Parametric instability of internal waves with rotation

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1 Introduction

Internal gravity waves carry a lot of energy in the ocean and are known to be unstable. Their eventual breakdown thus has a significant effect in the oceans because the waves cause substantial mixing. Internal wave driven mixing not only affects the local circulation patterns but also energitically controls the global-scale meridional overturning circulation as suggested by previous results, and hence is vital for understanding of past, present and future climates([2],[4]). A substantial portion of internal wave energy is in the form of an internal tide. The internal tide is created by the interaction of the barotropic (astronomical) tide with rough topography. A variety of recent results have mapped out global locations of efficient internal tide generation. However, once generated, the internal tide does not necessarily break down and produce mixing near the generation site. Observations from the Hawaiian Ocean Mixing Experiment show that a low mode internal tide can propagate up to thousands of kilometers from it's generation site. The global pattern of tidal mixing is therefore strongly influenced by the dynamical instabilities that can efficiently drain energy from a propagating wave. Recent 3-D simulations of the mode1(M2) internal tide done by Mackinnon and Winters [5] suggests that the M2 tide loses energy at a particular latitude where the tidal frequency is twice the local inertial frequency. This raises a lot of interesting questions regarding the instabilities. For example, what is the growth rate of the instabilities at the crtical latitude?.What are the dynamic nature of those instabilities?.

Resonant wave triad interactions offer a framework to investigate the wave interactions causing the instability of the primary M2 tide. Neef and Mackinnon [3] have modelled the phenomenon as a system of one primary (tidal) wave and two close to subharmonic (M2/2) waves and show that as one approaches the critical latitude the rates of tansfer of energy get larger.However the above approach suffers from the drawback that at the critical latitude the frequency of the subharmonic waves appropaches the local inertial frequency. Therefore they cannot be treated as propagating waves anymore. In fact, the MacKinnon and Winters simulations show that the growing instabilities at this latitude look like pure inertial oscillations.

Motivated by their results, we have undertaken a series of analysis to look at the interaction between a large-scale propagating tide and small-scale near-inertial oscillations at a latitude where the tidal frequency is twice the local inertial frequency. our primary goal would be to study the growth rates of instabilities that drain energy away from the primary M2 internal tide at the critical latitude. In sections 2 and 3 we do a linear stability analysis on the primary M2 internal tide within the framework of Boussinesq model with constant stratification frequency and study the growth rate of the instabilities at the critical latitude. In section 5 we propose an analytic approach that models the interaction of the primary internal tide with the instabilities and derive the growth rates for the fastest growing perturbations.

2 Numerical approach

We consider the inviscid Boussinesq equation with rotation and with constant stratification frequency N.

$$\frac{Du}{Dt} - fv = p_x \tag{1}$$

$$\frac{Dv}{Dt} + fu = -p_y \tag{2}$$

$$\frac{Dw}{Dt} - b = -p_z \tag{3}$$

$$u_x + v_y + w_z = 0 \tag{4}$$

$$\frac{Db}{Dt} + N^2 w = 0 \tag{5}$$

where (u, v, w) is the velocity,p the pressure , b the buoyancy and

$$\frac{D}{Dt} = \partial_t + u\partial_x + v\partial_y + w\partial_z$$

Linearising equations (1)-(4) we get

$$u_t - fv = -p_x \tag{6}$$

$$v_t + fu = -p_y \tag{7}$$

$$w_t + b = -p_z \tag{8}$$

$$b_t + N^2 w = 0 \tag{9}$$

$$\nabla.\mathbf{u} = \mathbf{0} \tag{10}$$

The M2 internal tide given by

$$U = a\cos(mx - \sigma t)\cos nz \tag{11}$$

$$V = \frac{af}{\sigma}\sin(mx - \sigma t)\cos nz \tag{12}$$

$$W = a\sin(mx - \sigma t)\sin nz \tag{13}$$

$$B = \frac{-N^2}{\sigma}\cos(mx - \sigma t)\sin nz \tag{14}$$

is a solution of (6)-(10). where σ is the M2 tidal frequency = 1/(12.4 hrs),$n = 2\pi/$ocean depth = $2\pi/4000$,m is a typical horizontal wavenumber = $2\pi/150km$ for midlatitudes and 'a' is a typical tidal velocity, on the order of 5 cm/s. Motivated by the tide propagating north from Hawaii, we're considering a tide propagating purely northward with infinite zonal wavenumber - hence there is no y dependence.

3 Floquet system governing the instability of the M1 internal tide

To investigate the stability of M2 internal tide we consider perturbations around this state.Let

$$u = U + u', v = V + v', w = W + w', b = B + b', p = P + p'. \quad (15)$$

Linearising the equations (1)-(5) for small perturbations u',v',w',b',p' gives (after dropping ' notation uniformly),

$$\begin{aligned} \tfrac{Du}{Dt} + uU_x + wU_z - fv &= -p_x \\ \tfrac{Dv}{Dt} + uV_x + wV_z + fu &= -p_y \\ \tfrac{Dw}{Dt} + b &= -p_z \\ \tfrac{Db}{Dt} + uB_x + wB_z + &= 0 \\ u_x + v_y + w_z &= 0 \end{aligned}$$

Where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z} \quad (16)$$

We now introduce the following scalings

$$\begin{aligned} (x, y, z) &= (mx^*, my^*, nz^*) \\ (U, V) &= \frac{\sigma}{m}(U^*, V^*) \\ W &= \frac{\sigma}{n}W^* \\ B &= \frac{\sigma^2}{n}B^* \\ (u, v) &= \frac{\sigma}{m}(u^*, v^*) \\ w &= \frac{\sigma}{n}w^* \\ b &= \frac{N^2}{n}b^* \\ P &= \frac{\sigma^2}{m^2}p^* \end{aligned}$$

The equations of perturbation become

$$\frac{Du}{Dt} - a\sin(x-t)\cos(z)u - a\cos(x-t)\sin zw - \frac{f}{\sigma}v = -p_x \quad (17)$$

$$\frac{Du}{Dt} + a\frac{f}{\sigma}\cos(x-t)\cos(z)u - a\frac{f}{\sigma}\sin(x-t)\sin(z)w + \frac{f}{\sigma}u = -p_y \quad (18)$$

$$\frac{Dw}{Dt} - a\cos(x-t)\sin(z)u - a\sin(x-t)\cos(z)w - \frac{N^2}{\sigma^2} = -\frac{n^2}{m^2}p_z \quad (19)$$

$$\frac{Db}{Dt} - a\sin(x-t)\sin(z)u + a\cos(x-t)\cos(z)w + w = 0 \quad (20)$$

$$u_x + v_y + w_z = 0 \quad (21)$$

$$\frac{D}{Dt} = \partial_t + a\cos(x-t)\cos z\partial_x + a\frac{f}{\sigma}\sin(x-t)\cos z\partial_y + a\sin(x-t)\sin z\partial_z$$

We now try a solution for (17)-(21) of the form

$$\begin{aligned} u &= u(t)\cos\gamma z e^{\imath(\alpha x+\beta y)} \\ v &= v(t)\cos\gamma z e^{\imath(\alpha x+\beta y)} \\ w &= w(t)\sin\gamma z e^{\imath(\alpha x+\beta y)} \\ b &= b(t)\cos\gamma z e^{\imath(\alpha x+\beta y)} \\ p &= p(t)\sin\gamma z e^{\imath(\alpha x+\beta y)} \end{aligned}$$

where the z dependence is chosen to meet the rigid lid top and boundary conditions.After substituting for u,v,w,b,p and taking

$$\partial_x(17) + \partial_y(18) + \partial_z(19)$$

we solve for p in terms of u,v,w,b and back substitute in equations (17)-(20).This gives us a system ordinary differential equation

$$X_t = AX \tag{22}$$

$$\begin{bmatrix} L(\alpha\beta f + a(\alpha^2 + \imath(\imath+\beta f)(\beta^2+\gamma^2 k))\cos z \\ \sin(t-x) - a\beta^2\gamma\sin(t-x)\sin z\tan\gamma z - \\ a\gamma^3 k\sin(t-x)\sin z\tan\gamma z + a\alpha\cos(t-x) \\ ((2\beta f - \imath\beta^2 + \gamma^2 k))\cos z + 2\imath\gamma\sin z\tan\gamma z)) \\ -(L(\alpha^2 f + f\gamma^2 k + \imath a\alpha\beta(2\imath+\beta f)\cos z\sin(t-x) - \\ a\alpha\beta\gamma\sin(t-x)\sin z\tan\gamma z + a\cos(t-x) \\ ((-\imath\alpha^2\beta + f(\alpha^2-\beta^2+\gamma^2 k))\cos z - 2\imath\beta\gamma\sin z\tan\gamma z))) \\ -(L\csc\gamma z(-\imath\gamma k\sin(\gamma z)(\beta f + a\alpha(2-\imath\beta f)\cos z\sin(t-x) + \\ a\alpha\gamma\sin(t-x)\sin z\tan\gamma z) + a\cos(t-x)(\alpha^2+\beta^2+\gamma^2 k)\cos(\gamma z) \\ \sin z + \gamma k\sin(\gamma z)(\alpha^2 - 2\imath\beta f)\cos z + 2\gamma\sin z\tan\gamma z)))) \\ a\cot(\gamma z)\sin(t-x)\sin z \end{bmatrix}$$

$$\begin{bmatrix} L(\beta^2 f + f\gamma^2 k + \imath a\alpha^2\beta\cos(t-x)\cos z - \imath a\alpha\beta^2 f\cos z\sin(t-x) + a\alpha\beta\gamma\sin(t-x)\sin z\tan\gamma z) \\ \imath L(-(a\alpha(\alpha^2+\gamma^2 k)\cos(t-x)\cos z) + a\beta f(\alpha^2+\gamma^2 k)\cos z\sin(t-x) + \\ \imath(\alpha\beta f + a\gamma(\alpha^2+\gamma^2 k)\sin(t-x)\sin z\tan\gamma z)) \\ \gamma k L(-\imath\alpha f - a\alpha\beta\cos(t-x)\cos z + a\beta^2 f\cos z\sin(t-x) + \imath a\beta\gamma\sin(t-x)\sin z\tan\gamma z) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} aL(\alpha \sin(t-x)(-((-2\imath+\beta f)\gamma \cos z) + (2\beta f - \imath\gamma^2)\sin z \tan\gamma z) + \\ \cos(t-x)(\alpha^2\gamma\cos z + (-\alpha^2+\beta^2+\gamma^2 k)\sin z \tan\gamma z)) \\ -(aL(\alpha\beta\cos(t-x)(-(\gamma\cos z) + 2\sin z\tan\gamma z) + \sin(t-x)(\beta(-2\imath+\beta f)\gamma\cos z + \\ (\imath\beta\gamma^2 + f(\alpha^2-\beta^2+\gamma^2 k))\sin z\tan\gamma z))) \\ -(aL\csc\gamma z(-(\gamma(\alpha^2+\beta^2+\gamma^2 k)cos(\gamma z)\sin(t-x)\sin z) + \\ \imath\sin\gamma z(\alpha\cos(t-x)((\alpha^2+\beta^2)\cos z + 2\gamma k\sin z\tan\gamma z) - \\ \sin(t-x)((-\imath\beta^2+\beta^3 f+\alpha^2(-\imath+\beta f)+\imath\gamma^2 k)\cos z + \gamma(2\beta f - \imath\gamma^2)k\sin z\tan\gamma z)))) \\ -1 + a\cos(t-x)\cos z \end{bmatrix}$$

$$\begin{bmatrix} \frac{\imath\alpha F^2\gamma L}{\sigma^2} \\ \frac{\imath\beta F^2\gamma L}{\sigma^2} \\ \frac{(\alpha^2+\beta^2)F^2 L}{\sigma^2} \\ a(-\imath\alpha\cos(t-x)\cos z + \\ \sin(t-x)(\imath\beta f\cos z + \\ \gamma\cot(\gamma z)\sin z)) \end{bmatrix}$$

and
$X = [u, v, w, b]^t$.

Since the matrix A is periodic with period 2π we can use floquet analysis to determine the stability/instability of the perturbations. The Floquet multipliers of the system then correspond to the growth rate of the perturbations. For a given x,y and z and for a given value of α, β, γ we can find the floquet multipliers which in turn determine the growth rate of the instabilities. From Floquet theory we know that there exists a constant matrix E which satisfies

$$X(\tau + 2\pi) = X(\tau)E, \forall\tau \tag{23}$$

The floquet multipliers are now defined as

$$\mu = \log\lambda$$

where λ is the eigen value of E.The real part of μ corresponds to the growth rate.If the absolute value of an eigen value is greater than unity then the perturbation is unstable. To determine E numerically let W(0) be the 4x4 identity matrix whose columns are the initial conditions for the system. Integrating (22) numerically from 0 to 2π let us denote by $W(2\pi)$ the matrix whose colums represent the solution of the system at time $t = 2\pi$. From (23) we get $E = W(0)^{-1}W(2\pi)$.ie,

$$E = W(2\pi)$$

4 Numerical results and discussion

Since we are interested in what the fastest growing modes are at the crtitical latitude we analyse the growth rates of the perturbations at the critical latitude. This corresponds to

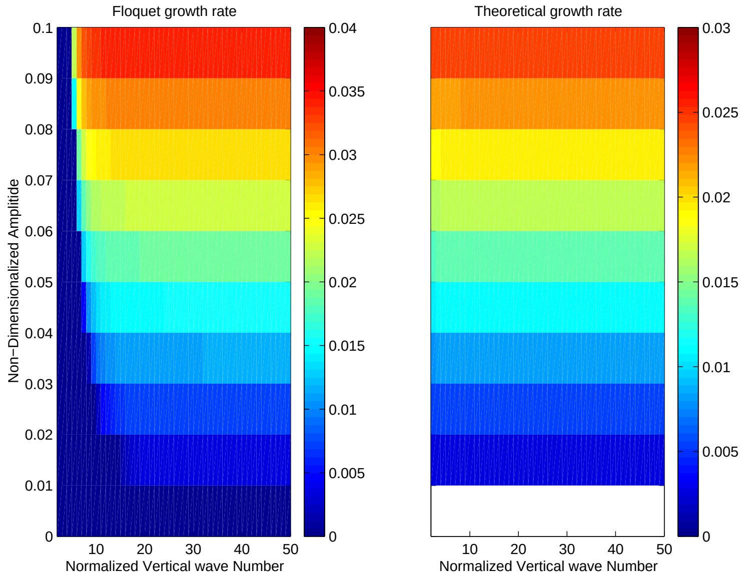

Figure 1: Theoretical growth rate vs Numerical growth rate

the case when $\frac{f}{\sigma} = 0.5$. We expect the growth rates to vary with depth. Since the waves and perturbations are uniform in the x,y direction we can fix $x = 0 = y$ and analyse the growth rate for different values of z. For the numerical calculation we fixed $\alpha = 1, \beta = 0$ and computed the growth rates numerically.The results are shown in figure 1 for z=0.02 fixed. The growth rate increases with the increase in amplitude of base wave. Also for a fixed amplitude the growth rate increases asymptotically to a constant value with increasing verical wave number for a given amplitude. The highest growth rates corresponds to wave numbers greater than ten. Infact for a fixed amplitude the growth rate shows a sudden increase for wave numbers greater than ten which suggests that the fastest growing instabilities have vertical scales which are ten times smaller than that of the original base wave. This agrees with the numerical simulations done by Winters and Mackinnon as well. However detailed numerical calculations need to be done to verify the results for a range of parameters α, β, γ and compare it with the theoretical prediction.

5 Analytical study of growth rate of instability at the critical latitude

We now do an analytical study of the growth rate of perturbations at the critical latitude,near the inertial frequency. We now introduce the following scalings $(u, v) = Rf(u^*, v^*), b = N^2Hb^*, p = N^2H^2p^*, t^* = ft$ where $R = \frac{NH}{f}$ is the Rossby deformation radius and H the depth of the ocean. The scales for x, y are given by $(x^*, y^*) = (\frac{x}{R}, \frac{x}{R})$ respectively. Numerical results, and observations from the Hawaii Ocean Mixing Experiment (HOME) both suggest that the strongest instabilities occur near the boundaries, where the tidal velocities are largest. Hence the scale for z is given by

$$z = \frac{Z}{\epsilon H} \tag{24}$$

where $\epsilon^2 = \frac{a\sigma}{mNH}$ is small. We also use the hydrostatic approximation $N^2 \gg \sigma^2$. After dropping the * notation for convenience the perturbation equations become,

$$\frac{Du}{Dt} + auU_x + awU_z - v = -p_x$$
$$\frac{Dv}{Dt} + auV_x + awV_z + u = -p_y$$
$$-\frac{p_z}{\epsilon} + b = 0$$
$$\frac{Db}{Dt} + uB_x + wB_z + = 0$$
$$u_x + v_y + w_z = 0$$

Since the vertical scale of the perturbations is small compared to the horizontal scale(as suggested by the scale for z in (24)) we let $w = \epsilon w, b = \epsilon b$.Then p scales with ϵ^2 and the equations can be written as

$$\tfrac{Du}{Dt} + auU_x + a\epsilon wU_z - v = -\epsilon^2 p_x$$
$$\tfrac{Dv}{Dt} + auV_x + a\epsilon wV_z + u = -\epsilon^2 p_y$$
$$-p_z + b = 0$$
$$u_x + v_y + w_z = 0$$
$$\tfrac{Db}{Dt} + auB_x + awB_z + \epsilon w = 0$$

If we choose $a = \epsilon^2 a_2$ where a_2 is an O(1) constant,then the equations can be written as

$$\tfrac{Du}{Dt} + \epsilon^2 a_2 uU_x + \epsilon^3 a_2 wU_z - v = -\epsilon^2 p_x$$
$$\tfrac{Dv}{Dt} + \epsilon^2 a_2 uV_x + \epsilon^3 awV_z + u = -\epsilon^2 p_y$$
$$-p_z + b = 0$$
$$u_x + v_y + w_z = 0$$
$$\tfrac{Db}{Dt} + \epsilon a_2 uB_x + \epsilon^2 a_2 wB_z + \epsilon w = 0$$

where,

$$\frac{D}{Dt} = \partial_t + \epsilon^2 a_2 U\partial_x + \epsilon^2 a_2 V\partial_y + \epsilon a_2 W\partial_z \tag{25}$$

We can further approximate $\cos \epsilon z \approx 1$, $\sin \epsilon z \approx \epsilon z$ in lieu of the fact that the interaction takes place near the boundaries .

Numerical experiments reveal that the evolution of the instabilities takes place over several cycles of the primary M2 tide. Hence we assume that the variables depend on both the fast time scale t and the slow time scale t_2. The appropriate slow time scale for the problem is $t_2 = \epsilon^2 t$. Note that the advective derivative now becomes

$$\frac{D}{Dt} = \partial_t + \epsilon^2 a_2 \left[\partial_{t_2} U \partial_x + V \partial_y + W \partial_z\right]$$

We now inroduce the complex field ,

$$q = u + iv$$

and

$$\xi = x + iy$$

The solution to the 0'th order equations can now be written from Ben-Jelloul,Young(97) [1] as follows

$$\begin{aligned} q_0 &= A_{zz} e^{-\imath t} \\ w_0 &= -M_\xi e^{-\imath t} + c.c \\ b_0 &= \imath M_\xi e^{-\imath t} + c.c \\ p_0 &= \imath A_\xi e^{-\imath t} + c.c \end{aligned}$$

where

$$M = A_z \tag{26}$$

O(ϵ^2)

$$q_{2,t_2} + iq_2 = -R_0$$

R_0 contains resonant terms i.e, terms proportional to $e^{-\imath t}$ and non-resonant terms. In order to be consistent with our assumptions we require that there are no resonantly growing terms which would cause the perturbation q_2 to grow larger than q_0. Hence equating the coefficient of R_0 that is proportional to $e^{-\imath t}$ to zero gives us an slow time evolution equation for A which is,

$$A_{zz,t_2} + \frac{\imath}{2} \nabla^2 A + \imath \gamma e^{imx} A^*_{zz} = 0 \tag{27}$$

where

$$\gamma = a_2 \frac{(1 + \hat{v})m}{4}$$

We try a solution A of the form,

$$A = \sin{(kz)} \tilde{A}(x, t_2).$$

then A_0 and A_1 satisfies,

$$\tilde{A}(x, t_2) = A_0(t) + A_1(t) e^{imx}$$

and A_0, A_1 satisfies,

$$\begin{aligned} \dot{A}_0 + i\gamma \dot{A}_1^* &= 0 \\ \ddot{A}_1 + i\left(\frac{m^2}{2k^2}\right)\dot{A}_1 - \gamma^2 A_1 &= 0 \end{aligned} \tag{28}$$

The solution of (28) is a linear combination of $e^{s_1 t}$ and $e^{s_2 t}$ where s_1 and s_2 are given by

$$s_1, s_2 = \frac{-i\frac{m^2}{2k^2} \pm \sqrt{4\gamma^2 - (\frac{m^2}{2k^2})^2}}{2}$$

The maximum of the real part of s_1 and s_2 determine the growth rate of the perturbations in slow time scale.

The predicted theoretical growth results suggests that the growth rate depends only on the vertical wave number k and the amplitude a_2 of the perturbation. Also the form of the solution given by (28) suggests to us that the fastest going perturbations have the same horizontal structure as that of the base wave. The growth rate increases with the increase of the amplitude and the vertical wave number. For a fixed amplitude the growth rate (see fig 1)shows a sudden increase for wave numbers greater than ten which illustrates that the vertical structure of fastest growing perturbationsare almost a factor of ten times smaller as compared to the base wave.

6 Conclusion

We have developed an anlaytic and a numerical model to study the growth rate of instabilities at the critical latitude.Both the thoeretical and numerical model predicts that the time period for the instabilities is roughly around 10-12 days which is relatively short. This has resulted in a significant improvement in our understanding of the role that internal tides plays in mixing. Most previous results(for example [2]) examine the role of internal tides for deep water mixing in the ocean. However, our new results and also other recent numerical results [5], now suggest that strong instabilities of the internal tide near the surface may lead to elevated tidal mixing in the upper ocean. Mixing in the upper ocean affects the rate heat, gasses and nutrients are moved around and stored on really short timescales (months to years), and so understanding upper ocean mixing is more relevant to ecosystem and climate evolution on short time scales whereas deep water mixing is generally important on longer time scales. Another fact that also emerges from our studies is that this is a very rich problem and offers a lot of scope for further work. An useful addition would be to carry out the anlaysis when the base wave is a plane wave which are exact solutions to (1)-(5). Also considering Beta plane effect and viscous effects will be an useful advance as well.

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