Lecture 6

Bounds on Turbulent Transport

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1 Introduction to the Background Method

The background method is a mathematical technique for deriving rigorous bounds on the energy dissipation rate in Navier-Stokes and similar problems.

2 Momentum Transport Across a Shear Flow

To introduce the general idea of the background method we are going to consider the example of momentum transport across a shear layer. Consider a flow between two two *finite* plates. The bottom plate is at rest while the top plate is moving with speed U_*, see Fig. 1. We introduce Cartesian coordinates with unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ so that the upper plate is moving in the x direction, the lower plate is in the $y = 0$ plane, the upper plate is in the $y = h$ plane. The boundary conditions for this problem are periodic in x and z. The area of the plates is A; eventually we will take $A \to \infty$ to describe the problem in an infinite domain.

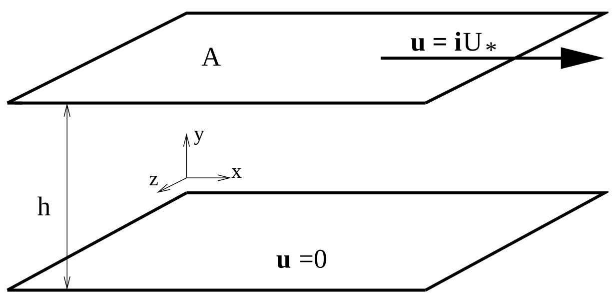

Figure 1: The Plane Couette flow.

The incompressible Navier-Stokes equations are

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = \nu \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0. \tag{1}$$

We start by asking the question: What is the vertical flux of horizontal momentum across the layer?

Dimensional analysis shows

$$
\begin{aligned}
\text{Momentum flux} &= \frac{\text{horizontal momentum}}{\text{time} \times \text{area}} \\
&= \frac{\text{horizontal force}}{\text{area}} = \frac{F}{A} \\
&= \text{wall shear stress} = \tau.
\end{aligned}
\tag{2}
$$

The dimensions of the quantities are

$$
[\tau] = \left[\frac{F}{A}\right] = \frac{ML}{T^2}\frac{1}{L^2} = M\frac{1}{T^2}\frac{1}{L}. \tag{3}
$$

The system parameters' dimensions are $[U_*] = L/T$, $[h] = L$, $[A] = L^2$, $[\nu] = L^2/T$, and $[\rho] = M/L^3$, where M denotes the dimension of mass. With these we can express the space-time averaged momentum flux as

$$
\tau = \rho h A \left(\frac{U_*}{h}\right)^2 \left(\frac{h}{A}\right) \times \beta\left(\frac{U_* h}{\nu}, \frac{A}{h^2}\right) = \rho U_*^2 \times \beta(Re, a)\,, \tag{4}
$$

where $Re = \frac{U_* h}{\nu}$ and $a = \frac{A}{h^2}$. We are interested in the function $\beta(Re, a)$, the dimesionless function of two dimensionless variables that fully describe the system.

An alternative version of the question could be posed in the following way. Define the time averaged dissipation rate per unit mass ϵ, which is also equal to the time averaged power input required to maintain the flow. Quantitatively

$$
\epsilon = \frac{FU_*}{\rho A h} = \frac{\tau}{\rho}\frac{U_*}{h}. \tag{5}
$$

Remembering the expression for τ in (4), we can write

$$
\epsilon = \frac{U_*^3}{h} \times \beta(Re, a)\,. \tag{6}
$$

Therefore a bound on the energy dissipation rate gives also a bound on the momentum transport.

In what follows we derive bounds on ϵ. First let us guess what we might find. For laminar flow we can expect the following dependence

$$
\tau \sim \rho\nu\frac{U_*}{h} \Rightarrow \beta \sim \frac{1}{Re}. \tag{7}
$$

For turbulent flow we do not expect dependence on the viscosity as we take $\nu \to 0$ (due to the cascade picture of energy transport across length scales), so we can write

$$\epsilon \sim U_*^2 \frac{U_*}{h} \Rightarrow \beta \sim Re^0. \tag{8}$$

Further we consider the exact stationary solution of the Navier-Stokes equations for plane Couette flow, see Fig. 1:

$$\mathbf{u}_{st} = \mathbf{i}\frac{U_*}{h}y, \quad p_{st} = \text{const}, \quad \tau_{st} = \rho\nu\frac{U_*}{h} \quad \beta_{st} = \frac{1}{Re}. \tag{9}$$

Now we set up a minimization problem. Let us multiply the Navier-Stokes equations (1) by $\rho\mathbf{u}$ and integrate over the volume. We have

$$\frac{d}{dt}\int \frac{1}{2}\rho|\mathbf{u}|^2\, d^3x = -\int \rho\nu|\nabla\mathbf{u}|^2\, d^3x + U_* \int_{\text{top plate}} \rho\nu\frac{\partial u_1}{\partial y}\, dx\, dz. \tag{10}$$

The only surface integral that survives after we integrate by parts is over the top plate of the volume. The term on the left-hand side of (10) is the kinetic energy of the fluid. The first term on the right-hand side is the instantaneous bulk dissipation rate, and the second term on the right-hand side is the input power (equal to $U_*F(t)$, where $F(t)$ is the instantaneous force applied to sustain the motion of the upper plate.) Suppose the kinetic energy behaves as $o(t)$ for large times, then its long time average vanishes, and we arrive at the following definition of the space-time averaged dissipation energy

$$\epsilon = \left\langle \nu|\nabla\mathbf{u}|^2 \right\rangle. \tag{11}$$

Therefore it is obvious that

$$\epsilon \geq \min_{\substack{\nabla\cdot\mathbf{u}=0 \\ \mathbf{u}|_{y=0}=0 \\ \mathbf{u}|_{y=h}=U_*}} \left\langle \nu|\nabla\mathbf{u}|^2 \right\rangle. \tag{12}$$

To put this in a variational frame, we consider the functional

$$\mathcal{F}[\mathbf{u}] = \int \left(\nu|\nabla\mathbf{u}|^2 - 2q(x)\nabla\cdot\mathbf{u}\right)\, d^3x, \tag{13}$$

where $q(x)$ is a Lagrange multiplier, enforcing the divergence-free constraint, which plays the role of a pressure. Variation of the above functional with respect to $\mathbf{u}$ and q and equating the results to zero yields

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\delta\mathcal{F}}{\delta\mathbf{u}} = -\nu\Delta\mathbf{u} + \nabla q, \\ 0 &= -\frac{1}{2}\frac{\delta\mathcal{F}}{\delta q} = \nabla\cdot\mathbf{u}. \end{aligned} \tag{14}$$

Thus we obtain the stationary Stokes equations as the Euler-Lagrange equations. The solution is given by (9) where p_{st} is substituted by q_{st}. We would like to know next if this solution really gives a minimum. The answer is affirmative and the proof follows from the sequence of realtions below.

Define

$$\mathbf{u} = \mathbf{u}_{st} + \mathbf{v}, \tag{15}$$

where $\mathbf{v}$ is the fluctuating deviation of $\mathbf{u}$ from plane Couette flow, satisfying

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v}|_{y=0} = \mathbf{v}|_{y=h} = 0. \tag{16}$$

The gradient of $\mathbf{u}$ is given by

$$\nabla \mathbf{u} = \mathbf{ij}\frac{U_*}{h} + \nabla \mathbf{v}, \tag{17}$$

from which follows

$$|\nabla \mathbf{u}|^2 = \frac{U_*^2}{h^2} + 2\frac{U_*}{h}\frac{\partial v_1}{\partial y} + |\nabla \mathbf{v}|^2. \tag{18}$$

If we find ϵ from formula (11), we obtain, after space-time averaging (noting that the cross term vanishes),

$$\epsilon = \nu\frac{U_*^2}{h^2} + \left\langle \nu|\nabla \mathbf{v}|^2 \right\rangle \geq \epsilon_{st}, \tag{19}$$

which shows that indeed the solution (9) gives a minimum of ϵ. Note that plane Couette flow is a solution for all Re and a so this lower bound is sharp. And sometimes this lower bound is also an upper bound.

We ask the question of when this solution is absolutely stable. Let us consider the equations for $\mathbf{v}$. They follow from the Navier-Stokes equations after the substitution of (15)

$$\begin{aligned}
&\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u}_{st} + \mathbf{u}_{st} \cdot \nabla \mathbf{v} + \frac{1}{\rho}\nabla p = \nu \Delta \mathbf{v}, \\
&\nabla \cdot \mathbf{v} = 0.
\end{aligned} \tag{20}$$

We will prove the following statement: Plane Couette flow is absolutely stable–and hence the unique time asymptotic flow–for sufficiently low Re. To see that, multiply the first of Eqs. (20) by $\mathbf{v}$ and integrate over the volume

$$\frac{d}{dt}\frac{1}{2}\int |\mathbf{v}|^2\, d^3x = -\int \left(\nu|\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla \mathbf{u}_{st})_{\text{sym}} \cdot \mathbf{v}\right)\, d^3x \tag{21}$$

and $(\nabla \mathbf{u}_{st})_{\rm sym} = 1/2\left((\nabla \mathbf{u}_{st}) + (\nabla \mathbf{u}_{st})^{tr}\right)$. On the left-hand side of this equation is time derivative of the perturbation energy $\mathcal{E}(t) = \int |\mathbf{v}|^2\, d^3x$. On the right-hand side we have a quadratic in the perturbation $\mathbf{v}$ form.

We note the inequality (sometimes referred to as Poincaré 's inequality)

$$\int |\nabla \mathbf{v}|^2 \geq \frac{\pi^2}{h^2} \int |\mathbf{v}|^2. \tag{22}$$

Then we calculate

$$\begin{aligned} &\left| \int \mathbf{v} \cdot (\nabla \mathbf{u}_{st})_{\rm sym} \cdot \mathbf{v}\, d^3x \right| = \left| \int \frac{U_*}{h} v_1 v_2\, d^3x \right| \leq \\ &\frac{U_*}{h} \int \frac{1}{2}(v_1^2 + v_2^2)\, d^3x \leq \frac{U_*}{2h} \int |\mathbf{v}|^2\, d^3x. \end{aligned} \tag{23}$$

So for the perturbation energy we have

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &\leq -2\left(\nu \frac{\pi^2}{h^2} - \frac{U_*}{2h}\right) \mathcal{E}(t) = \\ &= -\frac{\nu}{h^2}\left(2\pi^2 - Re\right) \mathcal{E}(t), \end{aligned} \tag{24}$$

and finally using Gronwall's lemma we have

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{-\frac{\nu}{h^2}\left(2\pi^2 - Re\right)t} \to 0 \quad \text{if} \quad Re < 2\pi^2 \approx 20, \tag{25}$$

which proves the assertion.

A more precise calculation shows that the critical value of the Reynolds number for this kind of energy stability is $Re_E \approx 82.6$. To see how we can get a more precise value consider

$$\frac{d\mathcal{E}(t)}{dt} = -2\left[\frac{\int \left(\nu |\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla \mathbf{u}_{st})_{\rm sym} \cdot \mathbf{v}\right)\, d^3x}{\int |\mathbf{v}|^2\, d^3x}\right] \mathcal{E}(t) \leq -2\lambda_0\, \mathcal{E}(t), \tag{26}$$

where

$$\lambda_0 = \min_{\substack{\nabla \cdot \mathbf{v} = 0 \\ \int |\mathbf{v}|^2\, d^3x = 1 \\ \mathbf{v}|_{y=0} = \mathbf{v}|_{y=h} = 0}} \int \left(\nu |\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla \mathbf{u}_{st}) \cdot \mathbf{v}\right)\, d^3x. \tag{27}$$

Define the functional

$$\mathcal{F}[\mathbf{v}] = \int \left[\nu |\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla \mathbf{u}_{st}) \cdot \mathbf{v} - 2p(x) \nabla \cdot \mathbf{v} - \lambda \left(|\mathbf{v}|^2 - \frac{1}{Ah}\right)\right] d^3x. \tag{28}$$

Upon variation with respect to $\mathbf{v}$ we obtain the eigenvalue problem

$$\lambda \mathbf{v} = -\nu \Delta \mathbf{v} + \nabla p + (\nabla \mathbf{u}_{st}) \cdot \mathbf{v}. \tag{29}$$

The region in the *Re*–*a* phase plane where the lowes eigenvalue is positive defines the parameter region of energy stability of plane Couette flow. In applications to bound other solutions we need to generalize the method described in this section and this is done next.

3 "Background" Method

In the previous section we saw that the stationary solution (9) only exists as an absolutely stable solution for sufficiently low Reynolds numbers. For high Reynolds numbers we can not use it to put an upper bound on the energy dissipation rate. However, we are going to present a more general technique that mimics to some extent what we did in the previous section.

Decompose a general solution of the Navier-Stokes equation as

$$\mathbf{u} = \mathbf{i}U(y) + \mathbf{v}(x,t). \tag{30}$$

We call the vector field $\mathbf{i}U(y)$ a "background" field. The other part of the decomposition is a "fluctuating" field. The purpose of the background [1] field is to "absorb" the boundary conditions, whereas the fluctuating part satisfies homogeneous boundary conditions:

$$U(0) = 0, \quad U(h) = U_*, \quad \mathbf{v}|_{y=0} = \mathbf{v}|_{y=h} = 0. \tag{31}$$

Next, from (1) and (30) we derive

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + U(y)\partial_x \mathbf{v} + \mathbf{i}\, v_2 U'(y) + \nabla p = \nu \Delta \mathbf{v} + \mathbf{i}\, \nu U''(y) \tag{32}$$

and for the fluctuation energy evolution

$$\frac{d}{dt}\frac{1}{2}\int |\mathbf{v}|^2\, d^3x = -\nu \int |\nabla \mathbf{v}|^2\, d^3x - \int U'(y) v_1 v_2\, d^3x - \nu \int U'(y)\frac{\partial v_1}{\partial y}\, d^3x. \tag{33}$$

As before, note that $\nabla \mathbf{u} = \nabla \mathbf{v} + \mathbf{j}\mathbf{i}U'(y)$ and derive

$$\frac{1}{2}\nu \int |\nabla \mathbf{u}|^2\, d^3x = \frac{1}{2}\nu \int |\nabla \mathbf{v}|^2\, d^3x + \frac{A}{2}\nu \int_0^h u'(y)^2\, dy + \nu \int U'(y)\frac{\partial v_1}{\partial y}\, d^3x. \tag{34}$$

Adding (33) and (34) we get

[1] From now on we drop the quotes on the words background and fluctuating but we should keep in mind that the background field is not (necessarily) a mean flow.

$$\frac{d}{dt}\int |\mathbf{v}|^2\, d^3x + \nu \int |\nabla \mathbf{u}|^2\, d^3x = -\int \left(\nu|\nabla \mathbf{v}|^2 + 2U'(y)v_1v_2\right)\, d^3x + A\nu \int_0^h U'(y)^2\, dy. \quad (35)$$

The terms in the formula above can be identified as follows. The second term on the left-hand side is the total instantaneous dissipation rate, the second term on the right-hand side is the dissipation rate in the background flow, and the first term on the right-hand side is a quadratic form which we denote by $Q_u\{\mathbf{v}\}$.

The key point is: If we can find a background profile $U(y)$ so that $Q_U\{\mathbf{v}\} > 0$, i.e., so that $Q_U\{\mathbf{v}\} \geq c\int |\mathbf{v}|^2\, d^3x$ with $c > 0$, then

a) We are convinced that the kinetic energy is uniformly bounded in time (even as $t \to \infty$) because then

$$\frac{d}{dt}\int |\mathbf{v}|^2\, d^3x \leq -c\int |\mathbf{v}|^2\, d^3x + A\nu \int_0^h U'(y)^2\, dy, \quad (36)$$

from where after integrating, we deduce

$$\int |\mathbf{v}|^2\, d^3x \leq e^{-ct}\int |\mathbf{v}(x,0)|^2\, d^3x + \frac{1}{c}\left(1 - e^{-ct}\right) A\nu \int_0^h U'(y)^2\, dy; \quad (37)$$

b) The background flow produces an upper bound on ϵ, for then the time averaged equation (35) gives

$$\epsilon = \left\langle \nu|\nabla \mathbf{u}|^2 \right\rangle \leq \frac{1}{h}\int_0^h U'(y)^2\, dy. \quad (38)$$

3.1 Trial background Method

Lets take the background profile to be the piecewise-linear velocity profile given by the one shown in the Fig. 2. We can make the following estimate: using the fundamental theorem of calculus and the Cauchy-Schwarz inequality:

$$|v_i| = \left|\int_0^y \frac{\partial v_i}{\partial y}(y')dy'\right| = \left|\int_0^y 1\cdot \frac{\partial v_i}{\partial y}(y')dy'\right| \leq \sqrt{y}\left|\int_0^y \left(\frac{\partial v_i}{\partial y}(y')\right)^2 dy'\right|^{1/2}. \quad (39)$$

This implies

$$\left|\int_0^h U'(y)v_1v_2dx^3\right| \leq \frac{U_*}{2\delta}\int dxdz\int_0^\delta y\left(\int_0^{h/2}\left(\frac{\partial v_1}{\partial y}\right)^2 dy'\right)^{1/2}\left(\int_0^{h/2}\left(\frac{\partial v_2}{\partial y}\right)^2 dy'\right)^{1/2} dy +$$

$$\frac{U_*}{2\delta}\int dxdz\int_{h-\delta}^h (h-y)\left(\int_{h/2}^h\left(\frac{\partial v_1}{\partial y}\right)^2 dy'\right)^{1/2}\left(\int_{h/2}^h\left(\frac{\partial v_2}{\partial y}\right)^2 dy'\right)^{1/2} dy. \quad (40)$$

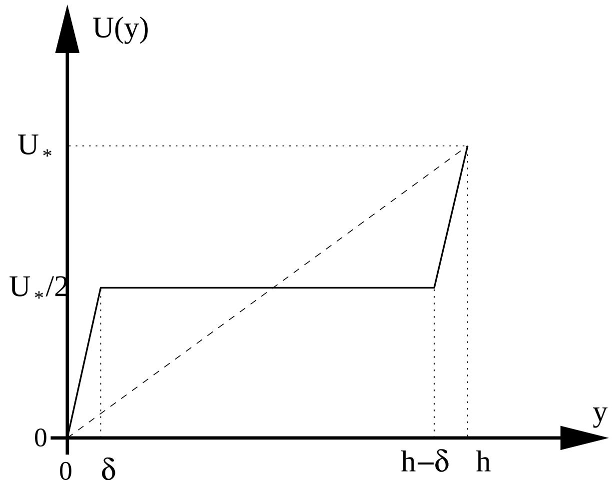

Figure 2: Trial function for the background profile.

Including all the other terms in $|\nabla \mathbf{v}|^2$ we obtain

$$\left|\int U'(y) v_1 v_2 dx^3\right| \leq \frac{U_*}{2\delta}\frac{\delta^2}{2}\frac{1}{2}\int |\nabla \mathbf{v}|^2 dx^3 = \frac{U_*\delta}{8}\int |\nabla \mathbf{v}|^2 dx^3. \tag{41}$$

This implies for Q :

$$Q_U\{\mathbf{v}\} = \int \left(\nu|\nabla \mathbf{v}|^2 + 2U' v_1 v_2\right) dx^3 \geq \int \nu |\nabla \mathbf{v}|^2 dx^3 - 2\left|\int U' v_1 v_2 dx^3\right|$$

$$\geq \int \nu|\nabla \mathbf{v}|^2 dx^3 - \frac{U_*\delta}{4}\int |\nabla \mathbf{v}|^2 dx^3 \geq \left(\nu - \frac{U_*\delta}{4}\right)\int |\nabla \mathbf{v}|^2 dx^3 \geq \frac{\pi^2}{h^2}\left(\nu - \frac{U_*\delta}{4}\right)\int |\mathbf{v}|^2 dx^3. \tag{42}$$

So $Q_U\{v\} \geq 0$ if we choose $\delta \leq 4\nu/U_* = 4h/Re$. This is the maximum value of δ that our estimates allow us to use, and gives a bound on the maximum possible energy dissipation rate for the set of background functions U that we have chosen. Using this value of δ we obtain

$$\epsilon \leq \frac{\nu}{h}\int_0^h U'(y)^2 dy = \frac{1}{8}\frac{U_*^3}{h} \quad \Rightarrow \quad \beta \leq \frac{1}{8} \tag{43}$$

4 Variational Problem for Optimal Background

We can pose the following question: What is the optimal background velocity profile that gives the smallest possible bound

$$\epsilon \leq \min_U \left\{\frac{1}{h}\int_0^h \nu U'(y)^2 dy\right\}. \tag{44}$$

under the constraints $Q_U\{\mathbf{v}\} \geq 0$, $U(0) = 0$ and $U(h) = U_*$, where

$$Q_U\{\mathbf{v}\} = \int \left[\nu|\nabla\mathbf{v}|^2 + 2v_1v_2U'(y)\right]dx^3. \tag{45}$$

The constrain $Q_U\{\mathbf{v}\} \geq 0$ is equivalent to the spectral constraint $\lambda_U \geq 0$ where

$$\lambda_U = \min_{\mathbf{v}} \frac{\int \left[\nu|\nabla\mathbf{v}|^2 + 2v_1v_2U'(y)\right]dx^3}{\int |\mathbf{v}|^2 dx^3} \tag{46}$$

under the constraints $\nabla\cdot\mathbf{v} = 0$ and $\mathbf{v} = 0$ at $y = 0, h$. λ is equal to the lowest eigenvalue of

$$\lambda\mathbf{v} = -\nu\Delta\mathbf{v} + \nabla p + \mathbf{i}U'(y)v_2 + \mathbf{j}U'(y)v_1 \tag{47}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{48}$$

We can then substitute the $Q_U(\mathbf{v}) \geq 0$ in (44) constraint with $\lambda_U \geq 0$

4.1 The Geometry of the Spectral Constraint

Let $U'(y) = U_*/h + \phi(y)$ so that $\int_0^h \phi(y)dy = 0$. Then equation (38) can be written as

$$\epsilon \leq \min_{\phi} \left[\nu\frac{U_*^2}{h_2} + \frac{\nu}{h}\int_0^h \phi(y)^2 dy\right] \tag{49}$$

with the constraints $\int_0^h \phi dy = 0$ and $\lambda_\phi \geq 0$ where we replace the label U with ϕ in the spectral constraint. There is one remark we want to make for the above minimization problem:

The set of functions $\phi(y)$ with $\lambda_\phi \geq 0$ is convex

Proof:

Concider two mean-zero functions $\phi_1(y)$ and $\phi_2(y)$. Then

$$\lambda_{\phi_1} \geq 0 \Leftrightarrow \text{ For every } \tilde{\mathbf{u}}, \quad \int \left(\frac{\nu}{2}|\nabla\tilde{\mathbf{u}}|^2 + \left(\frac{U_*}{h} + \phi_1(y)\right)\tilde{u}_1\tilde{u}_2\right)dx^3 \geq 0 \tag{50}$$

and

$$\lambda_{\phi_2} \geq 0 \Leftrightarrow \text{ For every } \tilde{\mathbf{u}}, \quad \int \left(\frac{\nu}{2}|\nabla\tilde{\mathbf{u}}|^2 + \left(\frac{U_*}{h} + \phi_2(y)\right)\tilde{u}_1\tilde{u}_2\right)dx^3 \geq 0. \tag{51}$$

Now let $0 < a < 1$. Using linearity in ϕ and the hypothesis that $\lambda_{\phi_1} \geq 0$ and $\lambda_{\phi_2} \geq 0$ we see that

$$\int \left(\frac{\nu}{2}|\nabla\tilde{\mathbf{u}}|^2 + \left(\frac{U_*}{h} + a\phi_1(y) + (1-a)\phi_2\right)\tilde{u}_1\tilde{u}_2\right)dx^3 \geq 0 \Leftrightarrow \lambda_{a\phi_1+(1-a)\phi_2} \geq 0. \tag{52}$$

This proves that the set $\Phi = \{\phi|\lambda_\phi > 0\}$ is convex. A sketch of the set Φ is shown in figure 3, where the curve indicates the functions $\phi(y)$ that have $\lambda_\phi = 0$.

4.2 Euler-Lagrange Equations for Optimal ϕ

It is clear from Fig. 3 that the condition for the unique minimizing ϕ (the ϕ closest in norm to $\phi = 0$) is

$$\phi(y) = \gamma \mathcal{P} \left\{ \frac{\delta \lambda_\phi}{\delta \phi} \right\} \tag{53}$$

where $\mathcal{P}$ stands for projection onto mean zero function space. (The vector ϕ that is the minimum must be parallel to the gradient of λ_ϕ at $\lambda_\phi = 0$.) The proportionality factor γ is a Lagrange multiplier. To evaluate $\delta\lambda/\delta\phi$ we begin from (48):

$$\lambda \mathbf{v} = -\nu \Delta \mathbf{v} + \nabla p + \mathbf{i}(U_* + \phi)v_2 + \mathbf{j}(U_* + \phi)v_3. \tag{54}$$

A change in ϕ to $\phi + \delta\phi$ implies a change in λ to $\lambda + \delta\lambda$ and a change in the eigenfunction $\mathbf{v}$ to $\mathbf{v} + \delta\mathbf{v}$. To first order

$$\delta\lambda \mathbf{v} + \lambda \delta \mathbf{v} = -\nu \Delta \delta \mathbf{v} + \nabla \delta p + \mathbf{i}(U_* + \phi)\delta v_2 + \mathbf{j}(U_* + \phi)\delta v_3 + \delta\phi(\mathbf{i}v_2 + \mathbf{j}v_3). \tag{55}$$

Take the dot product with the original eigenfunction $\mathbf{v}$ and note that (using $\int |\mathbf{v}|^2 dx^3 = 1$,

$$\delta\lambda = \int \delta\phi 2 v_1 v_2 dx^3 = A \int_0^h \delta\phi \overline{v_1 v_2} dy \tag{56}$$

so that

$$\frac{\delta\lambda}{\delta\phi}(y) = 2A\overline{v_1 v_2}(y) \tag{57}$$

where we introduced the overbar for the horizontal average over x and z. The projection then in (53) then gives

$$\phi(y) = \gamma(\overline{v_1 v_2}(y) - \langle v_1 v_2 \rangle) \tag{58}$$

where the $2A$ factor has been absorbed into the Lagrange multiplier.

The nonlinear equations we are therefore called to solve for the optimal bound are

$$\begin{aligned} 0 &= -\nu \Delta v + \nabla p + \mathbf{i}(\frac{U^*}{h} + \phi)v_2 + \mathbf{j}(\frac{U^*}{h} + \phi)v_1 && (59) \\ 0 &= \nabla \cdot v && (60) \\ \phi &= \gamma(\overline{v_1 v_2} - \langle v_1 v_2 \rangle) && (61) \end{aligned}$$

where γ is determined by the normalization condition $\int |\mathbf{v}|^2 dx^3 = 1$.

We also have to note that γ **is a scalar only if the isospectral surface** $\lambda_\phi = 0$ **is smooth.**

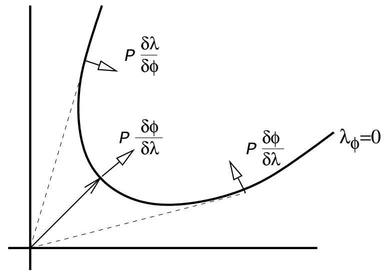

Figure 3: The space Φ. The curve denotes where $\lambda_\phi = 0$.

4.3 Structure of the Optimal Bound

Here we describe a general formulation of the Euler-Lagrange equations (61). The translation invariance in the $(x-z)$ plane allows us to write $\mathbf{v}$ as

$$\mathbf{v} = \sum_\alpha \hat{\mathbf{v}}^{(\alpha)}(y) e^{i(\alpha_x x + \alpha_z z)}, \tag{62}$$

where $\alpha = \mathbf{i}\alpha_1 + \mathbf{j}\alpha_2$ and the incompressibility condition now gives $\partial_y \hat{v}_2 + i\alpha \cdot \hat{\mathbf{v}} = 0$. We can write Q_ϕ as

$$Q_\phi\{\mathbf{v}\} = \sum_\alpha Q_\phi^{(\alpha)}\{\hat{\mathbf{v}}^{(\alpha)}(y)\} \tag{63}$$

where

$$Q_\phi^{(\alpha)}\{\hat{\mathbf{v}}\} = A \int_0^h \left[\nu \left| \frac{d\hat{\mathbf{v}}}{dy} \right|^2 + |\alpha|^2 |\hat{\mathbf{v}}|^2 + \left(\frac{U_*}{h} + \phi(y) \right) (\hat{v}_1 \hat{v}_2^* + \hat{v}_1^* \hat{v}_2) \right] dy \tag{64}$$

Since we want Q_ϕ to be positive we must demand

$$Q_\phi^{(\alpha)} \geq 0 \quad \forall \alpha \tag{65}$$

Note that if we drop the incompressibility condition $|\alpha_1| < |\alpha_2|$ would imply $Q^{(\alpha_1)} < Q^{(\alpha_2)}$ which does *not* generally hold for the incompressible case. The set of zero mean functions with positive λ can now be written as

$$\Phi = \{\phi | \lambda_\phi \geq 0\} = \cap_\alpha \{\phi | \lambda_\phi^{(\alpha)} \geq 0\}. \tag{66}$$

4.3.1 Single Wavenumber Case (Smooth)

First we examine the simplest possible case where the minimum bound comes from a single mode with wave number α. (See Fig. 4) The optimal ϕ is given by

$$\phi(y) = \gamma \Re \left\{ \hat{v}_1^{(\alpha)}(y) \hat{v}_2^{(\alpha)}(y) * - \frac{1}{h} \int_0^h \hat{v}_1^{(\alpha)}(y') \hat{v}_2^{(\alpha)}(y')^* dy' \right\} \tag{67}$$

and the equations that $\hat{\mathbf{v}}$ must satisfy are:

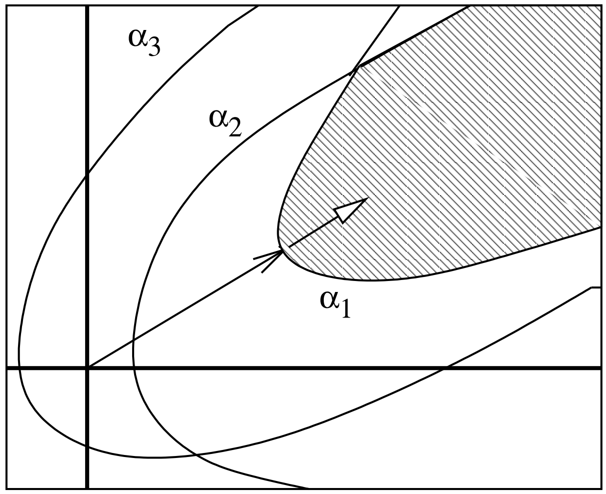

Figure 4: The space Φ in the case that the minimum ϕ is given by a single mode.

$$0 = -\nu\Delta\hat{\mathbf{v}}^{(\alpha)} + \nabla p + \mathbf{i}(\frac{U^*}{h} + \phi)\hat{v}_2^{(\alpha)} + \mathbf{j}(\frac{U^*}{h} + \phi)\hat{v}_1^{(\alpha)} \tag{68}$$

$$0 = \nabla \cdot \hat{\mathbf{v}}^{(\alpha)} \tag{69}$$

$$1 = \int |\hat{\mathbf{v}}^{(\alpha)}|^2 d^3x \tag{70}$$

4.3.2 Two Wavenumber Case

The next case we examine is when the minimum is obtained at the intersection of the curves $\lambda_\phi^{(\alpha_1)} = 0$ and $\lambda_\phi^{(\alpha_2)} = 0$. (See figure 5.)

The optimal ϕ in this case is given as a linear combination of $\mathcal{P}(\delta\lambda^1/\delta\phi)$ and $\mathcal{P}(\delta\lambda^2/\delta\phi)$

$$\phi(y) = \gamma_1\Re\left\{\hat{v}_1^{(\alpha_1)}(y)\hat{v}_2^{(\alpha_1)}(y) * -\frac{1}{h}\int_0^h \hat{v}_1^{(\alpha_1)}(y')\hat{v}_2^{(\alpha_1)}(y')^* dy'\right\}$$

$$+\gamma_2\Re\left\{\hat{v}_1^{(\alpha_2)}(y)\hat{v}_2^{(\alpha_2)}(y) * -\frac{1}{h}\int_0^h \hat{v}_1^{(\alpha_2)}(y')\hat{v}_2^{(\alpha_2)}(y')^* dy'\right\} \tag{71}$$

where γ_1 and γ_2 are to be determined from the normalization conditions and each $\hat{\mathbf{v}}^{(\alpha_i)}$ must obey (70).

4.3.3 General Situation

For the more general case the solution will be given by

$$\phi(y) = \sum_{n=1}^{N} \gamma_n\Re\left\{\hat{v}_1^{(\alpha_n)}(y)\hat{v}_2^{(\alpha_n)}(y) * -\frac{1}{h}\int_0^h \hat{v}_1^{(\alpha_n)}(y')\hat{v}_2^{(\alpha_n)}(y')^* dy'\right\} \tag{72}$$

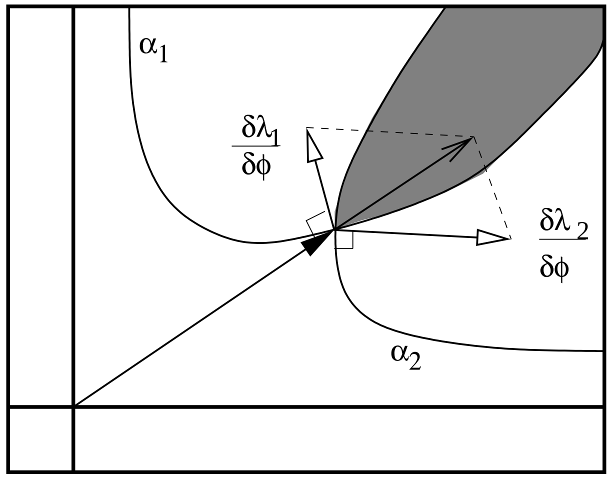

Figure 5: The space Φ in the case that the minimum ϕ is given by two modes.

where each $\hat{v}^{(\alpha_{\mathbf{n}})}$ satisfies

$$0 = -\nu\Delta\hat{\mathbf{v}}^{(\alpha_{\mathbf{n}})} + \nabla p + \mathbf{i}(\frac{U^*}{h} + \phi)\hat{v}_2^{(\alpha_{\mathbf{n}})} + \mathbf{j}(\frac{U^*}{h} + \phi)\hat{v}_1^{(\alpha_{\mathbf{n}})} \quad (73)$$

$$0 = \nabla\hat{\mathbf{v}}^{(\alpha_{\mathbf{n}})} \quad (74)$$

$$1 = \int |\hat{v}^{(\alpha_{\mathbf{n}})}|^2 d^3x \quad (75)$$

and all the γ_n are given by the normalization condition.

4.4 Results and Reality

Fig. (6) summarizes the results that have been obtained by solving the Euler Lagrange equations. The straight line $\sim Re^{-1}$ gives the results of the laminar flow which is an absolute minimum. For higher Re the energy dissipation rate in the flow is bounded from above by the curve shown in the figure. The crosses represent experimental measurements on a turbulant shear layer. The experimental results still show a weak (logarithmic) dependence on the Reynolds number which is not captured by the bounding method. Perhaps further physical information given to the analysis would improve the bound. (We note that the graph is just a sketch.)

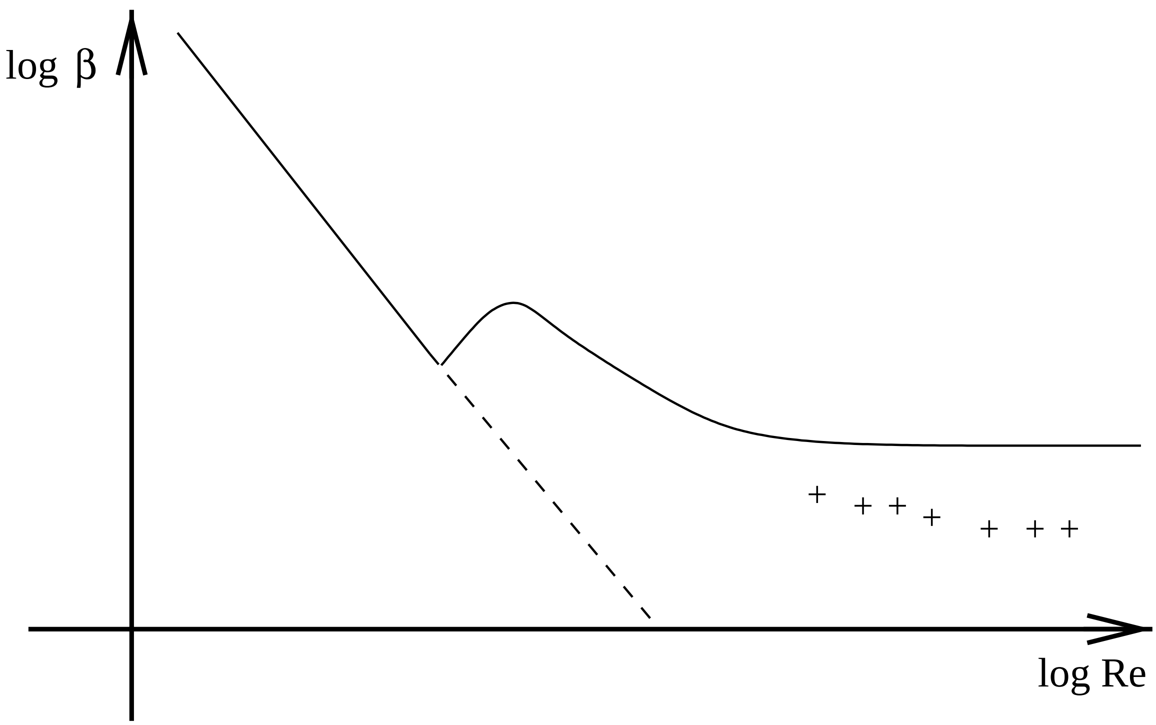

Figure 6: A sketch of the bound and the experimental data.