High-order Boussinesq models for internal interfacial waves

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1 Introduction

The irrotational flow of an incompressible homogenous inviscid fluid is generally a three-dimensional problem. The main issue of Boussinesq-type equations is to reduce the description to a two-dimensional one by introducing a polynomial approximation to the vertical distribution of the flow into integral conservation laws, while accounting for non-hydrostatic effects due to the vertical acceleration of water. After solving the two-dimensional problem we can easily find the flow properties everywhere in the three-dimensional domain using the polynomial approximation. In this work we apply this method to the flow of internal interfacial waves between two incompressible fluids. In the first part of the work (sections 2 and 3) both fluids are homogeneous and the flow is irrotational either with a rigid-lid or a free surface. In the second part (section 4) the fluids are both exponentially stratified and the flow is irrotational with a rigid-lid (a similar problem with a free surface could be solved as well by using the same techniques). In the third part of the work (section 5) the method is applied to the rotational flow, which is presented by the surface quasi-geostrophic model. Finally, in the last part of the work (section 6) a new layered model based on the surface quasi-geostrophic model is constructed and its dimension is reduced by the Boussinesq method.

2 Internal interfacial waves between two unstratified layers with a rigid-lid

2.1 An infinite series solution for the Laplace equation in each layer over a horizontal bottom

The equations governing the irrotational flow of an incompressible inviscid fluid in the lower layer over a horizontal bottom are:

$$\nabla^2\Phi^{(1)} + \Phi^{(1)}_{zz} = 0 \qquad -h_1 < z < \eta \tag{1}$$

$$\eta_t + \nabla\Phi^{(1)}\nabla\eta - \Phi^{(1)}_z = 0 \qquad z = \eta \tag{2}$$

$$\Phi^{(1)}_t + \frac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 + \frac{1}{2}\left(\Phi^{(1)}_z\right)^2 + g\eta - \frac{P}{\rho_1} = 0 \qquad z = \eta \tag{3}$$

$$\Phi^{(1)}_z = 0 \qquad z = -h_1 \tag{4}$$

The equations governing the irrotational flow of an incompressible inviscid fluid in the upper layer under a rigid lid are:

$$\nabla^2\Phi^{(2)} + \Phi^{(2)}_{zz} = 0 \qquad \eta < z < h_2 \tag{5}$$

$$\eta_t + \nabla\Phi^{(2)}\nabla\eta - \Phi^{(2)}_z = 0 \qquad z = \eta \tag{6}$$

$$\Phi^{(2)}_t + \frac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \frac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta - \frac{P}{\rho_2} = 0 \qquad z = \eta \tag{7}$$

$$\Phi^{(2)}_z = 0 \qquad z = h_2 \tag{8}$$

By the use of (2), (3), (6), and (7) we can construct two interface conditions on $z = \eta$:

$$\rho_1\left(\Phi^{(1)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(1)}_z\right)^2 + g\eta\right) = \rho_2\left(\Phi^{(2)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta\right) \tag{9}$$

$$-\nabla\Phi^{(1)}\nabla\eta + \Phi^{(1)}_z = -\nabla\Phi^{(2)}\nabla\eta + \Phi^{(2)}_z \tag{10}$$

Here Φ is the velocity potential, $h_1$and h_2 are the water depths of the two layers, P is the pressure and η the interface elevation. The horizontal gradient operator relates Φ to the horizontal velocity, u:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathbf{u} = (u, v). \tag{11}$$

For convenience we denote $\Phi_z = W$ and use the hat ($\hat{}$) and tilde ($\tilde{}$) signs to denote the value on $z = 0$ and on $z = \eta$ respectively.

One of the main ideas of Boussinesq-type theories is to reduce the three-dimensional description to a two-dimensional one. The first step towards such a reduction is to introduce an expansion of the velocity potential as a power of series in the vertical coordinates:

$$\Phi(x, y, z, t) = \sum_{n=0}^{\infty} z^n \phi_n(x, y, t). \tag{12}$$

By substituting this expansion into (1) we find

$$\Phi^{(1)}(x, y, z, t) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{z^{2n}}{(2n)!}\nabla^{2n}\phi^{(1)}_0 + \frac{z^{2n+1}}{(2n+1)!}\nabla^{2n}\phi^{(1)}_1\right). \tag{13}$$

This is a series solution with two unknown functions $\phi^{(1)}_0$and $\phi^{(1)}_1$. Note that the velocities in layer 1 at the undisturbed interface are given by

$$\hat{\mathbf{u}}^{(1)} = \nabla\hat{\Phi}^{(1)} = \nabla\phi_0^{(1)}, \quad \hat{W}^{(1)} = \phi_1^{(1)} \tag{14}$$

Now by the use of (13) and (14) the horizontal bottom condition (4) can be expressed as

$$L_c\left\{\hat{W}^{(1)}\right\} + L_s \cdot \left\{\nabla\hat{\Phi}^{(1)}\right\} = 0 \tag{15}$$

with

$$L_c = \sum_{n=0}^{\infty}(-1)^n \frac{h_1^{2n}}{(2n)!}\nabla^{2n}, \qquad L_s = \sum_{n=0}^{\infty}(-1)^n \frac{h_1^{2n+1}}{(2n+1)!}\nabla^{2n+1} \tag{16}$$

where ∇ is the gradient operator when applied to a scalar, and the divergence when applied to a vector. This equation defines a relation between $\hat{W}^{(1)}$ and $\hat{\Phi}^{(1)}$ which is of infinite order in $h\nabla$. The series are convergent if $\Phi^{(1)}$ has a Fourier transform, since they correspond to the analytic functions $\sinh(kh)$ and $\cosh(kh)$ where ik is the Fourier symbol of ∇.

Following Rayleigh (1876) , we may use symbolic notation of Taylor series operators by which (16) can be given in the compact form

$$L_c = \cos(h_1\nabla), \qquad L_s = \sin(h_1\nabla) \tag{17}$$

so that (15) becomes

$$\cos(h_1\nabla)\hat{W}^{(1)} + \sin(h_1\nabla)\nabla\hat{\Phi}^{(1)} = 0 \tag{18}$$

and (13) and its z derivative become

$$\Phi^{(1)}(x,y,z,t) = \cos(z\nabla)\hat{\Phi}^{(1)} + \tfrac{\sin(z\nabla)}{\nabla}\hat{W}^{(1)}, \tag{19}$$

$$W^{(1)}(x,y,z,t) = -\sin(z\nabla)\nabla\hat{\Phi}^{(1)} + \cos(z\nabla)\hat{W}^{(1)}. \tag{20}$$

Using (18) we can easily construct a Dirichlet to Neumann relation

$$\hat{W}^{(1)} = -\tan(h_1\nabla)\nabla\hat{\Phi}^{(1)} \quad @z = 0 \tag{21}$$

and define its operator as

$$G_1 = -\tan(h_1\nabla)\nabla. \tag{22}$$

By applying the same techniques we can construct a similar Dirichlet to Neumann relation for the second layer

$$\hat{W}^{(2)} = \tan(h_2\nabla)\nabla\hat{\Phi}^{(2)} \quad @z = 0 \tag{23}$$

2.2 Constructing the accurate equations for the linear problem

For small amplitude waves in the sense that the ration between the amplitude to the wave length (wave slope) is small the problem becomes linear. We can see that the kinematic and dynamic interface conditions (10) and (9) become

$$\hat{W}^{(1)} = \hat{W}^{(2)} \quad @z = 0 \tag{24}$$

$$\rho_1 \left(\hat{\Phi}_t^{(1)} + g\eta \right) = \rho_2 \left(\hat{\Phi}_t^{(2)} + g\eta \right) \quad @z = 0 \tag{25}$$

By the use of (21), (23) and (24) we can construct a relation between $\hat{\Phi}^{(1)}$ and $\hat{\Phi}^{(2)}$

$$\hat{\Phi}^{(2)} = -\tfrac{\tan(h_1 \nabla)}{\tan(h_2 \nabla)} \hat{\Phi}^{(1)} \quad @z = 0. \tag{26}$$

Let us define this operator as

$$G_2 = -\tfrac{\tan(h_1 \nabla)}{\tan(h_2 \nabla)}. \tag{27}$$

By using the linear version of (2), (26), (27) and the derivative of (25) with respect to time we get

$$(\rho_1 - \rho_2 G_2)\, \hat{\Phi}_{tt}^{(1)} = (\rho_2 - \rho_1)\, g \hat{W}^{(1)} \quad @z = 0 \tag{28}$$

By the use of (21) and (22) equation (28) becomes

$$(\rho_1 - \rho_2 G_2)\, \hat{\Phi}_{tt}^{(1)} = (\rho_2 - \rho_1)\, g G_1 \hat{\Phi}^{(1)} \quad @z = 0 \tag{29}$$

By substituting the linear relations $\nabla = ik$ and $\frac{\partial}{\partial t} = i\omega$ and the definitions of the differential operators (22) and (27) into (29) we get

$$\omega^2 = \frac{(\rho_1 - \rho_2)\, g \tanh(kh_1) \tanh(kh_2)}{\rho_1 \tanh(kh_2) + \rho_2 \tanh(kh_1)} \tag{30}$$

We can see that equation (30) is exactly the well known linear dispersion relation for this problem.

2.3 Constructing approximate equations for the linear problem

One way to use the above equations for solving a general wave problem is by using Fourier and Laplace transforms with the exact linear dispersion (30). A solution using that method have two downsides. The first is the difficulty of computing its integrals and the other downside is that this type of linear solution doesn't help us construct the solution of the nonlinear problem.

We introduce here another method for solving the linear problem. In this method we first need to approximate the infinite differential operators (22) and (27) to finite operators. The accuracy of this solution will relate to the accuracy of the approximations that are used. The simplest way of doing that is by truncating the Taylor series representing these operators. The higher the order of derivatives kept in the Taylor expansions the higher the accuracy. But, there is a better way of approximating these operators.

Padé approximation approximates these operators by a ratio of two power series. It has double the accuracy of the Taylor approximation, while using the same order of derivatives. For clarity we give both the Taylor and the Padé approximations. Taylor (2) represents the Taylor approximation up to the order of ∇^2 (including) and Padé (2,2) represents the Padé approximation up to the order of ∇^2 both in the numerator and the denominator.

Note that the two operators being approximated here are even operators. Therefore, for the Taylor approximation the lowest order term neglected is actually of $O\left(\nabla^4\right)$ and for the Padé approximation it is of $O\left(\nabla^8\right)$.

The Taylor and Padé approximations for relation (21) are

$$\hat{W}^{(1)} = -h_1 \nabla^2 \hat{\Phi}^{(1)} \qquad \text{Taylor (2)} \tag{31}$$

$$\left(1 - \tfrac{1}{3} h_1^2 \nabla^2\right) \hat{W}^{(1)} = -h_1 \nabla^2 \hat{\Phi}^{(1)} \qquad \text{Padé (2,2).} \tag{32}$$

The Taylor and Padé approximations for the coupling relation (26) are

$$\hat{\Phi}^{(2)} = -\tfrac{h_1}{h_2}\left(1 + \tfrac{1}{3}\left(h_1^2 - h_2^2\right)\nabla^2\right)\hat{\Phi}^{(1)} \qquad \text{Taylor (2)} \tag{33}$$

$$\left(15 - \left(6h_1^2 + h_2^2\right)\nabla^2\right)\hat{\Phi}^{(2)} = -\tfrac{h_1}{h_2}\left(15 - \left(h_1^2 + 6h_2^2\right)\nabla^2\right)\hat{\Phi}^{(1)} \qquad \text{Padé (2,2)} \tag{34}$$

Now we can use the Taylor (2) approximations to construct an approximated version of equation (29) by using the (31), (33) and the derivative of (25) with respect to time

$$\left(\left(\rho_1 h_2 + \rho_2 h_1\right) + \tfrac{1}{3}\rho_2 h_1\left(h_1^2 - h_2^2\right)\nabla^2\right)\hat{\Phi}_{tt}^{(1)} = \left(\rho_1 - \rho_2\right) h_1 h_2 g \nabla^2 \hat{\Phi}^{(1)} \tag{35}$$

and by using the Padé (2,2) approximations (32) and (34) we can construct in the same manner the following equation

$$\left(\rho_1 h_2 - \rho_2 h_1\left(15 - \left(h_1^2 + 6h_2^2\right)\nabla^2\right)\right)\hat{\Phi}_{tt}^{(1)} = \left(\rho_2 - \rho_1\right) g h_2\left(15 - \left(6h_1^2 + h_2^2\right)\nabla^2\right)\hat{W}^{(1)}. \tag{36}$$

Notice that we are not substituting equation (32) into (36) because by doing so the order of differentiation will increase resulting in a more complicated method. Therefore, we should first use (32) for computing $\hat{W}^{(1)}$ and then only then substitute $\hat{W}^{(1)}$ into (36) and solve for $\hat{\Phi}^{(1)}$.

2.4 Constructing approximate equations for the nonlinear Dirichlet to Neumann problem

From the Laplace equation (1) we can replace horizontal by vertical differentiation to obtain for layer 1

$$\nabla^{2m}\Phi^{(1)} = (-1)^m \frac{\partial^{2m}\Phi^{(1)}}{\partial z^{2m}} \tag{37}$$

Now we look for $\Phi^{(1)}$ of the form

$$\Phi^{(1)}(x,y,z,t) = \sum_{n=0}^{\infty}(z+h_1)^n \phi_n^{(1)}(x,y,t). \tag{38}$$

From (38) it follows that

$$\frac{\partial^m \Phi^{(1)}}{\partial z^m} = \sum_{n=0}^{\infty} \frac{n!}{(n-m)!}(z+h_1)^{n-m}\phi_n^{(1)}(x,y,t) \tag{39}$$

By substituting (38) into the Laplace equation (1) and collecting equal powers of z we obtain a recursive relation

$$\phi_{n+2}^{(1)} = -\frac{\nabla^2\phi_n^{(1)}}{(n+1)(n+2)}. \tag{40}$$

On the horizontal bottom, boundary condition (4) leads to

$$\phi_1^{(1)} = 0. \tag{41}$$

This implies from (40) that all $\phi_n^{(1)}$'s with odd n vanish

$$\phi_1^{(1)} = \phi_3^{(1)} = \phi_5^{(1)} = ... = 0. \tag{42}$$

Now by the use of (38), (37) and (39) we can write $\hat{\Phi}^{(1)}$, $\nabla^2\hat{\Phi}^{(1)}$, $\hat{W}^{(1)}$ and $\nabla^2\hat{W}^{(1)}$, which relate to the properties of the flow on $z=0$ as

$$\begin{aligned} \hat{\Phi}^{(1)} &= \sum_{n=0}^{3} h_1^{2n}\phi_{2n}^{(1)} & \hat{W}^{(1)} &= \sum_{n=0}^{3} 2n h_1^{2n-1}\phi_{2n}^{(1)} \\ \nabla^2\hat{\Phi}^{(1)} &= -\sum_{n=0}^{3}\frac{(2n)!}{(2n-2)!}h_1^{2n}\phi_{2n}^{(1)} & \nabla^2\hat{W}^{(1)} &= -\sum_{n=0}^{3}\frac{(2n)!}{(2n-3)!}h_1^{2n}\phi_{2n}^{(1)} \end{aligned} \tag{43}$$

Notice that (38) was truncated to use only 4 base functions. This enables us to solve for them using the 4 equations written in (43). Had we wanted to use more base functions for higher accuracy, we would have needed to increase the number of equations by adding equations for higher order of differentiation to equation set (43). By solving equation set (43) for the base functions $\phi_{2n}^{(1)}$, $n = 0..3$ in terms of $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$ and their second horizontal derivatives we get

$$
\begin{aligned}
\phi_0^{(1)} &= \frac{1}{48}\left(-9h_1^2\nabla^2\hat{\Phi}^{(1)} + h_1^3\nabla^2\hat{W}^{(1)} - 33h_1\hat{W}^{(1)} + 48\hat{\Phi}^{(1)}\right)\\
\phi_2^{(1)} &= \frac{1}{16}\left(7\nabla^2\hat{\Phi}^{(1)} - h_1\nabla^2\hat{W}^{(1)} + \frac{15}{h_1}\hat{W}^{(1)}\right)\\
\phi_4^{(1)} &= \frac{1}{16}\left(-\frac{5}{h_1^2}\nabla^2\hat{\Phi}^{(1)} + \frac{1}{h_1}\nabla^2\hat{W}^{(1)} - \frac{5}{h_1^3}\hat{W}^{(1)}\right)\\
\phi_6^{(1)} &= \frac{1}{48}\left(\frac{3}{h_1^4}\nabla^2\hat{\Phi}^{(1)} - \frac{1}{h_1^3}\nabla^2\hat{W}^{(1)} + \frac{3}{h_1^5}\hat{W}^{(1)}\right)
\end{aligned} \tag{44}
$$

Using the solution (44) in (38) on $z = \eta$ we get a relation involving $\hat{\Phi}^{(1)}$, $\hat{W}^{(1)}$ and $\tilde{\Phi}^{(1)}$:

$$
\begin{aligned}
\tilde{\Phi}^{(1)} = \frac{\eta^3\left(\eta + 2h_1^2\right)^3}{48h_1^3}\nabla^2\hat{W}^{(1)} + \frac{\eta^2\left(\eta + 2h_1^2\right)^2\left(\eta^2 + 2\eta h_1 - 2h_1^2\right)}{16h_1^4}\nabla^2\hat{\Phi}^{(1)}\\
+\frac{16h_1^5\eta + 10h_1^2\eta^4 + 6h_1\eta^5 + \eta^6}{16h_1^5}\hat{W}^{(1)} + \hat{\Phi}^{(1)}
\end{aligned} \tag{45}
$$

Equations (15) and (45) are two equations for $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$. We can solve for $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$using a numerical method such as finite differences. The set of equation will have the form

$$
\begin{bmatrix} A_{\text{padé}}^{(1)} & -B_{\text{padé}}^{(1)} \\ A_{\text{nl}}^{(1)} & B_{\text{nl}}^{(1)} \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{(1)} \\ \hat{W}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \tilde{\Phi}^{(1)} \end{bmatrix}. \tag{46}
$$

Here $A_{\text{padé}}^{(1)}$ and $B_{\text{padé}}^{(1)}$ are the finite difference matrices representing the Dirichlet to Neumann relation (32), $A_{\text{nl}}^{(1)}$ and $B_{\text{nl}}^{(1)}$ are the finite difference matrices representing the relation between the undisturbed potential and vertical velocity and the potential on the interfacial wave. Now by using $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$ we can find the base functions and from the base functions we can find the properties of the flow at every point in the layer.

Now let us present a way to calculate the properties of the flow in the upper layer. We can easily find $\tilde{W}^{(2)}$ using (10). Applying the same technique described in subsection 2.1 we can construct an equivalent version of equations (19) and (20) for the upper layer

$$
\Phi^{(2)}(x,y,z,t) = \cos(z\nabla)\hat{\Phi}^{(2)} + \tfrac{\sin(z\nabla)}{\nabla}\hat{W}^{(2)}, \tag{47}
$$
$$
W^{(2)}(x,y,z,t) = -\sin(z\nabla)\nabla\hat{\Phi}^{(2)} + \cos(z\nabla)\hat{W}^{(2)}. \tag{48}
$$

On $z = \eta$we can use equations (47) and (48) to solve $\hat{\Phi}^{(2)}$ and $\hat{W}^{(2)}$ and substitute the solution into (47) and (48) to get $\Phi^{(2)}$ and $W^{(2)}$ at every point in the layer.

2.5 Constructing the equations for the nonlinear time marching

An analytical relation between $\tilde{\Phi}_t^{(1)}$ and $\tilde{\Phi}_t^{(2)}$must be developed in order to enable marching $\tilde{\Phi}$ in time using the dynamical interface boundary condition (9). By applying the solution

(44) to the z derivative of (38) on $z=\eta$ we get a second relation between $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$with respect to $\tilde{W}^{(1)}$. By using this relation, equation (45) and equation (15) we can eliminate $\hat{\Phi}^{(1)}$ and $\hat{W}^{(1)}$and obtain an analytical Dirichlet to Neumann relation on the interfacial wave

$$\left(1-\left(\frac{1}{2}\eta^2+\eta h_1\right)\nabla^2\right)\tilde{\Phi}^{(1)}=-\left(\eta+h_1\right)\nabla^2\tilde{W}^{(1)} \qquad \text{Taylor (2)} \tag{49}$$

$$\left(1-\left(\frac{1}{2}\eta^2+\eta h_1+\frac{3}{7}h_1^2\right)\nabla^2\right)\tilde{\Phi}^{(1)}=-\left(\eta+h_1\right)\nabla^2\tilde{W}^{(1)} \qquad \text{Padé (2,2)} \tag{50}$$

A similar relation can be constructed for the second layer

$$\left(1-\left(\frac{1}{2}\eta^2-\eta h_2\right)\nabla^2\right)\tilde{\Phi}^{(2)}=-\left(\eta-h_2\right)\nabla^2\tilde{W}^{(2)} \qquad \text{Taylor (2)} \tag{51}$$

$$\left(1-\left(\frac{1}{2}\eta^2-\eta h_2+\frac{3}{7}h_2^2\right)\nabla^2\right)\tilde{\Phi}^{(2)}=-\left(\eta-h_2\right)\nabla^2\tilde{W}^{(2)} \qquad \text{Padé (2,2)} \tag{52}$$

Now by substituting equations (49) and (51) into the kinematic interface boundary condition (10) we get a relation between $\tilde{\Phi}^{(1)}$ and $\tilde{\Phi}^{(2)}$ for Taylor (2) approximation

$$\begin{aligned}&(\eta-h_2)\left(1-\nabla^2\eta\nabla-\left(\frac{1}{2}\eta^2+\eta h_1+2\nabla^2\eta\right)\nabla^2\right)\tilde{\Phi}^{(1)}\\&=(\eta+h_1)\left(1-\nabla^2\eta\nabla-\left(\frac{1}{2}\eta^2-\eta h_2+2\nabla^2\eta\right)\nabla^2\right)\tilde{\Phi}^{(2)}\end{aligned} \tag{53}$$

deriving this relation with respect to time yields a relation between $\tilde{\Phi}_t^{(1)}$ and $\tilde{\Phi}_t^{(2)}$
For Taylor (2) approximation

$$\begin{gathered}(\eta-h_2)\left(1-\nabla^2\eta\nabla-\left(\tfrac{1}{2}\eta^2+\eta h_1+2\nabla^2\eta\right)\nabla^2\right)\tilde{\Phi}_t^{(1)}+\\\left(-\eta_t+\left(\nabla^3\eta_t\left(\eta-h_2\right)+\nabla^3\eta\eta_t\right)\nabla\right)\tilde{\Phi}^{(1)}\\+\left(2\nabla^3\eta_t\left(\eta-h_2\right)+\left(2\nabla^2\eta+\tfrac{3}{2}\eta^2-h_2\eta-h_1\left(-2\eta+h_2\right)\right)\eta_t\right)\nabla^2\tilde{\Phi}^{(1)}\\=(\eta+h_1)\left(1-\nabla^2\eta\nabla-\left(\tfrac{1}{2}\eta^2-\eta h_2+2\nabla^2\eta\right)\nabla^2\right)\tilde{\Phi}_t^{(2)}+\\\left(-\eta_t+\left(\nabla^3\eta_t\left(\eta+h_1\right)+\nabla^3\eta\eta_t\right)\nabla\right)\tilde{\Phi}^{(2)}+\\\left(2\nabla^3\eta_t\left(\eta+h_1\right)+\left(2\nabla^2\eta+\tfrac{3}{2}\eta^2+h_1\eta-h_2\left(2\eta+h_1\right)\right)\eta_t\right)\nabla^2\tilde{\Phi}^{(2)}\end{gathered} \tag{54}$$

which together with the dynamical interface boundary condition (9) enable us to solve for $\tilde{\Phi}_t^{(1)}$ and $\tilde{\Phi}_t^{(2)}$ and then propagate them in time. The above derivatives should be handle with care because $\tilde{\Phi}^{(1)}$ and $\tilde{\Phi}^{(2)}$ are located on the interfacial wave, which means that in each location they are located on a different elevation (η). The horizontal derivatives can be locally related to the tangential ($\nabla\tilde{\Phi}\cdot\hat{t}$) derivatives, the elevation (η) and the vertical derivatives $\hat{W}^{(1)}$ and $\hat{W}^{(2)}$ using this relation

$$\nabla\tilde{\Phi}=(\nabla\tilde{\Phi}\cdot\hat{t})+\left(\tilde{W}-\frac{\nabla\eta}{|\langle 1,\nabla\eta\rangle|}\left(\nabla\tilde{\Phi}\cdot\hat{t}\right)\right)\frac{\nabla\eta}{|\langle 1,\nabla\eta\rangle|} \tag{55}$$

3 Internal interfacial waves between two unstratified layers with a free surface

3.1 An infinite series expansion for Φ in each layer over a horizontal bottom

The equations governing the irrotational flow of an incompressible inviscid fluid in the lower layer over a horizontal bottom are shown in equations (1)-(4). The equations governing the irrotational flow of an incompressible inviscid fluid in the second layer under a free surface boundary condition are:

$$\nabla^2\Phi^{(2)} + \Phi^{(2)}_{zz} = 0 \qquad \eta < z < h_2 \tag{56}$$

$$\eta_t + \nabla\Phi^{(2)}\nabla\eta - \Phi^{(2)}_z = 0 \qquad z = \eta \tag{57}$$

$$\Phi^{(2)}_t + \frac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \frac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta - \frac{P}{\rho_2} = 0 \qquad z = \eta \tag{58}$$

$$\xi_t + \nabla\Phi^{(2)}\nabla\eta - \Phi^{(2)}_z = 0 \qquad z = h_2 + \xi \tag{59}$$

$$\Phi^{(2)}_t + \frac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \frac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\xi = 0 \qquad z = h_2 + \xi \tag{60}$$

By the use of (2), (3), (57), and (58) we can construct two interface boundary conditions on $z = \eta$:

$$\rho_1\left(\Phi^{(1)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(1)}_z\right)^2 + g\eta\right) = \rho_2\left(\Phi^{(2)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta\right) \tag{61}$$

$$-\nabla\Phi^{(1)}\nabla\eta + \Phi^{(1)}_z = -\nabla\Phi^{(2)}\nabla\eta + \Phi^{(2)}_z \tag{62}$$

Here Φ is the velocity potential, $h_1$and h_2 are the water depths of the two layers, P is the pressure, η is the interface elevation and ξ is the free surface elevation. The horizontal gradient operator relates the horizontal velocity $\mathbf{u}$ to Φ:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathbf{u} = (u, v) = \nabla\Phi \tag{63}$$

For convenience we denote

$$\begin{aligned} \hat{\hat{\Phi}}^{(2)} &= \Phi^{(2)}(z = h_1)\,, \quad \tilde{\tilde{\Phi}}^{(2)} = \Phi^{(2)}(z = h_2 + \xi) \\ \hat{\hat{W}}^{(2)} &= W^{(2)}(z = h_1)\,, \quad \tilde{\tilde{W}}^{(2)} = W^{(2)}(z = h_2 + \xi) \end{aligned} \tag{64}$$

The work on the Dirichlet to Neumann relation that has been done for the first layer in the previous section holds for this section. Its linear part can be summarize in equations (15) and (17). For the second layer we again introduce an expansion of the velocity potential as a power of series in z:

$$\Phi^{(2)}(x, y, z, t) = \sum_{n=0}^{\infty} \bar{z}^n \phi^{(2)}_n(x, y, t) \tag{65}$$

Here $\bar{z} = z - h_2$. Now by substituting this expansion into (56) we find

$$\Phi^{(2)} = \cos(\bar{z}\nabla)\hat{\hat{\Phi}}^{(2)} + \frac{\sin(\bar{z}\nabla)}{\nabla}\hat{\hat{W}}^{(2)} \tag{66}$$

By using the above equation we can construct a linear relation between the interfacial wave and the free surface

$$\hat{W}^{(2)} = \sin(h_2\nabla)\nabla\hat{\hat{\Phi}}^{(2)} + \cos(h_2\nabla)\hat{\hat{W}}^{(2)} \tag{67}$$

3.2 Constructing accurate equations for the linear problem

For the linear problem that the kinematic and dynamic interface boundary conditions (62) and (61) become

$$\hat{W}^{(1)} = \hat{W}^{(2)} \quad @z = 0, \tag{68}$$

$$\rho_1\left(\hat{\Phi}_t^{(1)} + g\eta\right) = \rho_2\left(\hat{\Phi}_t^{(2)} + g\eta\right) \quad @z = 0. \tag{69}$$

The free surface boundary conditions (59) and (60) become

$$\xi_t - \Phi_z^{(2)} = 0 \qquad @z = h_2, \tag{70}$$

$$\Phi_t^{(2)} + g\xi = 0 \qquad @z = h_2. \tag{71}$$

By substituting (70) into the time derivative of (71) we get

$$\hat{\hat{\Phi}}_{tt}^{(2)} = -g\hat{\hat{W}}^{(2)} \quad @z = h_2. \tag{72}$$

Substituting (68) into (67) yields

$$\hat{\hat{W}}^{(2)} = -\tan(h_2\nabla)\nabla\hat{\hat{\Phi}}^{(2)} + \frac{1}{\cos(h_2\nabla)}\hat{W}^{(1)} \tag{73}$$

By substituting (21) into (73) and then into (66) on $z = 0$ and deriving twice with respect to time we get

$$\hat{\Phi}_{tt}^{(2)} = \frac{1}{\cos(h_2\nabla)}\hat{\hat{\Phi}}_{tt}^{(2)} + \tan(h_1\nabla)\tan(h_2\nabla)\hat{\hat{\Phi}}_{tt}^{(1)} \tag{74}$$

By using the linear version of (57), (72), (74) and the derivative of (69) with respect to time we get

$$\left(\rho_1 - \rho_2\tan(h_1\nabla)\tan(h_2\nabla)\right)\hat{\Phi}_{tt}^{(1)} = -\rho_2\cos^{-1}(h_2\nabla)\hat{\hat{W}}^{(2)} + \left(\rho_2 - \rho_1\right)g\hat{W}^{(1)} \tag{75}$$

For solving the linear problem we need to solve for $\hat{W}^{(1)}$ using (21), then solve for $\hat{\hat{W}}^{(2)}$ using (73). After this we can use equations (72) and (75) in order to find the potentials $\hat{\hat{\Phi}}^{(2)}$ and $\hat{\Phi}^{(1)}$.

3.3 Constructing the equations for solving the Dirichlet to Neumann relation for the nonlinear problem

For the first layer we can use the same equations (15) and (45) that have been developed in section 2.4 and in the same way find $\tilde{W}^{(1)}$. For the second layer let us use (66) and (62) to obtain

$$-\left(\nabla\eta\cos\left(\bar{\eta}\nabla\right)\nabla+\sin\left(\bar{\eta}\nabla\right)\right)\hat{\hat{\Phi}}^{(2)}-$$
$$\left(\nabla\eta\sin\left(\bar{\eta}\nabla\right)\nabla-\cos\left(\bar{\eta}\nabla\right)\right)\hat{\hat{W}}^{(2)}=-\nabla\tilde{\Phi}^{(1)}\nabla\eta+\tilde{W}^{(1)}. \tag{76}$$

Here $\bar{\eta}=\eta-h_2$. By using this equation with (66) on $z=h_2+\xi$ we can solve for $\hat{\hat{\Phi}}^{(2)}$and $\hat{\hat{W}}^{(2)}$.

3.4 Constructing equations for the nonlinear time marching

Again an analytical relation between $\tilde{\Phi}_t^{(1)}$ and $\tilde{\Phi}_t^{(2)}$must be developed in order to get equations for $\tilde{\Phi}$ using the dynamical interface boundary condition (61). Now by using (66) we get three equations relating $\tilde{\Phi}^{(2)}$, $\tilde{W}^{(2)}$ and $\tilde{\tilde{\Phi}}^{(2)}$ to $\hat{\hat{\Phi}}^{(2)}$and $\hat{\hat{W}}^{(2)}$. Eliminating $\hat{\hat{\Phi}}^{(2)}$ and $\hat{\hat{W}}^{(2)}$ from these three equations leaves a relation between $\tilde{W}^{(2)}$, $\tilde{\Phi}^{(2)}$ and $\tilde{\tilde{\Phi}}^{(2)}$

$$\tilde{W}^{(2)}=-\cot\left(\left(h_2+\xi-\eta\right)\nabla\right)\nabla\tilde{\Phi}^{(2)}+\sin^{-1}\left(\left(h_2+\xi-\eta\right)\nabla\right)\nabla\tilde{\tilde{\Phi}}^{(2)}. \tag{77}$$

By using (13) we can define $\tilde{\Phi}^{(1)}$ and $\tilde{W}^{(1)}$ with respect to $\hat{\Phi}^{(1)}$and $\hat{W}^{(1)}$together with (15) we can find a relation between $\tilde{W}^{(1)}$ and $\tilde{\Phi}^{(1)}$:

$$\tilde{W}^{(1)}=\frac{\cos\left(\left(h_1-\eta\right)\nabla\right)}{\cos\left(\left(h_1+\eta\right)\nabla\right)}\nabla\tilde{\Phi}^{(1)}. \tag{78}$$

Now let us substitute these two relations (77), (78) into the kinematic interface boundary condition (62) and derive it with respect to time to get

$$\begin{aligned}
\big(\nabla\eta_t-\eta_t\cos\left(2h_1\nabla\right)\sin^{-2}\left(\left(h_1+\eta\right)\nabla\right)\nabla\big)\,\tilde{\Phi}^{(1)}&-\\
\big(\nabla\eta_t-\left(\xi_t-\eta_t\right)\sin^{-2}\left(\left(h_2+\xi-\eta\right)\nabla\right)\nabla\big)\,\tilde{\Phi}^{(2)}&+\\
\left(\xi_t-\eta_t\right)\cot\left(\left(h_2+\xi-\eta\right)\nabla\right)\sin^{-1}\left(\left(h_2+\xi-\eta\right)\nabla\right)\nabla\tilde{\tilde{\Phi}}^{(2)}&+\\
\sin^{-1}\left(\left(h_2+\xi-\eta\right)\nabla\right)\tilde{\tilde{\Phi}}_t^{(2)}&+\\
\big(\nabla\eta+\cos^{-1}\left(\left(h_1+\eta\right)\nabla\right)\sin\left(\left(h_1-\eta\right)\nabla\right)\big)\,\tilde{\Phi}_t^{(1)}&-\\
\left(\nabla\eta+\cot\left(\left(h_2+\xi-\eta\right)\nabla\right)\right)\tilde{\Phi}_t^{(2)}&=0
\end{aligned} \tag{79}$$

Equation (79) together with equations (60) and (61) is a set of equations for $\tilde{\Phi}^{(1)}$, $\tilde{\Phi}^{(2)}$ and $\tilde{\tilde{\Phi}}^{(2)}$ and equations (2) and (59) are equations for η and ξ.

4 Internal interfacial waves between two exponentially stratified layers with a rigid-lid

4.1 An infinite series solution for the Laplace equation in each layer over a horizontal bottom

The equations governing the irrotational flow of an incompressible inviscid fluid in the lower layer over a horizontal bottom are:

$$\nabla^2\Phi^{(1)} + \Phi^{(1)}_{zz} + \frac{\frac{d\rho_1}{dz}}{\rho_1}\Phi^{(1)}_z = 0 \qquad -h_1 < z < \eta \tag{80}$$

$$\eta_t + \nabla\Phi^{(1)}\nabla\eta - \Phi^{(1)}_z = 0 \qquad z = \eta \tag{81}$$

$$\Phi^{(1)}_t + \frac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 + \frac{1}{2}\left(\Phi^{(1)}_z\right)^2 + g\eta - \frac{P}{\rho_1} = 0 \qquad z = \eta \tag{82}$$

$$\Phi^{(1)}_z = 0 \qquad z = -h_1 \tag{83}$$

The equations governing the irrotational flow of an incompressible inviscid fluid in the upper layer under a rigid-lid are:

$$\nabla^2\Phi^{(2)} + \Phi^{(2)}_{zz} + \frac{\frac{d\rho_2}{dz}}{\rho_2}\Phi^{(2)}_z = 0 \qquad \eta < z < h_2 \tag{84}$$

$$\eta_t + \nabla\Phi^{(2)}\nabla\eta - \Phi^{(2)}_z = 0 \qquad z = \eta \tag{85}$$

$$\Phi^{(2)}_t + \frac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \frac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta - \frac{P}{\rho_2} = 0 \qquad z = \eta \tag{86}$$

$$\Phi^{(2)}_z = 0 \qquad z = h_2 \tag{87}$$

By the use of (81), (82), (85), and (86) we can construct two interface conditions on $z = \eta$:

$$\begin{aligned}\rho_1\left(\Phi^{(1)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(1)}_z\right)^2 + g\eta\right) &= \\ \rho_2\left(\Phi^{(2)}_t + \tfrac{1}{2}\left(\nabla\Phi^{(2)}\right)^2 + \tfrac{1}{2}\left(\Phi^{(2)}_z\right)^2 + g\eta\right)\end{aligned} \tag{88}$$

$$-\nabla\Phi^{(1)}\nabla\eta + \Phi^{(1)}_z = -\nabla\Phi^{(2)}\nabla\eta + \Phi^{(2)}_z \tag{89}$$

Here Φ is the velocity potential, h_1 and h_2 are the water depths of the two layers, P is the pressure and η the interface elevation. The horizontal gradient operator relates Φ to the horizontal velocity, $\mathbf{u}$:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathbf{u} = (u, v). \tag{90}$$

For convenience we denote $\Phi_z = W$ and use the hat ($\hat{}$) and tilde ($\tilde{}$) signs to denote the value on $z = 0$ and on $z = \eta$ respectively.

For an exponentially stratified fluid the ratios $\frac{\frac{d\rho_1}{dz}}{\rho_1}$ and $\frac{\frac{d\rho_2}{dz}}{\rho_2}$ are constant and will be regarded as small parameters

$$\varepsilon_1 = \frac{\frac{d\rho_1}{dz}}{\rho_1}, \qquad \varepsilon_2 = \frac{\frac{d\rho_2}{dz}}{\rho_2}.$$

Notice that for linearly stratified fluid ε_1 and ε_2 will no longer be constants but functions of z. Nevertheless, their z derivatives will be of $O\left(\varepsilon^2\right)$ and every additional z derivative will further more increase the order, so if we choose to have an accuracy of $O\left(\varepsilon\right)$ the following derivations hold for linearly stratified fluids as well.

Now let us introduce an expansion of the velocity potential as a power series in the vertical coordinates with a small perturbation related to the stratification:

$$\begin{aligned}\Phi^{(1)}(x,y,z,t) &= \Phi_0^{(1)}(x,y,z,t) + \varepsilon_1\Phi_1^{(1)}(x,y,z,t) = \\ &= \textstyle\sum_{n=0}^{\infty} z^n \left(\phi_{0,n}(x,y,t) + \varepsilon_1\phi_{1,n}(x,y,t)\right).\end{aligned} \tag{91}$$

By substituting (91) into (80) and collecting equal powers of z we obtain recursive relations

$$\phi_{0,n+2}^{(1)} = -\frac{\nabla^2\phi_{0,n}^{(1)}}{(n+1)(n+2)}, \qquad \phi_{1,n+2}^{(1)} = -\frac{\nabla^2\phi_{1,n}^{(1)}}{(n+1)(n+2)} - \frac{\phi_{0,n+1}^{(1)}}{(n+2)}. \tag{92}$$

Now by substituting (92) into (91) we get

$$\begin{gathered}\Phi^{(1)}(x,y,z,t) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{z^{2n}}{(2n)!}\nabla^{2n}\phi_{0,0}^{(1)} + \frac{z^{2n+1}}{(2n+1)!}\nabla^{2n}\phi_{0,1}^{(1)}\right) + \\ \sum_{n=0}^{\infty}(-1)^n\varepsilon_1\left(\frac{z^{2n}}{(2n)!}\nabla^{2n}\phi_{1,0}^{(1)} - \frac{z^{2n+1}n}{(2n+1)!}\nabla^{2n}\phi_{0,0}^{(1)} + \frac{z^{2n+1}}{(2n+1)!}\nabla^{2n}\phi_{1,1}^{(1)} + \frac{z^{2n}n}{(2n)!}\nabla^{2n-2}\phi_{0,1}^{(1)}\right)\end{gathered} \tag{93}$$

This is a series solution with 4 unknown functions $\phi_{0,0}^{(1)}$ and $\phi_{0,1}^{(1)}$, $\phi_{1,0}^{(1)}$ and $\phi_{1,1}^{(1)}$. Note that the velocities in layer 1 at the undisturbed interface $z=0$ are given by

$$\begin{aligned}\hat{\mathbf{u}}^{(1)} &= \nabla\hat{\Phi}_0^{(1)} + \varepsilon_1\nabla\hat{\Phi}_1^{(1)} = \nabla\phi_{0,0}^{(1)} + \varepsilon_1\nabla\phi_{1,0}^{(1)}, \\ \hat{W}^{(1)} &= \hat{W}_0^{(1)} + \varepsilon_1\hat{W}_1^{(1)} = \phi_{0,1}^{(1)} + \varepsilon_1\phi_{1,1}^{(1)}. \qquad @z=0\end{aligned} \tag{94}$$

By using Rayleigh's notation and (94) we can write equation (93) as

$$\begin{gathered}\Phi^{(1)}(x,y,z,t) = \left(\cos(z\nabla) + \tfrac{1}{2}\varepsilon_1 z\cos(z\nabla) + \varepsilon_1\tfrac{\sin(z\nabla)}{2\nabla}\right)\hat{\Phi}_0^{(1)} + \\ \left(\tfrac{\sin(z\nabla)}{\nabla} - \varepsilon_1\tfrac{z\sin(z\nabla)}{2\nabla}\right)\hat{W}_0^{(1)} + \varepsilon_1\cos(z\nabla)\hat{\Phi}_1^{(1)} + \varepsilon_1\tfrac{\sin(z\nabla)}{\nabla}\hat{W}_1^{(1)}\end{gathered} \tag{95}$$

Equation (95) enables us to write $\nabla\Phi^{(1)}$ and $W^{(1)}$:

$$\nabla\Phi^{(1)} = \left(\left(1+\tfrac{1}{2}\varepsilon_1 z\right)\cos(z\nabla)\nabla + \tfrac{1}{2}\varepsilon_1\sin(z\nabla)\right)\hat{\Phi}_0^{(1)} +$$
$$\left(1-\tfrac{1}{2}\varepsilon_1 z\right)\sin(z\nabla)\hat{W}_0^{(1)} + \varepsilon_1\cos(z\nabla)\nabla\hat{\Phi}_1^{(1)} + \varepsilon_1\sin(z\nabla)\hat{W}_1^{(1)} \tag{96}$$

$$W^{(1)} = \left(\varepsilon_1\cos(z\nabla) - \left(1+\tfrac{1}{2}\varepsilon_1 z\right)\sin(z\nabla)\nabla\right)\hat{\Phi}_0^{(1)} +$$
$$\left(\left(1-\tfrac{1}{2}\varepsilon_1 z\right)\cos(z\nabla) - \tfrac{1}{2}\varepsilon_1\tfrac{\sin(z\nabla)}{\nabla}\right)\hat{W}_0^{(1)} - \varepsilon_1\sin(z\nabla)\nabla\hat{\Phi}_1^{(1)} + \varepsilon_1\cos(z\nabla)\hat{W}_1^{(1)} \tag{97}$$

Now with the use of (97) and (94) the horizontal bottom condition (83) can be expressed as

$$\begin{aligned} O\left(\varepsilon_1^0\right): &\quad \hat{W}_0^{(1)} = -\tan(h_1\nabla)\nabla\hat{\Phi}_0^{(1)} \\ O\left(\varepsilon_1^1\right): &\quad \hat{W}_1^{(1)} = -\tan(h_1\nabla)\nabla\hat{\Phi}_1^{(1)} - \tfrac{1}{2}h_1\tan^2(h_1\nabla)\hat{\Phi}_0^{(1)} \end{aligned} \tag{98}$$

A similar relation can be found for the upper layer

$$\begin{aligned} O\left(\varepsilon_2^0\right): &\quad \hat{W}_0^{(2)} = \tan(h_2\nabla)\nabla\hat{\Phi}_0^{(2)} \\ O\left(\varepsilon_2^1\right): &\quad \hat{W}_1^{(2)} = \tan(h_2\nabla)\nabla\hat{\Phi}_1^{(2)} + \tfrac{1}{2}h_2\tan^2(h_2\nabla)\hat{\Phi}_0^{(2)} \end{aligned} \tag{99}$$

4.2 Constructing the accurate equations for the linear problem

For small amplitude waves in the sense that the ration between the amplitude to the wave length (wave slope) is small the problem becomes linear. We can see that the kinematic and dynamic interface conditions (89) and (88) become

$$\hat{W}^{(1)} = \hat{W}^{(2)} \qquad @z=0 \tag{100}$$
$$\rho_1\left(\hat{\Phi}_t^{(1)} + g\eta\right) = \rho_2\left(\hat{\Phi}_t^{(2)} + g\eta\right) \qquad @z=0 \tag{101}$$

By the use of (98), (99) and (100) we can construct relations between $\hat{\Phi}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$, $\hat{\Phi}_0^{(2)}$ and $\hat{\Phi}_1^{(2)}$

$$\begin{aligned} \hat{\Phi}_0^{(2)} &= -\tfrac{\tan(h_1\nabla)}{\tan(h_2\nabla)}\hat{\Phi}_0^{(1)}, \\ \hat{\Phi}_1^{(2)} &= \tfrac{h_2\tan(h_1\nabla)}{2\nabla}\hat{\Phi}_0^{(1)} - \tfrac{\varepsilon_1}{\varepsilon_2}\left(\tfrac{\tan(h_1\nabla)}{\tan(h_2\nabla)}\hat{\Phi}_1^{(1)} + \tfrac{h_1\tan^2(h_1\nabla)}{2\tan(h_2\nabla)\nabla}\hat{\Phi}_0^{(1)}\right) \quad @z=0 \end{aligned} \tag{102}$$

By using (102), the linear version of (82) and the time derivative of (101) we get

$$O\left(\varepsilon_1^0\right): \quad \left(\rho_1 + \rho_2\tfrac{\tan(h_1\nabla)}{\tan(h_2\nabla)}\right)\tfrac{\partial^2}{\partial t^2}\hat{\Phi}_0^{(1)} = \left(\rho_2-\rho_1\right)g\hat{W}_0^{(1)}$$

$$\begin{aligned} O\left(\varepsilon_1^1\right): &\quad \left(\rho_1 + \rho_2\tfrac{\varepsilon_1}{\varepsilon_2}\tfrac{\tan(h_1\nabla)}{\tan(h_2\nabla)}\right)\tfrac{\partial^2}{\partial t^2}\hat{\Phi}_1^{(1)} = \\ &\left(\rho_2-\rho_1\right)g\hat{W}_1^{(1)} - \left(\tfrac{h_2\tan(h_1\nabla)}{2\nabla} + \tfrac{\varepsilon_1}{\varepsilon_2}\tfrac{h_1\tan^2(h_1\nabla)}{2\tan(h_2\nabla)\nabla}\right)\tfrac{\partial^2}{\partial t^2}\hat{\Phi}_0^{(1)} \end{aligned} \tag{103}$$

Now by using equations (103) and (98) we can solve $\hat{\Phi}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$, $\hat{W}_0^{(1)}$ and $\hat{W}_1^{(1)}$.

4.3 Constructing approximate equations for the nonlinear Dirichlet to Neumann problem

Equation (80) allows us to replace horizontal differentiation by vertical ones in order to obtain

$$\nabla^{2m}\Phi^{(1)} = (-1)^m \left(\frac{\partial^{2m}\Phi^{(1)}}{\partial z^{2m}} + m\varepsilon \frac{\partial^{2m-1}\Phi^{(1)}}{\partial z^{2m-1}} \right) \tag{104}$$

By extending one of the main Boussinesq concepts we look for $\Phi^{(1)}$ of the form

$$\Phi^{(1)}(x, y, z, t) = \sum_{n=0}^{\infty} (z + h_1)^n \left(\phi_{0,n}(x, y, t) + \varepsilon_1 \phi_{1,n}(x, y, t)\right). \tag{105}$$

From (105) it follows that

$$\frac{\partial^m \Phi^{(1)}}{\partial z^m} = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} (z + h_1)^{n-m} \left(\phi_{0,n}(x, y, t) + \varepsilon_1 \phi_{1,n}(x, y, t)\right). \tag{106}$$

Now by using (104) and (106) we get

$$\begin{gathered}\nabla^{2m}\Phi^{(1)} = (-1)^m \textstyle\sum_{n=2m}^{\infty} \frac{n!}{(n-2m)!} (z + h_1)^{n-2m} \phi_{0,n} + \\ \varepsilon_1 (-1)^m \left(\textstyle\sum_{n=2m}^{\infty} \frac{n!}{(n-2m)!} (z + h_1)^{n-2m} \phi_{1,n} + m \textstyle\sum_{n=2m-1}^{\infty} \frac{n!}{(n-2m+1)!} (z + h_1)^{n-2m+1} \phi_{0,n} \right) + O\left(\varepsilon_1^2\right),\end{gathered} \tag{107}$$

$$\begin{gathered}\nabla^{2m}W^{(1)} = (-1)^m \textstyle\sum_{n=2m+1}^{\infty} \frac{n!}{(n-2m-1)!} (z + h_1)^{n-2m-1} \phi_{0,n} + \\ \varepsilon_1 (-1)^m \left(\textstyle\sum_{n=2m+1}^{\infty} \frac{n!}{(n-2m-1)!} (z + h_1)^{n-2m-1} \phi_{1,n} + m \textstyle\sum_{n=2m}^{\infty} \frac{n!}{(n-2m)!} (z + h_1)^{n-2m} \phi_{0,n} \right) + O\left(\varepsilon_1^2\right).\end{gathered} \tag{108}$$

Next by substituting $m = 0, 1$ into equations (107) and (108) we get the following equations for $\hat{\Phi}^{(1)}$, $\nabla^2\hat{\Phi}^{(1)}$, $\hat{W}^{(1)}$ and $\nabla^2\hat{W}^{(1)}$:

$$O\left(\varepsilon_1^0\right): \quad \begin{aligned} \hat{\Phi}_0^{(1)} &= \textstyle\sum_{n=0}^{3} h_1^n \phi_{0,n}, \\ \hat{W}_0^{(1)} &= \textstyle\sum_{n=1}^{3} n h_1^{n-1} \phi_{0,n}, \\ \nabla^2\hat{W}_0^{(1)} &= -3!\phi_{0,3}, \\ \nabla^2\hat{\Phi}_0^{(1)} &= -\textstyle\sum_{n=2}^{3} \frac{n!}{(n-2)!} h_1^{n-2} \phi_{0,n}. \end{aligned} \qquad @z = 0 \tag{109}$$

$$O\left(\varepsilon_1^1\right): \quad \begin{aligned} \hat{\Phi}_1^{(1)} &= \textstyle\sum_{n=0}^{3} h_1^n \phi_{1,n}, \\ \hat{W}_1^{(1)} &= \textstyle\sum_{n=1}^{3} n h_1^{n-1} \phi_{1,n}, \\ \nabla^2\hat{\Phi}_1^{(1)} &= -\textstyle\sum_{n=2}^{3} \frac{n!}{(n-2)!} h_1^{n-2} \phi_{1,n} - \textstyle\sum_{n=1}^{3} \frac{n!}{(n-1)!} h_1^{n-1} \phi_{0,n}, \\ \nabla^2\hat{W}_1^{(1)} &= -3!\phi_{1,3} - \textstyle\sum_{n=2}^{3} \frac{n!}{(n-2)!} h_1^{n-2} \phi_{0,n}. \end{aligned} \qquad @z = 0 \tag{110}$$

Notice that (107) and (108) were truncated. Only the first 4 base functions were used in each order. This enables us to find these base functions using the 8 equations written in (109) and (110). Had we wanted to use more base functions for higher accuracy, we would have needed to increase the number of equations by substituting $m = 2$, $m = 3$ and so on into equations (107) and (108). That would also have increased the order of the derivatives, which would have been more complex to solve.

Note that the accuracy of this calculation is lower than the one presented in section 2.4 because we need to use the odd base functions as well as the even ones. Thus, for the same number of base functions the expansion of $\Phi^{(1)}$ shown in equation (105) gives us a power expansion up to z^3, whereas in section 2.4 it was up to z^6.

By solving equation sets (109) and (110) for the base functions $\phi_{0,n}^{(1)}$, $n = 0..3$ and $\phi_{1,n}^{(1)}$, $n = 0..3$ in terms of $\hat{\Phi}_0^{(1)}$, $\hat{W}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$ and $\hat{W}_1^{(1)}$ and their second horizontal derivatives we get

$$
\begin{aligned}
O\left(\varepsilon_1^0\right): \qquad & \phi_{0,0}^{(1)} = \hat{\Phi}_0^{(1)} - h_1 \hat{W}_0^{(1)} - \tfrac{1}{2}\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{6}\nabla^2\hat{W}_0^{(1)} \\
& \phi_{0,1}^{(1)} = \hat{W}_0^{(1)} + h_1\nabla^2\hat{\Phi}_0^{(1)} - \tfrac{1}{2}h_1^2\nabla^2\hat{W}_0^{(1)} \\
& \phi_{0,2}^{(1)} = -\tfrac{1}{2}\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{2}h_1\nabla^2\hat{W}_0^{(1)} \\
& \phi_{0,3}^{(1)} = -\tfrac{1}{6}\nabla^2\hat{W}_0^{(1)}
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
O\left(\varepsilon_1^1\right): \quad & \phi_{1,0}^{(1)} = \hat{\Phi}_1^{(1)} - h_1\hat{W}_1^{(1)} - \tfrac{1}{6}h_1^2\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{6}h_1^2\nabla^2\hat{W}_0^{(1)} - \tfrac{1}{2}h_1^2\nabla^2\hat{\Phi}_1^{(1)} - \tfrac{1}{2}h_1^2\hat{W}_0^{(1)} \\
& \phi_{1,1}^{(1)} = \hat{W}_1^{(1)} + h_1\nabla^2\hat{\Phi}_1^{(1)} + h_1\hat{W}_0^{(1)} - \tfrac{1}{2}h_1\nabla^2\hat{W}_0^{(1)} + \tfrac{1}{2}h_1\nabla^2\hat{\Phi}_0^{(1)} \\
& \phi_{1,2}^{(1)} = -\tfrac{1}{2}\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{2}\nabla^2\hat{W}_0^{(1)} - \tfrac{1}{2}\hat{W}_0^{(1)} - \tfrac{1}{2}\nabla^2\hat{\Phi}_1^{(1)} \\
& \phi_{1,3}^{(1)} = -\tfrac{1}{6h_1}\nabla^2\hat{W}_0^{(1)} + \tfrac{1}{6h_1}\nabla^2\hat{\Phi}_0^{(1)}
\end{aligned} \tag{112}
$$

Substituting the solutions (111) and (112) into (91) on $z = \eta$ gives us relations involving $\hat{\Phi}_0^{(1)}$, $\hat{W}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$, $\hat{W}_1^{(1)}$, $\tilde{\Phi}_0^{(1)}$ and $\tilde{\Phi}_1^{(1)}$:

$$
\tilde{\Phi}_0^{(1)} = \hat{\Phi}_0^{(1)} - (h_1 - \eta)\,\hat{W}_0^{(1)} - \tfrac{1}{2}(h_1 - \eta)^2\,\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{6}(h_1 - \eta)^3\,\nabla^2\hat{W}_0^{(1)} \tag{113}
$$

$$
\begin{aligned}
\tilde{\Phi}_1^{(1)} &= \hat{\Phi}_1^{(1)} - (h_1 - \eta)\,\hat{W}_1^{(1)} - \tfrac{1}{2}(h_1 - \eta)^2\,\nabla^2\hat{\Phi}_1^{(1)} - \\
&- \tfrac{1}{2}(h_1 - \eta)^2\,\hat{W}_0^{(1)} - \tfrac{1}{6h_1}(h_1 - \eta)^3\,\nabla^2\hat{\Phi}_0^{(1)} + \tfrac{1}{6h_1}(h_1 - \eta)^3\,\nabla^2\hat{W}_0^{(1)}
\end{aligned} \tag{114}
$$

Labels (98), (113) and (114) presents us with 4 equations for $\hat{\Phi}_0^{(1)}$, $\hat{W}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$ and $\hat{W}_1^{(1)}$. We can solve $\hat{\Phi}_0^{(1)}$, $\hat{W}_0^{(1)}$, $\hat{\Phi}_1^{(1)}$ and $\hat{W}_1^{(1)}$ using a numerical method such as finite differences. By implementing this method we receive 2 sets of linear equations, which take the form of

$$\begin{bmatrix} A^{(1)}_{0\,\text{padé}} & -B^{(1)}_{0\,\text{padé}} \\ A^{(1)}_{0\,\text{nl}} & B^{(1)}_{0\,\text{nl}} \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{(1)}_0 \\ \hat{W}^{(1)}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \tilde{\Phi}^{(1)}_0 \end{bmatrix}, \tag{115}$$

$$\begin{bmatrix} A^{(1)}_{1\,\text{padé}} & -B^{(1)}_{1\,\text{padé}} \\ A^{(1)}_{1\,\text{nl}} & B^{(1)}_{1\,\text{nl}} \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{(1)}_1 \\ \hat{W}^{(1)}_1 \end{bmatrix} = \begin{bmatrix} C^{(1)}_1 \hat{\Phi}^{(1)}_0 \\ \tilde{\Phi}^{(1)}_1 + C^{(1)}_2 \hat{\Phi}^{(1)}_0 + C^{(1)}_3 \hat{W}^{(1)}_0 \end{bmatrix}. \tag{116}$$

Here $A^{(1)}_{\text{padé}}$ and $B^{(1)}_{\text{padé}}$ are the finite difference matrices representing the Dirichlet to Neumann relation (label 98), $A^{(1)}_{\text{nl}}$and $B^{(1)}_{\text{nl}}$are the finite difference matrices representing the relation between the undisturbed potential and vertical velocity and the potential on the interfacial wave. $C^{(1)}_1$ and $C^{(1)}_2$ are the finite difference matrices representing the relation between $\hat{W}^{(1)}_1$ and $\hat{\Phi}^{(1)}_1$ to $\hat{\Phi}^{(1)}_0$. $C^{(1)}_3$ is the finite difference matrix representing the relation between $\hat{W}^{(1)}_1$ and $\hat{\Phi}^{(1)}_1$ to $\hat{W}^{(1)}_0$. By using $\hat{\Phi}^{(1)}_0$, $\hat{W}^{(1)}_0$, $\hat{\Phi}^{(1)}_1$ and $\hat{W}^{(1)}_1$ we can find the base functions using (111) and (112), which gives us the properties of the flow at every point in the layer.

Now let us present a way to calculate the properties of the flow in the upper layer. We can easily find $\tilde{W}^{(2)}_0$ and $\tilde{W}^{(2)}_1$ using (89). Applying the same technique described in subsection 2.1 we can construct an equivalent version of equations (95) and (97) for the upper layer

$$\begin{aligned}\Phi^{(2)} = \left(\cos(z\nabla) + \tfrac{1}{2}\varepsilon_2 z\cos(z\nabla) + \varepsilon_2 \tfrac{\sin(z\nabla)}{2\nabla}\right)\Phi^{(2)}_0 + \\ \left(\tfrac{\sin(z\nabla)}{\nabla} - \varepsilon_2 \tfrac{z\sin(z\nabla)}{2\nabla}\right) W^{(2)}_0 + \varepsilon_2 \cos(z\nabla)\Phi^{(2)}_1 + \varepsilon_2 \tfrac{\sin(z\nabla)}{\nabla} W^{(2)}_1,\end{aligned} \tag{117}$$

$$\begin{aligned}W^{(2)} = \left(-1 + \tfrac{1}{2}\varepsilon_2 z\right)\sin(z\nabla)\nabla\Phi^{(2)}_0 + \left(\left(1 - \tfrac{1}{2}\varepsilon_2 z\right)\cos(z\nabla) - \tfrac{1}{2}\varepsilon_2 z\tfrac{\sin(z\nabla)}{\nabla}\right) W^{(2)}_0 - \\ \varepsilon_2 \sin(z\nabla)\nabla\Phi^{(2)}_1 + \varepsilon_2\cos(z\nabla) W^{(2)}_1\end{aligned} \tag{118}$$

On $z = \eta$ we can use equations (117) and (118) to solve $\hat{\Phi}^{(1)}_0$, $\hat{W}^{(1)}_0$, $\hat{\Phi}^{(1)}_1$ and $\hat{W}^{(1)}_1$ and substitute the solution into (117) and (118) to get $\Phi^{(2)}$ and $W^{(2)}$ at every point in the layer.

4.4 Constructing the equations for the nonlinear time marching

An analytical relation between $\tilde{\Phi}^{(1)}_t$ and $\tilde{\Phi}^{(2)}_t$ must be developed in order to enable marching $\tilde{\Phi}$ in time using the dynamical interface boundary condition (88). Equations (95), (97) and (98) on $z = \eta$ are a set of 6 equations relating $\hat{\Phi}^{(1)}_0$, $\hat{W}^{(1)}_0$, $\hat{\Phi}^{(1)}_1$, $\hat{W}^{(1)}_1$, $\tilde{\Phi}^{(1)}_0$, $\tilde{W}^{(1)}_0$, $\tilde{\Phi}^{(1)}_1$ and $\tilde{W}^{(1)}_1$. By eliminating these 6 equations we get

$$\begin{aligned}\tilde{W}^{(1)}_0 &= -\tan\left(\left(h_1+\eta\right)\nabla\right)\nabla\tilde{\Phi}^{(1)}_0 \\ \tilde{W}^{(1)}_1 &= -\tan\left(\left(h_1+\eta\right)\nabla\right)\nabla\tilde{\Phi}^{(1)}_1 + \\ \tfrac{1}{4}\sec^2\left(\left(h_1+\eta\right)\nabla\right)&\left(1 - h_1 + (2+h_1)\cos\left(2h_1\nabla\right) + 2\cos\left(2\eta\nabla\right) + 2\eta\sin\left(2h_1\nabla\right)\nabla\right)\tilde{\Phi}^{(1)}_0\end{aligned} \tag{119}$$

Similar relations can be found for the upper layer

$$\tilde{W}_0^{(2)} = \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\tilde{\Phi}_0^{(2)} \tag{120}$$

$$\tilde{W}_1^{(2)} = \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\tilde{\Phi}_1^{(2)} +$$
$$\tfrac{1}{4}\sec^2\left(\left(h_2 - \eta\right)\nabla\right)\left(1 - h_2 + \left(2 + h_2\right)\cos\left(2h_2\nabla\right) + 2\cos\left(2\eta\nabla\right) - 2\eta\sin\left(2h_2\nabla\right)\nabla\right)\tilde{\Phi}_0^{(2)}$$

By using equations (119), (120) and (89) we get relations between $\tilde{\Phi}_0^{(1)}$, $\tilde{\Phi}_1^{(1)}$, $\tilde{\Phi}_0^{(2)}$and $\tilde{\Phi}_1^{(2)}$

$$-\nabla\eta\tilde{\Phi}_0^{(1)} - \tan\left(\left(h_1 + \eta\right)\nabla\right)\nabla\tilde{\Phi}_0^{(1)} = \nabla\eta\tilde{\Phi}_0^{(2)} + \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\tilde{\Phi}_0^{(2)}$$

$$\tfrac{1}{4}\sec^2\left(\left(h_1 + \eta\right)\nabla\right)\left(1 - h_1 + \left(2 + h_1\right)\cos\left(2h_1\nabla\right) + 2\cos\left(2\eta\nabla\right) + 2\eta\sin\left(2h_1\nabla\right)\nabla\right)\tilde{\Phi}_0^{(1)} -$$
$$\nabla\eta\tilde{\Phi}_1^{(1)} - \tan\left(\left(h_1 + \eta\right)\nabla\right)\nabla\tilde{\Phi}_1^{(1)} =$$
$$\tfrac{1}{4}\sec^2\left(\left(h_2 - \eta\right)\nabla\right)\left(1 - h_2 + \left(2 + h_2\right)\cos\left(2h_2\nabla\right) + 2\cos\left(2\eta\nabla\right) - 2\eta\sin\left(2h_2\nabla\right)\nabla\right)\tilde{\Phi}_0^{(2)} -$$
$$\nabla\eta\tilde{\Phi}_1^{(2)} + \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\tilde{\Phi}_1^{(2)} \tag{121}$$

By taking the time derivative of (121) we get

$$\eta_t\sec^2\left(\left(h_1 + \eta\right)\nabla\right)\nabla^2\tilde{\Phi}_0^{(1)} - \left(\nabla\eta + \tan\left(\left(h_1 + \eta\right)\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_0^{(1)} =$$
$$\eta_t\sec^2\left(\left(h_2 - \eta\right)\nabla\right)\nabla^2\tilde{\Phi}_0^{(2)} - \left(\nabla\eta - \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_0^{(2)}$$

$$-\eta_t\sec^2\left(\left(h_1 + \eta\right)\nabla\right)\nabla^2\tilde{\Phi}_1^{(1)} + \tfrac{1}{2}\eta_t\sec^2\left(\left(h_1 + \eta\right)\nabla\right)$$
$$\left(-\sin\left(2\eta\nabla\right) + \left(1 - h_1 + \left(2 + h_1\right)\cos\left(2h_1\nabla\right)\right)\tan\left(\left(h_1 + \eta\right)\nabla\right) + 2\sin\left(2h_1\nabla\right)\right)\nabla\tilde{\Phi}_0^{(1)} +$$
$$\left(\cos\left(2\eta\nabla\right) + 2\nabla\eta\sin\left(2h_1\nabla\right)\right)\tan\left(\left(h_1 + \eta\right)\nabla\right)\nabla\tilde{\Phi}_0^{(1)} +$$
$$\tfrac{1}{4}\sec^2\left(\left(h_1 + \eta\right)\nabla\right)\left(1 - h_1 + \left(2 + h_1\right)\cos\left(2h_1\nabla\right) + 2\cos\left(2\eta\nabla\right) + 2\eta\sin\left(2h_1\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_0^{(1)} -$$
$$\left(\nabla\eta + \tan\left(\left(h_1 + \eta\right)\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_1^{(1)} =$$
$$-\eta_t\sec^2\left(\left(h_2 - \eta\right)\nabla\right)\nabla^2\tilde{\Phi}_1^{(2)} + \tfrac{1}{2}\eta_t\sec^2\left(\left(h_2 - \eta\right)\nabla\right)$$
$$\left(-\sin\left(2\eta\nabla\right) + \left(1 - h_2 + \left(2 + h_2\right)\cos\left(2h_2\nabla\right)\right)\tan\left(\left(h_2 - \eta\right)\nabla\right) - 2\sin\left(2h_2\nabla\right)\right)\nabla\tilde{\Phi}_0^{(2)}$$
$$\left(\cos\left(2\eta\nabla\right) + 2\nabla\eta\sin\left(2h_2\nabla\right)\right)\tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\tilde{\Phi}_0^{(2)} +$$
$$\tfrac{1}{4}\sec^2\left(\left(h_2 - \eta\right)\nabla\right)\left(1 - h_2 + \left(2 + h_2\right)\cos\left(2h_2\nabla\right) + 2\cos\left(2\eta\nabla\right) - 2\eta\sin\left(2h_2\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_0^{(2)} -$$
$$\left(\nabla\eta - \tan\left(\left(h_2 - \eta\right)\nabla\right)\nabla\right)\tfrac{\partial}{\partial t}\tilde{\Phi}_1^{(2)} \tag{122}$$

Now by using equations (122), (81) and (88) we can march the problem in time.

5 Surface Quasi-Geostrophic Model

The quasi-geostrophic equation for constant stratification is

$$\left(\frac{\partial}{\partial t}+\Psi_x\frac{\partial}{\partial y}-\Psi_y\frac{\partial}{\partial x}\right)\left(\nabla^2\Psi+\frac{1}{S}\Psi_{zz}+\beta y\right)=0 \tag{123}$$

$$\mathbf{u}=(-\Psi_y,\Psi_x)\,,\qquad \Psi_z=\Theta \tag{124}$$

Here $S=\frac{N^2H^2}{f_0^2L^2}$, N is the buoyancy frequency, f_0 is the rotation frequency, H is the vertical length scale, L is the horizontal length scale and Ψ is the horizontal stream function and its z-derivative relates to the potential temperature Θ. Assuming zero initial potential vorticity equation (123) becomes

$$\nabla^2\Psi+\frac{1}{S}\Psi_{zz}+\beta y=0 \tag{125}$$

The Boundary conditions for this one layer model are

$$\Theta_t+\Psi_x\Theta_y-\Psi_y\Theta_x=0 \qquad @z=0 \tag{126}$$

$$\Psi_z=0 \qquad @z=-H \tag{127}$$

Equations (125), (126) and (127) represents the surface quasi-geostrophic model. We introduce an expansion of Ψ as a power series in z:

$$\Psi(x,y,z,t)=\sum_{n=0}^{\infty}z^n\psi_n(x,y,t)-\frac{1}{6}\beta y^3 \tag{128}$$

By substituting equation (128) into equation (125) and using equation (127) we get

$$\cos(\sqrt{S}H\nabla)\hat{\Theta}+\sin(\sqrt{S}H\nabla)\nabla\hat{\Psi}=0 \tag{129}$$

Therefore,

$$\begin{aligned}\hat{\Psi}_x&=-\cot\left(\sqrt{S}H\tfrac{\partial}{\partial x}\right)\hat{\Theta}\\ \hat{\Psi}_y&=-\cot\left(\sqrt{S}H\tfrac{\partial}{\partial y}\right)\hat{\Theta}\end{aligned} \tag{130}$$

As before, the hat sign ($\hat{}$) denotes the value on $z=0$. Now by applying (130) to (126) we get

$$\Theta_t-\Theta_y\cot\left(\sqrt{S}H\frac{\partial}{\partial x}\right)\Theta+\Theta_x\cot\left(\sqrt{S}H\frac{\partial}{\partial y}\right)\Theta=0 \qquad @z=0 \tag{131}$$

The infinite differential operators in equation (131) can now be approximated using Taylor approximation or Padé approximation to any required order and solved for Θ.

6 A new Layered Quasi-Geostrophic Model

6.1 Formulating the equations for the layered model

The surface quasi-geostrophic model regards the entire water depth as one stratified layer. In this new model the water depth is divided to two stratified layers with a density jump in the interface. This represents better the ocean's density structure and should give more accurate results than the one-layered representation. Due to the density jump in the interface there is a mechanism for internal interfacial Rossby waves to propagate. The equations governing the quasi-geostrophical flow of an incompressible inviscid fluid in the lower layer over a horizontal bottom are:

$$\nabla^2\Psi^{(1)} + \frac{1}{S_1}\Psi^{(1)}_{zz} + \beta y = 0 \tag{132}$$

$$\hat{\Theta}^{(1)}_t + \hat{\mathbf{u}}^{(1)} \cdot \nabla\hat{\Theta}^{(1)} = N_1^2\hat{W}^{(1)} \qquad @z = 0 \tag{133}$$

$$\eta_t + \hat{\mathbf{u}}^{(1)} \cdot \nabla\eta = \hat{W}^{(1)} \qquad @z = 0 \tag{134}$$

$$\Theta^{(1)} = 0 \qquad @z = -h_1 \tag{135}$$

The equations governing the quasi-geostrophical flow of an incompressible inviscid fluid in the upper layer under a rigid lid are:

$$\nabla^2\Psi^{(2)} + \frac{1}{S_2}\Psi^{(2)}_{zz} + \beta y = 0 \tag{136}$$

$$\hat{\hat{\Theta}}^{(2)}_t + \hat{\hat{\mathbf{u}}}^{(2)} \cdot \nabla\hat{\hat{\Theta}}^{(2)} = F \qquad @z = h_2 \tag{137}$$

$$\hat{\Theta}^{(2)}_t + \hat{\mathbf{u}}^{(2)} \cdot \nabla\hat{\Theta}^{(2)} = N_2^2\hat{W}^{(2)} \qquad @z = 0 \tag{138}$$

$$\eta_t + \hat{\mathbf{u}}^{(2)} \cdot \nabla\eta = \hat{W}^{(2)} \qquad @z = 0 \tag{139}$$

Here we added a surface potential temperature forcing function F to the surface boundary condition that is presented in equation (137). Using (133), (134), (138) and (139) we can write the potential temperature interfacial equation

$$\begin{aligned} &N_2^2\left(\hat{\Theta}^{(1)}_t + \hat{\mathbf{u}}^{(1)} \cdot \nabla\left(\hat{\Theta}^{(1)} - N_1^2\eta\right)\right) = \\ &N_1^2\left(\hat{\Theta}^{(2)}_t + \hat{\mathbf{u}}^{(2)} \cdot \nabla\left(\hat{\Theta}^{(2)} - N_2^2\eta\right)\right) \qquad @z = 0 \end{aligned} \tag{140}$$

Here,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{141}$$

$$\mathbf{u} = \left(-\Psi_y, \Psi_x\right), \qquad \Psi_z = \Theta. \tag{142}$$

Now in order for the problem to be well-posed we need to construct a pressure interfacial condition. Let us write the pressure formulation for each layer,

$$-p_s - p_2(z) = \rho_1\, g\,(h_2 - z) - \Delta\rho\, g\,(h_2 - z) - p_2^{,} \quad @\ 0 < z < h_2 \tag{143}$$

$$-p_s - p_1(z) = \rho_1\, g\,(h_2 - z) - \Delta\rho\, g\,(h_2 - \eta) - p_1^{,} \quad @\, -h_1 < z < 0. \tag{144}$$

Here $p_1^{,}$ and $p_2^{,}$ are the non-hydrostatic part of the pressure, p_s is the rigid-lid surface pressure and $\Delta\rho = \rho_2 - \rho_1$. By defining Ψ with p_s as the datum and using equations (143) and (144) we can write the pressure interfacial equation,

$$\hat{\Psi}^{(2)} + \Delta\rho\, g\eta = \hat{\Psi}^{(1)} \quad @z = 0. \tag{145}$$

By using (133) and (134) we can also construct an equation relating η to the potential temperature,

$$\eta_t = \tfrac{1}{N_1^2}\left(\hat{\Theta}_t^{(1)} + \hat{\mathbf{u}}^{(1)} \cdot \left(\hat{\Theta}^{(1)} - N_1^2\eta\right)\right) \quad @z = 0. \tag{146}$$

6.2 An infinite series solution for the quasi-geostrophic equation in each layer over a horizontal bottom

We introduce an expansion of Ψ as a power series in z:

$$\Psi(x, y, z, t) = \sum_{n=0}^{\infty} z^n \psi_n(x, y, t) - \frac{1}{6}\beta y^3 \tag{147}$$

By substituting equation (147) for the lower layer into equation (132) and using equation (135) we get

$$\cos(\sqrt{S_1}h_1\nabla)\hat{\Theta}^{(1)} + \sin(\sqrt{S_1}h_1\nabla)\nabla\hat{\Psi}^{(1)} = 0 \tag{148}$$

and by substituting equation (147) for the upper layer into equation (136) we get

$$\Psi^{(2)} = \tfrac{\sin\left(\sqrt{S_2}(z-h_2)\nabla\right)}{\nabla}\hat{\hat{\Theta}}^{(2)} + \cos\left(\sqrt{S_2}\,(z - h_2)\,\nabla\right)\hat{\hat{\Psi}}^{(2)} - \tfrac{1}{6}\beta y^3. \tag{149}$$

Now by using equation (149) we can formulate the relations between the flow properties of the upper layer on the interface and the flow properties on the surface,

$$\hat{\Theta}^{(2)} = \sqrt{S_2}\cos\left(\sqrt{S_2}h_2\nabla\right)\hat{\hat{\Theta}}^{(2)} + \sqrt{S_2}\sin\left(\sqrt{S_2}h_2\nabla\right)\nabla\hat{\hat{\Psi}}^{(2)} \tag{150}$$

$$\hat{\Psi}^{(2)} = -\tfrac{\sin\left(\sqrt{S_2}h_2\nabla\right)}{\nabla}\hat{\hat{\Theta}}^{(2)} + \cos\left(\sqrt{S_2}h_2\nabla\right)\hat{\hat{\Psi}}^{(2)} - \tfrac{1}{6}\beta y^3. \tag{151}$$

Let us now eliminate equations (145), (148), (150) and (151) to give us an equation containing only the potential temperature functions $\hat{\Theta}^{(1)}$,$\hat{\Theta}^{(2)}$and $\hat{\hat{\Theta}}^{(2)}$,

$$\sqrt{S_2}\frac{1}{\nabla}\cot\left(\sqrt{S_1}h_1\nabla\right)\hat{\Theta}^{(1)}+\frac{1}{\nabla}\cot\left(\sqrt{S_2}h_2\nabla\right)\hat{\Theta}^{(2)}-\sqrt{S_2}\frac{1}{\nabla}\csc\left(\sqrt{S_2}h_2\nabla\right)\hat{\hat{\Theta}}^{(2)}+ \sqrt{S_2}\left(\Delta\rho\, g\eta-\tfrac{1}{6}\beta y^3\right)=0. \tag{152}$$

The next step is to take the derivative of equation (152) with respect to time and also use equations (137) and (146) in order to get

$$\frac{1}{\nabla}\cot\left(\sqrt{S_2}h_2\nabla\right)\hat{\Theta}^{(2)}_t-\frac{1}{\nabla}\csc\left(\sqrt{S_2}h_2\nabla\right)\left(F-\hat{\hat{\mathbf{u}}}^{(2)}\cdot\nabla\hat{\hat{\Theta}}^{(2)}\right)= -\frac{\Delta\rho f}{N_1^2}\hat{\mathbf{u}}^{(1)}\cdot\nabla\left(\hat{\Theta}^{(1)}-N_1^2\eta\right)-\left(\frac{1}{\nabla}\cot\left(\sqrt{S_1}h_1\nabla\right)+\frac{\Delta\rho f}{N_1^2}\right)\hat{\Theta}^{(1)}_t. \tag{153}$$

At this point we have all the equations needed in order to construct the method. From initial conditions or from a prior time step we know $\hat{\Theta}^{(1)}$, $\hat{\hat{\Theta}}^{(2)}$ and η. By the use of equations (145), (148) and (151) we can find $\hat{\Psi}^{(1)}$, $\hat{\Psi}^{(2)}$ and $\hat{\hat{\Psi}}^{(2)}$. Afterwards, we can use equations (140), (153) and (146) in order to march $\hat{\Theta}^{(1)}$, $\hat{\hat{\Theta}}^{(2)}$ and η in time and the process can be repeated.

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