the black-body curve, this amounts to an increase of about half a degree in surface temperature. If, however, the relative humidity (RH) remains the constant, more water vapour will enter the atmosphere, and the rise in temperature becomes as large as 2 $^\circ$ C.

Although, the effect of CO_2 on the OLR is significant, the effects of water vapour and clouds are even greater (figure 11). For example, doubling the RH from 10 to 20% causes a 10 W/m^2 shift in the OLR, equivalent to nearly tripling the CO_2 in the atmosphere. Clouds, on the other hand, constitute a very delicate climate variable. By adding the condensed substance, in this case water, the opportunities for molecular collisions are vastly increased, thereby leading to a significant broadening of the absorption lines. In this regard, clouds act like greenhouse gases and one expects a cloudy climate to have a lower OLR. The height and water mass of a cloud largely determine its radiative effect, because its temperature is given by the moist adiabat. Idealized computations suggest that the greenhouse warming effect of clouds is minimal at the surface, but at 10 km, they can lower the OLR by 150 W/m^2.

Clouds, however, also reflect the incoming short-wave radiation back into space, thereby increasing the albedo of the planet. Calculating cloud albedo is nontrivial exercise. For

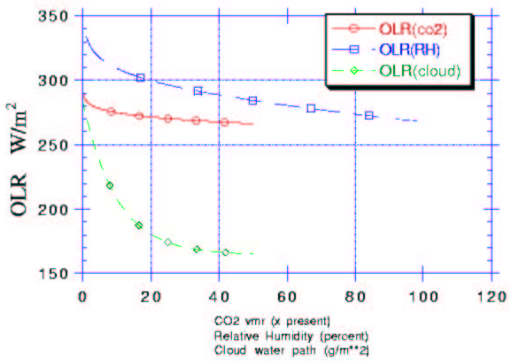

Figure 11: Relative effect of carbon dioxide, water vapor and clouds on OLR.

instance. the albedo dependends strongly on the size of the water droplets composing the cloud, so there is no simple correlation with the total mass of water. The average size of a cloud droplet is 10 micrometers. Changing from 8 to 12 micrometers, however, can result in albedo changes that are equivalent to a 20% reduction of the OLR.

Overall, clouds near the surface have a net cooling effect, while high clouds can be warming. Experimental evidence suggests that the net effect of clouds in the tropics is near zero, with a 100 W/m^2 jump in the OLR caused by increased absorption almost completely compensated by cloud albedo. In the extratropics (30° latitude and up) the net result is cooling, effectively lowering the OLR by approximately 15 W/m^2.

That the net effect of clouds must be computed from the close subtraction of two relatively large quantities makes the problem prone to severe error. The matter is complicated still further by the fact that cloud formation is also not particularly well understood: Nucleation sites are needed to begin condensation, creating a dependence on the concentration of airborne particles, such as dust and sulfate aerosols. (The dependence on sulfate aerosols is revealed in the higher rate of cloud cover over ship tracks, where aerosol pollutants are introduced to the atmosphere!) In the absence of nucleation centers, water vapor can become supersaturated in the atmosphere without forming any clouds. Suffice to say that clouds are the main uncertainty in climate modelling, from the toy system to the GCM. We need a theory of clouds if we are to make more progress in answering climate questions.

In summary, as the intensity of the OLR must balance incoming solar radiation, at least in the steady state climate, any change in the environment that lowers or raises the OLR will eventually manifest itself as a change in the surface temperature. The relationship between OLR and surface temperature is roughly linear over small variations, as shown in Fig. 12. All one needs therefore do is to calculate the linear coefficients and incorporate this into a model, as we do next.

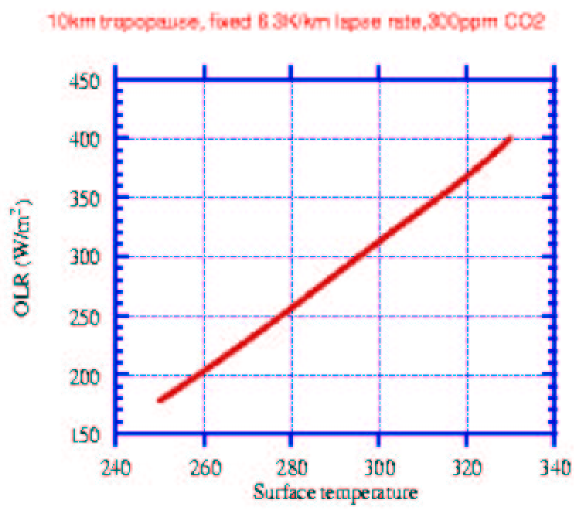

Figure 12: Dependence of OLR on Temperature

6 Ice albedo feedback

We are now ready to construct toy models; we illustrate with a simple model of the ice-albedo feedback effect. The albedo is quite complicated: Deserts have large albedos compared to forests and oceans, and fresh ice and snow is more reflective than older ice, on which dust and other debris may have collected. The average albedo of land and sea ice $\alpha_{ice} = 0.7$, whilst the average ice-free land/sea albedo $\alpha_0 = 0.1$. We consider just the average albedo of the entire surface of the Earth, $\alpha(T)$, as a function of the average surface temperature, T. The reflected light lies in the visible and so escapes immediately into space; the absorbed radiation is converted into the infra-red and percolates up through the atmosphere eventually providing the OLR. Thus the energy balance is

$$S_0 = \alpha(T)S_0 + OLR, \qquad \text{or} \qquad S(T) \equiv S_o[1 - \alpha(T)] = OLR, \tag{20}$$

where S_0 is the annual average incident radiation.

Let T_0 be the annual mean temperature necessary to sustain permafrost over the whole earth. Such a "snowball earth" may have existed in the neoprotozoic – about 600 million years ago (see lecture 10). We set T_1 to be the average temperature of a completely ice-free earth, as it was during the Eocene, about 55 million years ago, when lemurs roamed Spitzbergen and Crocodiles cavorted in the Hudson bay (as also discussed in lecture 10). We then make up a "plausible" function $f(T)$ to connect $\alpha(T)$ between its values for the permafrost and temperate earths:

$$
\begin{aligned}
\alpha(T) &= \alpha_{ice}, \quad T < T_0, && (21)\\
&= f(T), \quad T_0 < T < T_1 && (22)\\
&= \alpha_0, \quad T > T_0 && (23)
\end{aligned}
$$

The interpolating function $f(T)$ should have the features that it decrease sharply for T just above T_0, but more slowly for T near T_1. This is because near T_0, the albedo declines when the equatorial region becomes ice free; this region has both the largest area, per degree latitude, and experiences the strongest, annual average incident radiation. But approaching T_1, a slowly increasing albedo reflects the shrinking polar ice caps that have least area and weakest annual radiation.

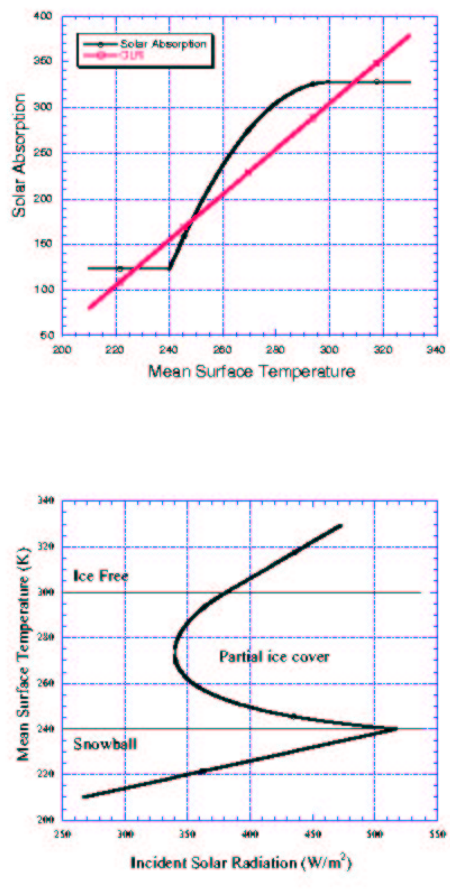

Figure 13: Top panel: absorbed solar radiation and the OLR. Lower panel: Bifurcation diagram.

In figure 13 we plot the OLR curve, which we approximate to be a straight line, and the net absorbed solar radiation $S_o(T)$. In this particular example, $T_o = 240K$, $T_1 = 300K$, and the greenhouse gas content gives an OLR intersecting the net incoming radiation curve at three points. This reveals three possible steady states. At the far left, we have a stable snowball earth, while at the far right we have the ice-free eocene climate. If we shift the OLR over to the left by, for example, decreasing the CO_2, the warmer steady state becomes a world like today, with a small polar ice cap. In the middle we have steady state characterized by large ice sheets; this state is an unstable equilibrium, as indicated as follows.

Away from the steady states the earth is not in balance with the incoming radiation. If we let M be the "thermal mass" of the earth, a fuzzy constant meant to characterize earth's heat capacity and the energy needed to melt ice sheets, and so on, then

$$M\frac{dT}{dt} = -(A + BT) + S_0[1 - \alpha(T)], \tag{24}$$

where the first term is the linearization of the OLR curve, and the latter is the net incident radiation. If the planet were placed immediately to the right of the middle steady state, which would correspond to adding a small warm anomaly, $\mathrm{d}T/\mathrm{dt}$ becomes positive, pushing the earth toward the warmer state. Conversely, if the planet were pushed leftward by a small cold deviation, the climate becomes pushed all the way to a snow ball. This is the mechanism of the large ice-sheet instability.

If we allow the incident solar flux to vary in some way, we can find transitions between the other two, stable equilibria; this is the content of the bifurcation diagram in Fig. 13. On the $x-$axis we plot the solar radiation constant, S_0, and on the $y-$axis, the global mean surface temperature, T. The curve plots all possible equilibria. The upper curve above 270 K represents the stable warm climate equilibrium. The curve below it, stopping at 240 K, shows the unstable, partial ice cover equilibria. Below 240 K, we have the snowball earth. We see that if the output of the sun falls below 340 W/m^2, the earth can fall from a stable warm climate into a snowball. Similarly, at 520 W/m^2, a stable snowball climate evaporates into a simmering tropical earth.

Figure 13 plots the equilibria against the incoming flux and so models the effect of secular variations in the solar constant, which could be brought about by, for example, the evolution of the sun. One could rather vary the greenhouse gas content of the atmosphere, instead of the solar flux, and obtain a similar bifurcation diagram. As increasing the CO_2 is approximately equivalent to raising the sun's radiation, the $x-$axis could equally well read $\ln CO_2$.

Finally, we close this lecture with a few remarks on some of the missing pieces in the climate puzzle. We have already mentioned that clouds are one of the main unknown ingredients to models. But we have also neglected vegetation, which can have a significant effect on, amongst other properties, the surface albedo. Unlike tundra, trees can offset the reflective effect of snow cover. Also, the thermal stratification of the atmosphere, a crucial part of the recipe for constructing the OLR, has been tacitly assumed to be given by the moist adiabat. This is certainly true for the tropics, but is not an accurate approximation for the mid-latitude atmosphere. Here, a significant role is played by fluid motion in determining the mean stratification (in particular, transport by turbulent eddies may play a key role), and there could be some, as yet unexplored, interesting interactions between the mid-latitude greenhouse effect and the atmospheric fluid dynamics.

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