Lecture 1

Hydrodynamic Stability
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1 Introduction

In many cases in nature, like in the Earth's atmosphere, in the interior of stars and planets, one sees the appearance of patterns that seem to be stable and persistent. Those patterns usually correspond to broken symmetries. The purpose of the following lectures is to explain the formation of these patterns and their persistence even under 'strong' turbulent conditions. This first lecture is dealing with onset of turbulence by determining the critical value of the control parameters that the laminar solution becomes unstable.

2 Linear theory

2.1 States of Minimum Energy

Continuous material systems are subject to conservation laws like mass, energy, momentum and angular momentum. The laws of thermodynamics also tell us that the mechanical energy has a tendency to be converted to thermal energy leading a system to a state of minimum mechanical energy, subject to other conservation laws. Systems that have reached this state are called equilibrium systems. As an example consider a rotational flow in a cylinder. To simplify the problem we consider that the flow has only a radial dependence e.g. $\omega = \omega(r)$. We can then ask the following question: Given an initial condition with angular momentum $\mathcal{A}$, what is the state of minimum mechanical energy our system can have keeping the angular momentum fixed? The kinetic energy $\mathcal{K}$ and the angular momentum can be expressed as functionals of the angular frequency as:

$$\mathcal{K}[\omega] = \pi \int_0^{r_0} (\omega(r) r)^2 r dr, \qquad \mathcal{A}[\omega] = 2\pi \int_0^{r_0} \omega(r) r^2 r dr = \mathcal{A}_0 \qquad \text{(fixed)} \tag{1}$$

To minimize the energy keeping the angular momentum fixed we have to minimize the functional:

$$\mathcal{F}[\omega, \lambda] = \mathcal{K}[\omega] + \lambda \left(\mathcal{A}[\omega] - \mathcal{A}_0 \right) \tag{2}$$

Where λ is a Lagrange multiplier and $\mathcal{A}_0$ the angular momentum of the flow. Evaluating the variation of $\mathcal{F}$ with respect to ω and λ we obtain

$$\delta\mathcal{F} = \delta K + \delta(\lambda\mathcal{A}) = \delta \left\{ 2\pi \int_0^{r_0} \frac{1}{2}\omega^2(r) r^3 dr + \lambda \left(2\pi \int_0^{r_0} \omega(r) r^3 dr - \mathcal{A}_0 \right) \right\} = 0 \tag{3}$$

$$\delta\mathcal{F} = 2\pi\left\{\int_0^{r_0}\left(\omega(r)r^3 dr + \lambda r^3\right)\delta\omega dr + \delta\lambda\left(2\pi\int_0^{r_0}\omega(r)r^3 dr - \mathcal{A}_0\right)\right\} = 0. \qquad (4)$$

In order for the variation to be equal to zero for every $\delta\omega$ and every $\delta\lambda$ we must have

$$\omega(r)r^3 + \lambda r^3 = 0 \qquad \text{and} \qquad \int_0^{r_0}\omega(r)r^3 dr = \mathcal{A}_0. \qquad (5)$$

which leads to

$$\omega(r) = -\lambda = \text{constant} = \omega_0 \qquad \text{with} \qquad \omega_0 = \frac{2\mathcal{A}_0}{\pi r_0^4} \qquad (6)$$

which is a rigid body rotation.

A similar example is if we consider a flow in a cylinder with the velocity being given by $\mathbf{u} = v(r)\mathbf{k}$ where $\mathbf{k}$ is the unit vector parallel to the axis of symmetry. The kinetic energy and the momentum are given by

$$\mathcal{K}[v] = \pi\int_0^{r_0} v^2(r)rdr, \qquad \mathcal{M}[v] = 2\pi\int_0^{r_0} v(r)rdr = \mathcal{M}_0 \qquad \text{(fixed)} \qquad (7)$$

To minimize the energy, keeping the momentum fixed, we define the functional

$$\mathcal{F}[\omega] = \mathcal{K}[v] + \lambda(\mathcal{M}[v] - \mathcal{M}_0). \qquad (8)$$

Varying it we get

$$\delta\mathcal{F} = \delta K + \delta(\lambda\mathcal{M}) = \delta\left\{\pi\int_0^{r_0} v^2(r)rdr + \lambda\left(2\pi\int_0^{r_0} v(r)rdr - \mathcal{M}_0\right)\right\} = 0 \qquad (9)$$

which leads to

$$v(r) + \lambda = 0 \text{ and } 2\pi\int_0^{r_0} v(r)rdr = \mathcal{M}_0 \qquad (10)$$

Which again leads to the motion of a rigid body with velocity given by $\mathbf{u} = \mathcal{M}_0/\pi r_0^2\mathbf{k}$.

The fact that the above states are of minimum energy indicates that they are stable. Any other state with more energy will not be stationery or stable and will decay to the solutions of minimum energy.

As a further example we will examine the flow of a fluid between two infinite parallel plates separated by a distance d. A uniform pressure gradient along one of the parallel directions is assumed to keep the flow from being non-zero. The only control parameter of the problem is given by the Reynolds number $Re = Ud/\nu$ where ν is the kinematic viscosity and U is the averaged velocity. The above system has a steady solution given by $\mathbf{u} = \left(Re(1/4 - z^2), 0, 0\right)$ (Poiseuille flow.). We want to examine for which values of the control parameter Re the Poiseuille flow is stable. It is typical that in stability problems like the one described above four regimes of the flow parameter Re can be distinguished, see Fig. 1.

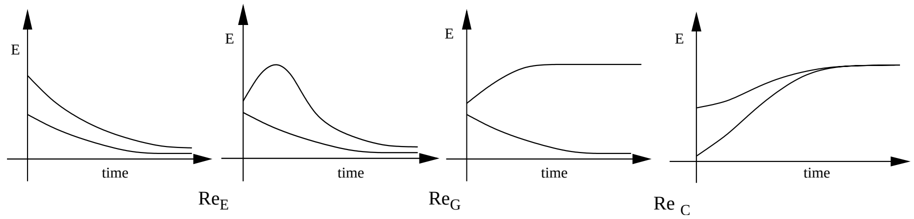

Figure 1: The behavior of perturbations on the laminar solution for different Reynolds numbers.

A) From 0 to Re_E all disturbances decay exponentially.

B) From Re_E to Re_G some infinitesimal or finite disturbances might grow for finite time but all disturbances decay exponentially for $t \to \infty$

C) From Re_G to Re_c infinitesimal disturbances decay exponentially but finite disturbances converge to a new solution.

D) From Re_c to ∞ infinitesimal disturbances grow exponentially.
The following paper is dedicated to estimating the values of Re_E, Re_c.

2.2 The Energy Method

We restrict ourselves to an incompressible fluid on a domain $\mathcal{D}$. The equations of motion are given by

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \mathbf{f} + \nu \nabla^2 \mathbf{v} && (11) \\ \nabla \cdot \mathbf{v} &= 0 && (12) \end{aligned}$$

with boundary conditions on $\partial\mathcal{D}$

$$\mathbf{v} = 0 \tag{13}$$

or

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ and } \mathbf{n} \times (\nabla \times (\mathbf{n} \times \mathbf{v})) = 0 \tag{14}$$

where $\mathbf{n}$ is the normal unit vector to $\partial\mathcal{D}$.

Denoting the stationary solution of maximum symmetry by $\mathbf{v}_s$ and writing the general solution as the stationary solution plus a perturbation $\mathbf{u}$

$$\mathbf{v} = \mathbf{v}_s + \mathbf{u}, \tag{15}$$

The Navier Stokes equation for the velocity $\mathbf{u}$ becomes

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{v}_s \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{v}_s &= -\nabla \pi + \nu \nabla^2 \mathbf{u} && (16) \\ \nabla \cdot \mathbf{u} &= 0 && (17) \end{aligned}$$

with the same boundary conditions as (13) or (14). Multiplying the above equation by $\mathbf{u}$ and taking the volume average we obtain

$$\frac{1}{2}\frac{d}{dt}\langle \mathbf{u}\cdot\mathbf{u}\rangle = -\langle|\nabla \mathbf{u}|^2\rangle - Re\langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_s\rangle \tag{18}$$

where we have used the boundary conditions to eliminate the surface terms. From the above equation it is obvious that if $Re\langle \mathbf{u}(\mathbf{u}\cdot\nabla)\mathbf{v_s}\rangle \geq 0$ then $\frac{1}{2}\frac{d}{dt}\langle \mathbf{u}\cdot\mathbf{u}\rangle \leq 0$ and therefor all perturbations decrease in amplitude with time. On the other hand, if $\mathrm{Re}\langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_s\rangle < 0$ we can define the functional

$$\mathcal{R}_E \equiv \frac{\langle|\nabla \hat{\mathbf{u}}|^2\rangle - 2\langle \pi\nabla\cdot\hat{\mathbf{u}}\rangle}{-\langle \hat{\mathbf{u}}\cdot(\hat{\mathbf{u}}\cdot\nabla)\mathbf{v}_s\rangle} \tag{19}$$

and look for its minimum.

Let

$$I_1 = \langle|\nabla\hat{\mathbf{u}}|^2\rangle,\ I_2 = -\langle \hat{\mathbf{u}}\cdot(\hat{\mathbf{u}}\cdot\nabla)\mathbf{v}_s\rangle \text{ and } I_3 = -2\langle \pi\nabla\hat{\mathbf{u}}\rangle. \tag{20}$$

Then

$$\delta\mathcal{R}_E = \frac{\delta I_1 + \delta I_3}{I_2} - \frac{(I_1+I_3)\delta I_2}{I_2^2} = \frac{\delta I_1 + \delta I_3}{I_2} - M\frac{\delta I_2}{I_2} = 0, \tag{21}$$

where $M = \min\{\mathcal{R}_E(u)\}$. Expressing the variations $\delta I_1, \delta I_2, \delta I_3$ as we did in the previous paragraph, we obtain

$$\frac{1}{2}M[\hat{u}_j\partial_j v_{s\,i} + \hat{u}_j\partial_i v_{s\,j}] = -\partial_i\pi + \partial_j\partial_j\hat{u} \tag{22}$$

and

$$\partial_i\hat{u}_i = 0. \tag{23}$$

Now, since M is the minimum of the functional $\mathcal{R}_E$ we have that for an arbitrary solution of (17) that

$$M \leq \frac{\langle|\nabla\mathbf{u}|^2\rangle - 2\langle\pi\nabla\mathbf{u}\rangle}{-\langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_s\rangle} \tag{24}$$

using the energy equation (18) we have that

$$\frac{1}{2}\frac{d}{dt}\langle \mathbf{u}\cdot\mathbf{u}\rangle \leq -(M - Re)\langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_s\rangle \tag{25}$$

and since $\langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{v_s}\rangle \leq 0$ we have that the perturbation can grow only if $Re \geq M$.

2.3 Linear Stability of Plane Couette Flow

As a special case illustrating the above general theory we take a flow between two parallel plates moving in opposite directions with relative velocity $\mathbf{U}_D$. The distance d between the plates can be used to define Reynolds number $Re = |\mathbf{U}_D|d/\nu$ and the solution can be written in dimensionless form

$$\mathbf{v}_s = -Re\, z\mathbf{i}, \tag{26}$$

where we have introduced Cartesian coordinates and the unit vectors in the directions of (x, y, z) are $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, respectively. The velocity $\mathbf{U}_D$ is in the direction of $\mathbf{i}$. For the solution of Eqs. (22), (23) we introduce the general representation

$$\tilde{\mathbf{u}} = \nabla \times (\nabla\varphi \times \mathbf{k}) + \nabla\psi \times \mathbf{k} \tag{27}$$

for a solenoidal vector field $\tilde{\mathbf{u}}$, where φ and ψ are some scalar functions. The z-components of curl and $(\text{curl})^2$ of Eq. (22) give

$$\nabla^4 \Delta_2 \varphi = \frac{1}{2} M (2\partial_x \partial_z \Delta_2 \varphi + \partial_y \Delta_2 \psi). \tag{28}$$

$$\nabla^2 \Delta_2 \psi = \frac{1}{2} M \partial_y \Delta_2 \varphi, \tag{29}$$

where $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$. The boundary conditions for this problem are

$$\varphi = \partial_z \varphi = \psi = 0. \tag{30}$$

If we only consider solutions independent of x, the function ψ can be eliminated from Eqs. (28), (29) to give

$$(\nabla^6 - \frac{1}{4} M_y^2 \partial^2_{yy}) \Delta_2 \varphi = 0 \quad \text{with} \quad \varphi = \partial_z \varphi = \nabla^4 \varphi = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{31}$$

Since this eigenvalue problem is similar to the problem of determining the critical Reynolds number in a fluid layer heated from below with rigid boundaries, we can use the latter fact to write

$$\frac{1}{4} M_y^4 = 1708 \quad \text{corresponding to} \quad \varphi = \cos(\alpha y)\, f(z) \quad \text{with} \quad \alpha_c = 3.116, \tag{32}$$

where α_c is the lowest eigenvalue. It can be prooven that more general solutions φ and ψ that depend on x and y do not yield values of M lower than M_y [1]. Therefore finally we have $Re_E = 2\sqrt{1708} \approx 82.6$ for the plane Couette flow. The values for various non-rotating systems have been determined experimentally and theoretically, and comparison with the linear theory is given [1] in Table 1.

[1]The maximum velocity and the channel width d (radius d in the case of pipe flow) have been used in the definition of Re.

	Re_E	Re_G (from exp.)	Re_c
Plane Couette Flow	82.6	≈ 1300	∞
Poiseuille Flow (Channel Flow)	99.2	≈ 2000	5772
Hagen–Poiseuille Flow (Pipe Flow)	81.5	≈ 2100	∞

Table 1: Reynolds Numbers for Shear Flows in Non–Rotating Systems.

2.4 Linear Stability of Circular Couette Flow

Consider the flow between coaxial cylinders with radii r_1 and r_2 ($> r_1$) that rotate with angular velocity Ω_1 and Ω_2, respectively. The basic solution of Eq. (12) for the azimuthal velocity v_φ is

$$v_\varphi = \frac{r_2^2\Omega_2 - r_1^2\Omega_1}{r_2^2 - r_1^2}\, r - \frac{r_1^2 r_2^2(\Omega_2 - \Omega_1)}{(r_2^2 - r_1^2)r} \tag{33}$$

and is called the circular Couette flow. For simplicity we restrict our analysis to the case $r_1 - r_2 \ll r_1$ and $0 < \Omega_1 - \Omega_2 \ll \Omega_1$. In this limiting case the solution (33) assumes the form of a plane Couette flow studied in the previous section, with angular velocity $\Omega_D = \frac{1}{2}(\Omega_1 + \Omega_2)$. The corresponding coordinate system is oriented so that the x-coordinate points in the azimuthal direction, the y-coordinate points in the axial direction, and the z-coordinate is pointed radially outward. The Reynolds number is defined by $Re = (\Omega_1 r_1 - \Omega_2 r_2)d/\nu$.

Next we study infinitesimal disturbances therefore neglecting the nonlinear term $\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}$ that enters Eq. (17), and add a Coriolis term

$$\frac{\partial}{\partial t}\tilde{\mathbf{u}} + \mathbf{v}_s \cdot \nabla\tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla\mathbf{v}_s + 2\Omega \times \tilde{\mathbf{u}} = -\nabla\pi + \nabla^2\tilde{\mathbf{u}}, \tag{34}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \tag{35}$$

where $\Omega = \Omega_D d^2/\nu$. Assuming time dependence of the form $\exp(\sigma t)$, boundary conditions $\tilde{\mathbf{u}} = 0$ at $z = \pm\frac{1}{2}$ and a representation for $\tilde{\mathbf{u}}$ in the form

$$\tilde{\mathbf{u}} = \nabla \times (\nabla \times \mathbf{k}\tilde{\varphi}) + \nabla \times \mathbf{k}\tilde{\psi}, \tag{36}$$

we obtain the following eigenvalue problem

$$\nabla^4\Delta_2\tilde{\varphi} - 2\Omega \cdot \nabla\Delta_2\tilde{\psi} = \mathbf{v}_s \cdot \nabla\nabla^2\Delta_2\tilde{\varphi} + \sigma\nabla^2\Delta_2\tilde{\varphi} - \mathbf{v}_s^{''} \cdot \nabla\Delta_2\tilde{\varphi} \tag{37}$$

$$\nabla^2\Delta_2\tilde{\psi} + 2\Omega \cdot \nabla\Delta_2\tilde{\varphi} = \mathbf{v}_s \cdot \nabla\nabla^2\Delta_2\tilde{\psi} + \sigma\Delta_2\tilde{\psi} + \mathbf{k} \cdot (\nabla\Delta_2\tilde{\varphi} \times \mathbf{v}_s^{'}). \tag{38}$$

Again, we are going to focus on disturbances which are x-independent and for which the imaginary part of σ vanishes. In this case the critical disturbances correspond to $\sigma = 0$ and Eqs. (37), (38) reduce to

$$\nabla^4 \partial^2_{yy} \tilde{\varphi} - 2\Omega \partial_y \partial^2_{yy} \tilde{\psi} = 0, \tag{39}$$

$$\nabla^2 \partial^2_{yy} \tilde{\psi} - (Re - 2\Omega) \partial_y \partial^2_{yy} \tilde{\varphi} = 0. \tag{40}$$

In the last formula we have used the expression (26) for $\mathbf{v}_s$. Then we observe that the above equations are identical with the ones without x-dependence, up to a numerical factor in the second term in Eq. (40). So, we can use the solution (32) to write

$$Re_y = 2\Omega + \frac{1708}{2\Omega}. \tag{41}$$

A calculation of the minimum of the above expression gives

$$Re_c = 2\sqrt{1708} \quad \text{corresponding to} \quad 2\Omega = \sqrt{1708}. \tag{42}$$

It can be shown that the energy stability limit coincides with the result just obtained. Therefore, at this point the stability problem is solved completely because of the relation $Re_E \leq Re_G \leq Re_c$ which in this problem attains strict equalities. We see that for large values of Ω Eq. (41) that yields

$$Re < 2\Omega \tag{43}$$

as a condition for stability.

This also can be shown to follow from the Rayleigh stability criterion, $\frac{d(\omega(r)r^2)^2}{dr} \geq 0$, which describes the condition for stability of rotating inviscid fluid to axisymmetric disturbances. In our case assumes the form

$$|\Omega_1 r_1^2| \leq |\Omega_2 r_2^2|. \tag{44}$$

Using the notation in Fig. 2 we can write (44) as

$$\left(\Omega_D + \frac{\Omega_1 - \Omega_2}{2}\right)\left(r_0 - \frac{d}{2}\right)^2 \leq \left(\Omega_D - \frac{\Omega_1 - \Omega_2}{2}\right)\left(r_0 + \frac{d}{2}\right)^2. \tag{45}$$

After expanding and regrouping we obtain

$$\frac{\Omega_1 - \Omega_2}{2}\left(\left(r_0 - \frac{d}{2}\right)^2 + \left(r_0 + \frac{d}{2}\right)^2\right) \leq 2\Omega_D\, r_0\, d. \tag{46}$$

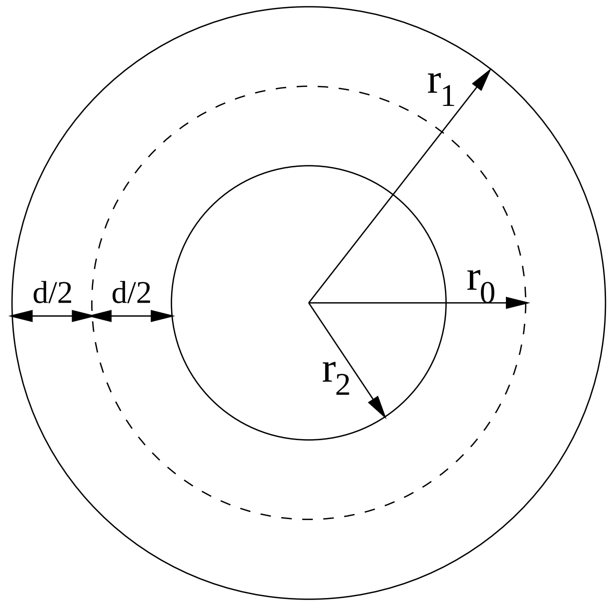

Figure 2: The asymptotic limit for large values of Ω leads to the inequality $Re < 2\Omega$ as a condition for stability.

Remembering the relation between Ω_D and Ω we finally obtain

$$Re\frac{\nu r_0}{d} < 2\Omega\frac{\nu r_0}{d} \tag{47}$$

from which our assertion follows.

References

[1] Busse, F. H. A property of the energy stability limit for plane parallel shear flow. *Arch. Rat. Mech. Anal.* **47**, pp. 28–35 (1972)