Lecture 4

Simple Approaches to Some Bounding
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1 On Some Properties of Good Chalk and People Working on Bounding Theory

2 Extremizing Functions and Functionals. Definitions and Simple Examples.

We all know that if $\Phi(x_i)$ is a differentiable function, then the **critical points** x_i that extremize the function can be determined from the conditions

$$\frac{\partial \Phi(x_i)}{\partial x_i} = 0.$$

But these simple conditions do not determine whether the critical values correspond to minima, maxima, or even guarantee an extremum. For example, consider the case shown in figure 1 of a horizontal inflection point and a monotonic function in a closed interval.

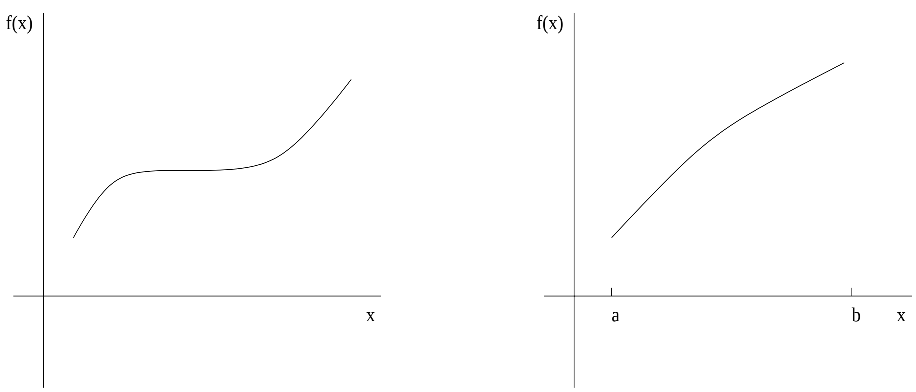

Figure 1: A horizontal inflection point and a monotonic function.

However, if the matrix of the second derivatives is positive definite at the critical point then there is indeed a (local) minimum, and if it is negative definite then we have a local maximum. But if this matrix is indefinite, it does not necessarily help us decide the character of the critical point. This is illustrated by the examples $f_1 = x^4, f_2 = -x^4, f_3 = x^3$, all of which have $f'(0) = f''(0) = 0$, though f_1 has a minimum, f_2 a maximum and f_3 neither at $x = 0$. (In the case of several variables, if the matrix of second derivatives has at least one positive and one negative eigenvalue, we can assert that the critical point is neither a minimum nor a maximum).

Another case where the above equations are not sufficient to determine the extrema of a function is when there is an imposed constraint. As an example consider the unit circle and

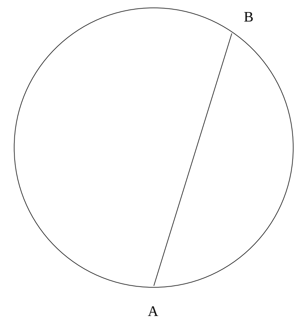

Figure 2: Points on a circle.

ask which two points $A = (x_1, y_1), B = (x_2, y_2)$ lying on it are furthest apart. The distance between them is given by

$$D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

and the constraint takes the form

$$x_i^2 + y_i^2 = 1, \text{ for } i = 1, 2.$$

The constraint can be automatically satisfied by letting $x_i = \cos\theta_i$, $y_i = \sin\theta_i$ and substituting into the expression for the distance. In this way there is no need to take into account the constraint explicitly. Then

$$D^2 = (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 = 2 - 2\cos(\theta_1 - \theta_2),$$

and maximum distance is obtained for $\theta_1 - \theta_2 = \pi$.

But when the constraint cannot be simply eliminated the method of Lagrange multipliers must be used. To find the extremum of $f(x_1, \ldots, x_n)$, subject to the constraints $g_i(x_1, \ldots, x_n) = 0$ for $i = 1, 2, \ldots, m$, one forms the function

$$\Phi(\mathbf{x}, \lambda) = f(x_i) - \sum_{i=1}^{m} \lambda_i g_i,$$

and solves

$$\frac{\partial\Phi(x_i, \lambda_j)}{\partial x_i} = 0, \quad \frac{\partial\Phi(x_i, \lambda_j)}{\partial\lambda_i} = g_i = 0.$$

This seems to be a simple method but let us consider why it works. Suppose that the extremum of f subject to the given constraints is at some point $\overline{x}_i$. If $g_j(x_i)$ is to remain zero then for small changes in the x_i one must have,

$$\nabla g_j(\overline{x}_i) \cdot dx_i = 0,$$

in addition to

$$df = \nabla f(\overline{x}_i) \cdot dx_i = 0.$$

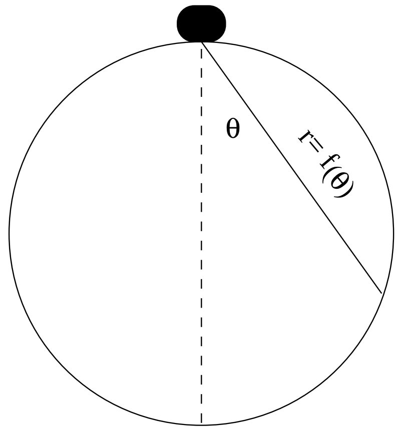

Figure 3: Shape of the Earth.

We may state this as "∇f should be orthogonal to any vector $\mathbf{dx}$ which is orthogonal to all the ∇g_i". In the language of linear algebra, ∇f should be in the orthogonal complement of the space $G^\perp$, which is itself the orthogonal complement of the space G generated by the ∇g_i. Since taking the orthogonal complement twice gets you back where you started (i.e. $(G^\perp)^\perp = G$), ∇f must be in the space generated by the ∇g_i, or $\nabla f = \sum_i^m \lambda_i \nabla g_i$ for some constants λ_i. This is the Lagrange multipliers rule.

More simply, if we maximize a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, then the admissible $\mathbf{dx}$'s satisfy $\mathbf{dx}\cdot\nabla g = 0$. That means $\mathbf{dx}$ can be any vector in the tangent plane to the surface $g = 0$ at the critical point and ∇f must be orthogonal to this tangent plane. Thus ∇f must be parallel to ∇g, i.e. $\nabla f = \lambda \nabla g$, yielding the Lagrange multipliers rule.

As an example let us consider the following problem. We ask to what shape one should transform the Earth in order to maximize his own weight, given that he cannot change his mass. Let us assume the Earth is incompressible, with an uniform density and search for an axisymmetric solution. Since it is incompressible, the Earth must have volume

$$V = \frac{4\pi a^3}{3}.$$

We introduce spherical polar coordinates with an origin at the position of the person, as shown in figure 3. In this coordinate system the volume is given by

$$V = \frac{2\pi}{3} \int_0^{\frac{\pi}{2}} f^3(\theta) \sin\theta \, d\theta = \frac{4\pi a^3}{3},$$

and the person's weight is

$$W = 2\pi \int_0^{\frac{\pi}{2}} \int_0^{f(r)} \left(\frac{Gm\rho}{r^2} \cos\theta \right) r^2 \sin\theta \, dr \, d\theta = 2\pi G m \rho \int_0^{\frac{\pi}{2}} f \sin\theta \cos\theta \, d\theta.$$

Thus the relevant functional is

$$\Phi = \int_0^{\frac{\pi}{2}} (f \sin\theta \cos\theta + \lambda f^3 \sin\theta) \, d\theta.$$

Setting its variation to zero gives

$$\delta\Phi = \int_0^{\frac{\pi}{2}} (\delta f \cos\theta + 3\lambda f^2 \delta f) \sin\theta \, d\theta = 0,$$

and so the minimum is obtained with

$$f^2 = -\frac{1}{3\lambda} \cos\theta,$$

which is not too different from a sphere (which has $f = 2a\cos\theta$) and increases the weight of the person by $\left(\frac{27}{25}\right)^{\frac{1}{3}} \approx 1.02$.

If we extend our considerations to functionals we arrive at the *Euler-Lagrange equations.* For the simpler case when

$$F(f) = \int L(f, \theta) \, d\theta,$$

the Euler-Lagrange equations following from $\delta F = 0$ are $L_1 \delta f = 0$, (where the subscript denotes differentiation with respect to the corresponding argument), but if the functional is of the type,

$$F(f) = \int L(f, f', \theta) \, d\theta,$$

then one has,

$$\int \left(L_1 \delta f + L_2 \delta f'\right) d\theta = 0.$$

After integrating the second term by parts,

$$\int \left(L_1 - \frac{dL_2}{d\theta}\right) \delta f \, d\theta + [L_2 \delta f]_{bs} = 0,$$

and assuming that $L_2 \delta f$ vanishes on the bounding surface, one obtains the *Euler-Lagrange equation*,

$$L_1 - \frac{dL_2}{d\theta} = 0.$$

In particular, if the functional is of the type

$$F(f) = \int_a^b L(f, f', \theta) \, d\theta,$$

we get

$$\delta F = \int_a^b L_1 \delta f + L_2 \delta f' = \int_a^b (L_1 - \frac{\partial}{\partial\theta} L_2) \delta f \, d\theta + [L_2 \delta f]_a^b.$$

This should be zero for all δf's that are admissible. In some cases the boundary term vanishes automatically, for example if f is given at $x = a$ and $x = b$. If there are no such conditions we may first of all take δf's that *are* in fact zero at $\theta = a$ and $\theta = b$, but are otherwise arbitrary. Then, assuming that $(L_1 - \frac{d}{d\theta}L_2)$ is continuous, we may conclude that $L_1 - \frac{d}{d\theta}L_2 = 0$ at all interior points of (a, b). Then, by taking δf that is zero at, say b but not zero at a, we conclude that $L_2(f(a), f'(a), a) = 0$. Similarly we may show that $L_2(f(b), f'(b), b) = 0$. In such cases the variational problem itself provides boundary conditions, so-called "natural boundary conditions", to supplement the Euler equation.

As an example, we minimize $\int_0^1 f'^2(x)\,dx$ subject to the two constraints $\int_0^1 f^2(x)\,dx = 1$, $\int_0^1 f(x)\,dx = 0$ (with *no* boundary conditions specified). To do this we consider the functional $\Phi = \int_0^1(f'^2 - \lambda_1 f^2 - \lambda_2 f)$. Then,

$$\delta\Phi = \int_0^1 (2f'\delta f' - 2\lambda_1 f\delta f - \lambda_2\delta f)dx = 2\left[f'\delta f\right]_0^1 - 2\int_0^1 [f'' + \lambda_1 f + \frac{1}{2}\lambda_2]\delta f\,dx,$$

so the necessary conditions for a minimum are :

- Euler-Lagrange equation $f'' + \lambda_1 f + \frac{1}{2}\lambda_2 = 0$,
- the natural boundary conditions $f'(0) = f'(1) = 0$,
- the constraints $\int_0^1 f^2 dx = 1, \int_0^1 f dx = 0$.

One could write down the general solution of the differential equation and use the two boundary conditions and the two constraints to determine the two λ's and the two arbitrary constants in the general solution. It is a little neater to note that integrating the Euler-Lagrange equation from 0 to 1 and using the natural boundary condition and the second constraint gives $\frac{1}{2}\lambda_2 = 0$, hence $\lambda_2 = 0$. We then see that $f = \sqrt{2}\cos(n\pi x)$ and $\lambda_1 = (n\pi)^2$, for some integer n that cannot be zero because of the second constraint. $n = 1$ gives the least-value of the integral – indeed if we integrate $f'(f'' + \lambda_1 f) = 0$ from 0 to 1, using the first constraint and the natural boundary conditions, we see that $\lambda_1 = \int_0^1 f'^2 dx$, i.e. λ_1 itself is the required minimum value π^2. (Note that without the second constraint the minimum value would be zero, achieved by $f = 1$.)

As a simple illustration the shortest path between two parabolas such as those shown in figure 4, is a straight line perpendicular to both curves. In this example the Euler-Lagrange equation shows that the path must be straight, while the natural boundary conditions show that it should be orthogonal to each of the two parabolas at its endpoints, which is pretty obvious geometrically.

3 Minimization of $\int f'^2$ given $\int f^2$

As an example, we consider the following problem:

Minimize $\int_0^L f'^2(x)dx$ subject to $\int_0^L f^2(x)dx = 1$ and $f(0) = f(L) = 0$.

The technique of Lagrange multipliers gives $\Phi = \int_0^L(f'^2 - \lambda f^2)dx$, which has

$$\delta\Phi = -2\int_0^L (f'' + \lambda f)\delta f dx$$

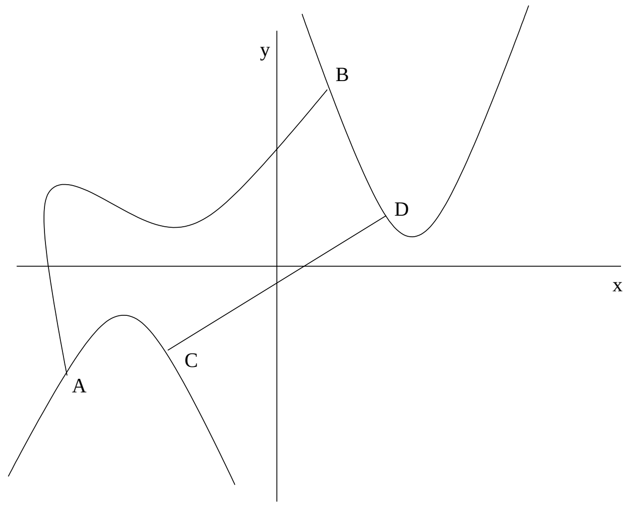

Figure 4: Distance between parabolas.

so the minimizing function satisfies $f'' + \lambda f = 0$, meaning that f is proportional to $\sin(n\pi x/L)$ for some integer n. $n = 1$ gives the minimal value. This yields the relation

$$\boxed{\int_0^L f'^2 dx \geq \frac{\pi^2}{L^2} \int_0^L f^2(x) dx \text{ for all functions } f \text{ with } f(0) = f(L) = 0.}$$

This derivation was not very rigorous, although it does give the correct answer.

The problem can also be tackled using Fourier series. Under rather mild restrictions, $f(x)$ has a Fourier sine series, $f(x) \sim \sum_1^\infty b_n \sin(n\pi x/L)$, and with a little more assumed about f, this series actually converges to f in the interior of $(0, L)$ – for instance if f is continuous and continuously differentiable there. Of course the sine series converges to 0 at $x = 0$ and L, which need not be the values of f at those points. For simplicity, however, we assume that f vanishes at these endpoints, and that f, f' and f'' are all continuous on $[0, L]$. Then not only does the sine series of f converge to f on $[0, L]$, but the cosine series of f' converges to f' on this interval, and this cosine series is *in fact* the same as the formal term-by-term derivative of the sine series of f. (This would *not* be true unles $f(0) = f(L) = 0$, however smooth f might be on $[0, L]$.) Thus

$$f(x) = \sum_1^\infty b_n \sin(n\pi x/L), \quad f'(x) = \sum_1^\infty n\pi b_n \cos(n\pi x/L)/L,$$

and

$$\int_0^L f^2(x) dx = \frac{L}{2} \sum_1^\infty b_n^2, \quad \int_0^L f'^2(x) dx = \frac{L}{2} \sum_1^\infty \left(\frac{n\pi}{L}\right)^2 b_n^2.$$

Thus $\int_0^L f'^2(x) dx \geq (\pi/L)^2 \int_0^L f^2(x) dx$, with equality only when all the b_n beyond b_1 are zero.

Another approach to this inequality that also avoids consideration of the Euler–Lagrange equation is the *method of multiplicative variation*: we set $f(x) = \sin(\pi x/L)g(x)$. Then we compute

$$
\begin{aligned}
\int_0^L f'^2 dx =& \int_0^L \left(\sin\frac{\pi x}{L}g'(x) + \frac{\pi}{L}\cos\frac{\pi x}{L}g(x)\right)^2 dx, \\
=& \int_0^L \sin^2\frac{\pi x}{L}g'^2 dx + \int_0^L 2\frac{\pi}{L}\cos\frac{\pi x}{L}\sin\frac{\pi x}{L}g(x)g'(x)dx + \int_0^L \left(\frac{\pi}{L}\right)^2\cos^2\frac{\pi x}{L}g^2 dx, \\
=& \int_0^L \sin^2\frac{\pi x}{L}g'^2 dx - \int_0^L \left(\frac{\pi}{L}\right)^2\left(\cos^2\frac{\pi x}{L} - \sin^2\frac{\pi x}{L}\right)g^2 dx \\
&\quad + \int_0^L \left(\frac{\pi}{L}\right)^2\cos^2\frac{\pi x}{L}g^2 dx, \\
=& \int_0^L \sin^2\frac{\pi x}{L}\,g'^2(x)dx + \frac{\pi^2}{L^2}\int_0^L f^2(x)dx, \\
\geq& \frac{\pi^2}{L^2}\int_0^L f^2 dx,
\end{aligned}
$$

with equality only when $g' \equiv 0$ everywhere on $(0, L)$ assuming it is continuous there.

A variation on this theme is the following little calculation:

$$
\begin{aligned}
0 \leq& \int_0^L \left(f'(x) - \frac{\pi}{L}\cot\frac{\pi x}{L}f(x)\right)^2 dx \\
=& \int_0^L f'^2 dx + \left(\frac{\pi}{L}\right)^2\int_0^L \cot^2\frac{\pi x}{L}f^2(x)dx - \int_0^L 2f(x)f'(x)\frac{\pi}{L}\cot\frac{\pi x}{L}dx \\
=& \int_0^L f'^2 dx + \left(\frac{\pi}{L}\right)^2\int_0^L \cot^2\frac{\pi x}{L}f^2(x)dx - \left[\frac{\pi}{L}f^2(x)\cot^2\frac{\pi x}{L}\right]_0^L \\
&\quad - \frac{\pi}{L}\int_0^L f^2(x)\frac{\pi}{L}\operatorname{cosec}\frac{\pi x}{L}dx \\
=& \int_0^L f'^2 dx - \left(\frac{\pi}{L}\right)^2\int_0^L f^2 dx
\end{aligned}
$$

In this argument we must assume that $f \to 0$ faster that $x^{1/2}$ as $x \to 0$ or than $(L-x)^{1/2}$ as $x \to L$. Some such hypothesis is needed to assume the existence of $\int_0^L f'^2(x)dx$.

Claim: Assume that f is continuous and differentiable on the interval $[-1, 1]$, $f(-1) = f(1) = 0$, $f(x) = \int_{-1}^x f'(t)dt$ and that $\int_{-1}^1 f'^2 dx$ exists. Then

$$\boxed{f^2(x) \leq (1-x^2)\langle f'^2\rangle,}$$

where $\langle g\rangle$ denotes $\int_{-1}^1 g(x)dx/2$ for any function g.

Proof:

$$\begin{aligned} f^2(x) =& \left(\int_{-1}^{x} f'(t)dt\right)^2, \\ \leq& \int_{-1}^{x} f'^2(t)dt \int_{-1}^{x} 1^2 dt, \\ =&(1+x)\int_{-1}^{x} f'^2(t)dt. \end{aligned}$$

Similarly

$$f^2(x) \leq (1-x)\int_{x}^{1} f'^2 dt,$$

and combining the two gives

$$f^2(x)\left(\frac{1}{1+x}+\frac{1}{1-x}\right) \leq 2\langle f'^2\rangle,$$

leading to the required result.

$\square$

Integrating both sides leads to

$$\langle f^2\rangle \leq \frac{2}{3}\langle f'^2\rangle.$$

However, we already know that the stronger relationship

$$\langle f^2\rangle \leq \frac{4}{\pi^2}\langle f'^2\rangle,$$

holds from the previous calculations, so the above method does not give the optimal estimate. Still, the *pointwise* estimate given by this result cannot be improved; it is only the *integrated* result that is less than optimal.

In two dimensions, if we wish to minimize $\int_A |\nabla f|^2 dA$ subject to $\int_A |f|^2 dA = 1$, where A is a region in $\mathbb{R}^2$, then the Lagrange method yields the relation $\nabla^2 f + \lambda f = 0$ for the minimizing function f. If A is a circle centred on the origin with radius 1 then this has solutions $J_m(j_{mn}r)e^{im\theta}$, where J_m is the mth order Bessel function of the first kind and j_{mn} is the nth positive root of J_m. Then $1 = \int_A |f|^2 dA = 2\pi\int_0^1 rJ_m^2(j_{mn}r)dr$ and so $\int_A |\nabla f|^2 dA = 2\pi j_{mn}\int_0^1 rJ_m^2(j_{mn}r)dr = j_{mn}^2$. So the minimum value of $\int_A |\nabla f|^2 dA$ is given by $j_{01}^2 \approx 5.78$.

4 The Dual Lagrange Problem

We have

$$L(\mathbf{x},\boldsymbol{\lambda}) = F(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i.$$

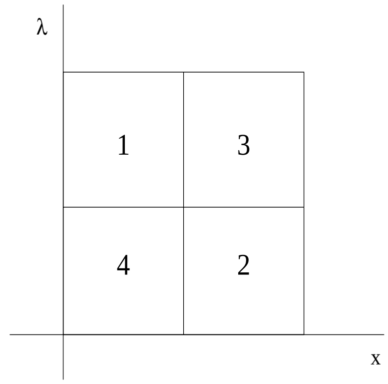

Figure 5: An example showing values of $L(\mathbf{x}, \boldsymbol{\lambda})$ for a case where $\sup_{\boldsymbol{\lambda}} \inf_{\mathbf{x}} L = 2$ and $\inf_{\mathbf{x}} \sup_{\boldsymbol{\lambda}} L = 3$.

Suppose that F has minimum value F_*, subject to the constraints $g_i = 0$ for $i = 1, 2, \ldots, m$, which is attained when $\mathbf{x} = \mathbf{x}_*$.

Now let

$$h(\boldsymbol{\lambda}) = \inf_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) \leq F_* \ \forall \boldsymbol{\lambda}.$$

The dual problem is given by maximizing $h(\boldsymbol{\lambda})$. Say this has value h_*. Then $h_* \leq F_*$, sometimes with equality, though not always. An example of a case where there is not equality is shown in figure 5.

4.1 Further examples

$$F(x, y) = x^2 + 2y^2, \text{ with } g = 1 - x^2 - y^2.$$

Then $L(x, y, \lambda) = x^2 + 2y^2 + \lambda(1 - x^2 - y^2)$. Thus the Euler–Lagrange equations for seeking a minimum of F with $g_1 = 0$ are $2x - 2x\lambda = 0$, $4y - 2y\lambda = 0$. Therefore either $\lambda = 1$, $x = \pm 1$ and $y = 0$ giving $F_* = 1$, or $\lambda = 2$, $x = 0$ and $y = \pm 1$ giving $F_* = 2$. Thus the minimum is $F_* = 1$, attained at $x = \pm 1$, $y = 0$.

Now consider

$$h(\lambda) = \inf_{x} L(x, \lambda) = \begin{cases} -\infty & \text{if } \lambda > 1 \\ 1 & \text{if } \lambda = 1 \text{ (at } y = 0) \\ 1 & \text{if } \lambda < 1 \text{ (at } x = y = 0) \end{cases}$$

Thus $\max_\lambda h(\lambda) = 1$ so that $h_* = 1 = F_*$. So for this example, $\max_\lambda h = \min_{\mathbf{x}, g_1(x,y)=0} F$. On the other hand consider again

$$F = x^2 + 2y^2, \text{ but with } g = 1 - x^4 - y^4.$$

For this case, $L(x, y, \lambda) = x^2 + 2y^2 + \lambda(1 - x^4 - y^4)$. Thus using the Lagrange multiplier rule to seek a minimum of F given $g = 0$ we get

$$2x - 4\lambda x^3 = 0, \quad 4y - 4\lambda y^3 = 0, \quad 1 - x^4 - y^4 = 0.$$

Therefore either

- $x \neq 0$. Then $\lambda = 1/2x^2$ and $y = 0$ or $y = \pm\sqrt{2}x$. If $y = 0$ then $x = \pm 1$ and $F_* = 1$ Otherwise, $x = \pm 5^{-1/4}$, $y = \pm 5^{-1/4}\sqrt{2}$ and $F_* = 5^{-1/2} + 4 \cdot 5^{-1/2} = \sqrt{5}$, or

- $x = 0$. Then the constraint requires $y = \pm 1$, (and hence $\lambda = 1$) and then $F = 2$, which is greater than 1.

Thus the least value of F is 1, obtained at $x \pm 5^{-1/4}$, $y = \pm 5^{-1/4}\sqrt{2}$.

However, Lagrange multipliers are not really needed for this problem: considering only x and $y \geq 0$ we could eliminate the constraint by setting $x = \sqrt{\cos\theta}$, $y = \sqrt{\sin\theta}$ and then $F = \cos\theta + 2\sin\theta$, $F' = 2\cos\theta - \sin\theta$. At the minimum, $\tan\theta = 2$ and hence $F = \sqrt{5}$. But at $\theta = 0$, $F = 1$ and at $\theta = \pi/2$, $F = 2$. Thus the minimum F is 1, attained at $x = \pm 1$, $y = 0$.

But what is $h(\lambda) = \inf_{x,y}(x^2 + 2y^2 + \lambda(1 - x^4 - y^4))$?

$$h(\lambda) = \begin{cases} -\infty & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ \lambda & \text{if } \lambda < 0 \end{cases}$$

Therefore $\max_\lambda h(\lambda) = 0$, so

$$\max_\lambda h(\lambda) < \min_{x,y,x^4+y^4=1} F.$$

There is a "duality gap".

Remark: If our original problem had been to *maximize* $F(\mathbf{x})$ subject to the constraints $g_1 = g_2 = \cdots = g_m = 0$ we would still have

$$L(\mathbf{x}, \boldsymbol{\lambda}) = F(\mathbf{x}) + \sum_{k=1}^{m} \lambda_k g_k(\mathbf{x}),$$

and the same Lagrange multiplier rule: look for x_* where

$$\frac{\partial F}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \text{ and } g_1 = g_2 = \cdots = g_m = 0.$$

If we have an x_* that maximises F subject to the constraint and consider

$$H(\boldsymbol{\lambda}) = \sup_{\mathbf{x}} \left(F(\mathbf{x}) + \sum_{k=1}^{m} \lambda_k g_k(\mathbf{x}) \right),$$

then we have $\min_{\boldsymbol{\lambda}} H(\boldsymbol{\lambda}) \geq F(\mathbf{x}_*)$, since whatever $\boldsymbol{\lambda}$ might be, there is an $\mathbf{x}$ (namely $\mathbf{x}_*$) that makes $L(\mathbf{x}, \boldsymbol{\lambda}) = F(\mathbf{x}_*)$, so $\sup_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) \geq F(\mathbf{x}_*)$, and so $\min_{\boldsymbol{\lambda}} H(\boldsymbol{\lambda}) \geq F(\mathbf{x}_*)$.

These maximum minimum dual problems are reminiscent of "Courant's maximum principle", a rather striking result about eigenvalues of symmetric or Hermitian matrices, Sturm–Liouville problems, etc. It will be recalled that the lowest eigenvalue λ_1 of a real symmetric matrix A may be characterized as the minimum of $x^T Ax/xTx$ for all non-zero vectors x, this minimum being achieved for $x = e$, the first eigenvector. Similarly the second eigenvalue λ_2 is the minimum of $x^T Ax/x^T x$ for all vectors x that are orthogonal to e, and the kth eigenvalue is the same minimum over all vectors orthogonal to $e_1, e_2, \ldots, e_{k-1}$. Courant pointed out that the kth eigenvalue can be described directly without explicit reference to the previous ones as follows: take an arbitrary set of $k-1$ vectors $v_1, \ldots, v_{k-1}$, and form $H(v_1, \ldots, v_{k-1}) = \min(x^T Ax/x^T x)$ for all non-zero vectors x that are orthogonal to $v_1, v_2, \ldots, v_{k-1}$. Then $\lambda_k = \max_{v_1,\ldots,v_{k-1}} H$. (See for instance Courant and Hilbert [1]).

References

[1] D. Hilbert and R. Courant, *Methods of Mathematical Physics Volume I* (John Wiley & Sons, USA, 1989).