# Transmission of Rossby Wave Energy onto Gentle Slopes

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## 1 Introduction

We have investigated the transmission of energy of barotropic Rossby waves from a flat bottom region onto a gentle slope, on a $\beta$ plane. We studied the simple case in which topography is a constant slope and both regions are large enough so that the plane wave approximation can be used. We have considered slopes gentle enough so that over one wavelength ($L$), the variation in depth ($\vec{\nabla H}\, L$) is small compared to the total depth $H$ or, more precisely, $\frac{h}{H}$ is order Rossby number ($\epsilon$). We also made use of the of the rigid lid approximation which renders the fluid horizontally non-divergent,

$$(Hu)_x + (Hv)_y = 0. \tag{1}$$

In our case, $H = H_0 - h$, where $H_0$ is the reference depth and h is the topographic elevation. A mass transport stream function can be defined:

$$Hu = -\psi_y, \qquad Hv = \psi_x, \tag{2}$$

which is used to write the linearized form of the potential vorticity equation:

$$\nabla_h \cdot \left(\frac{1}{H}\nabla_h \psi_t\right) + \left(\nabla_h \psi \times \nabla_h \left(\frac{f}{H}\right)\right) \cdot \hat{z} = 0. \tag{3}$$

For our constant slopes this becomes:

$$\psi_{xxt} + \frac{\psi_{xt}}{H} h_x + \psi_{yyt} + \frac{\psi_{yt}}{H_0} h_y - \frac{f}{H} h_x \psi_y + \frac{f}{H} h_y \psi_x + \beta \psi_x = 0, \tag{4}$$

where $h_x$ and $h_y$ are constants.

In the above equation we wish to retain only terms with similar orders of magnitude. If we call X and Y the horizontal length scales of the wave in the $x$ and $y$ directions respectively, and T its time scale, we get for the equation above:

$$\frac{\psi}{X^2 T} + \frac{\psi}{X^2 T}\frac{X h_x}{H} + \frac{\psi}{Y^2 T} + \frac{\psi}{Y^2 T}\frac{Y h_y}{H} - \frac{f\psi}{YX}\frac{X h_x}{H} + \frac{f\psi}{XY}\frac{Y h_y}{H} \tag{5}$$

The second and fourth terms are small compared to the first and third ones and are disregarded. The last two terms are not disregarded because we are working in the low frequency regime ($\omega << f$). We realized after the summer that this approximation is questionable and are still working on the consequences of removing it. We will however keep to it in this report for we still have a very limited understanding of the dynamics that would ensue otherwise.

With the approximation mentioned we get the equation:

$$\nabla_h^2 \psi_t + [\nabla \psi \times \nabla Q] \cdot \hat{z}, \tag{6}$$

where the gradient of ambient potential vorticity ($\vec{\nabla Q}$) is the constant vector: $\vec{\nabla Q} = \left(\frac{f}{H_0}\right) h_x \, \hat{x} + \left(\left(\frac{f}{H_0}\right) h_y + \beta\right) \hat{y}$.

If we choose the coordinate system so that the y-axis is aligned with $\vec{\nabla Q}$, (6) becomes

$$\nabla_h^2 \psi_t + |\vec{\nabla Q}| \psi_x = 0, , \tag{7}$$

which is the equation for planetary Rossby waves, with $\beta$ substituted by $|\vec{\nabla Q}|$. Because $h_x$ and $h_y$ are constant, equations (6) and (7) have constant coefficients and admit plane wave solutions. Substituting

$$A e^{i(kx+ly-\omega t)} \tag{8}$$

into (7), we get the dispersion relation:

$$\omega = \frac{-|\vec{\nabla Q}| k}{k^2 + l^2} \tag{9}$$

This expression shows that $k$ and $\omega$ must have opposite signs. The $k$-component of the phase velocity, the quotient $\frac{\omega}{k}$, is always negative, so phase velocity is always contained in the half plane to the left of the $\vec{\nabla Q}$. When $|\vec{\nabla Q}|$ points to the North, as is the case for purely planetary Rossby waves, or for waves on a meridionally oriented slope growing towards North, phase propagation always has a westward component.

As noted by Longuet-Higgins, (9) can be written:

$$\left(k - \left(\frac{-|\vec{\nabla Q}|}{2xs\omega}\right)\right)^2 + l^2 = \frac{|\vec{\nabla Q}|^2}{4\omega} \tag{10}$$

showing that the locus of points that satisfy this relation for fixed $\omega$ is a circle with centre $(\frac{-|\vec{\nabla Q}|}{2\omega}, 0)$, and radius $r = \frac{|\nabla Q|^2}{4\omega^2}$. If we choose the positive sign for $\omega$ we get the familiar illustration depicted in figure 1. We will make extensive use of this geometric interpretation, so we must look at it more closely.

First we notice that the vector defining the centre of the circle is $\frac{|\vec{\nabla Q}|}{2\omega}$, rotated by $\frac{\pi}{2}$ anti-clockwise. The centre vector and the radius having the same magnitude, the circle is tangent to the origin (this is a consequence of disregarding the displacement of the free surface, the geometrical effect of the free surface is to shorten the radius while leaving the centre vector unaltered). These circles are level curves of the function $\omega(k, l)$. The group velocity, which is the gradient in wave number space of this function:

$$\vec{c_g} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}\right) = \left(\frac{|\vec{\nabla} Q|(k^2 - l^2)}{(k^2 + l^2)^2}, \frac{2|\vec{\nabla Q}| kl}{(k^2 + l^2)^2}\right) \tag{11}$$

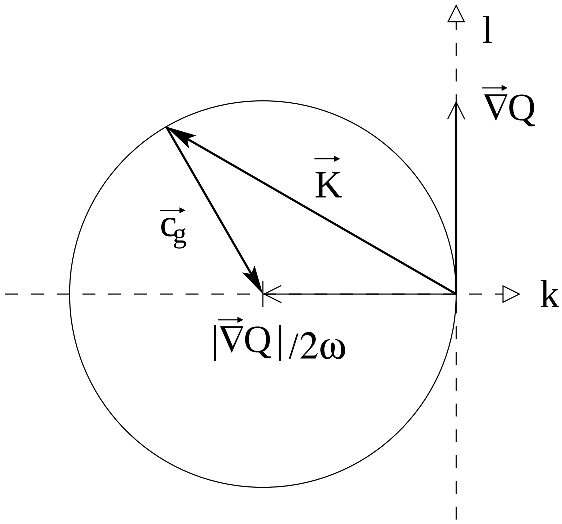


Figure 1: Longuet-Higgins' geometrical interpretation of the dispersion relation for Rossby waves on wave number space [1]. Points on the circumference are the values of $k$ and $l$ allowed by the dispersion relation. The vectors from the origin represent wave vectors and the ones from the origin to the centre give the direction of the group velocity.

must be radial. This becomes explicit if we rewrite (11) in polar coordinates:

$$(k,l) = K(\cos\alpha, \sin\alpha) \tag{12}$$

$$c_g = \frac{|\vec{\nabla}Q|}{K^2}(\cos 2\alpha, \sin 2\alpha) = \frac{|\vec{\nabla}Q|}{K^2}\vec{r}, \tag{13}$$

where $\alpha$ is the angle measured clockwise from the x-axis and $K^2 = k^2 + l^2$. We see that the group velocity grows with $|\vec{\nabla}Q|$ and decreases quadratically with the wave number. Figure (2) illustrates clearly why the group velocity is smaller for larger $K^2$.

We see from (13) that wave vectors lying on the right half of the circle have group velocities with a westward component, while those on the other half have group velocities with an eastward component and that the eastward moving waves are slower than the westward moving ones.

### 1.1 Energy flux

If we multiply equation (9) by $\psi$ and rearrange terms, we get the equation of conservation of energy ([3])

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\psi)^2}{2}\right] + \nabla\cdot\left[-\psi\nabla\psi_t - \frac{1}{2}|\vec{\nabla}Q|\psi^2\hat{x}\right] = 0. \tag{14}$$

The first term is the time variation of the kinetic energy, and the second is the divergence of a vector, called the energy flux vector ($\vec{S}$). By substituting the plane wave solution (8) into (14) and averaging over a wave period ($\langle\rangle$) we see that

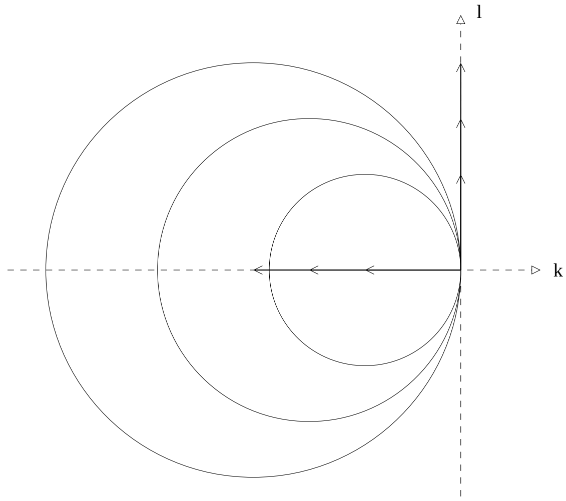


Figure 2: Level curves of $\omega(k,l)$.

$$\left\langle \vec{S} \right\rangle = \vec{c}_g \frac{A^2 K^2}{4} = \vec{c}_g \left\langle E \right\rangle , \tag{15}$$

the mean energy flux is the mean energy multiplied by the group velocity. The group velocity however is inversely proportional to the mean energy (11), so

$$\left\langle |\vec{S}| \right\rangle = A^2 |\vec{\nabla} Q|, \tag{16}$$

the magnitude of the energy flux for barotropic Rossby waves with a rigid lid is a function only of the gradient of ambient potential vorticity of the medium [1, 2].

We now proceed to formally solve the problem of energy transmission of Rossby waves from a flat bottom onto a straight slope. For clarity we will initially restrict ourselves to dealing with a slope in the zonal direction, an x-slope. In interpreting the formal solution to this and to the more general problem of a slope with arbitrary orientation, the geometrical interpretation of the dispersion relation reviewed above will become very useful.

# 2 Solution for the x-slope problem

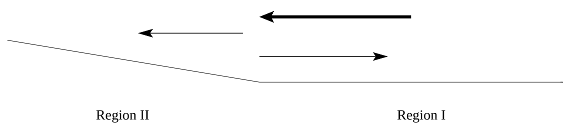


Figure 3: x-slope with impinging, reflected (region I) and transmitted (region II) waves

We want to find solutions to the equation:

$$\nabla^2\psi_t + \beta\psi_x = 0; \quad x > 0 \quad \text{region I} \tag{17}$$

$$\nabla^2\psi_t + h_x\psi_y + \beta\psi_x = 0; \quad x < 0 \quad \text{region II} \tag{18}$$

where $x = 0$ is the boundary between flat bottom and slope. We note that the boundary has been set perpendicular to $\vec{\nabla} h$. If the slope were to have a component along the boundary, there would be a step of variable height between the regions. We wish to restrict ourselves to the cases where $\vec{\nabla} h$ is discontinuous but not $h$, so our slopes will always be perpendicular to the boundary. The gradients of ambient potential vorticity for regions I and II are illustrated in figure 4. We remark again, for it will be relevant later, that they have the same along boundary component.

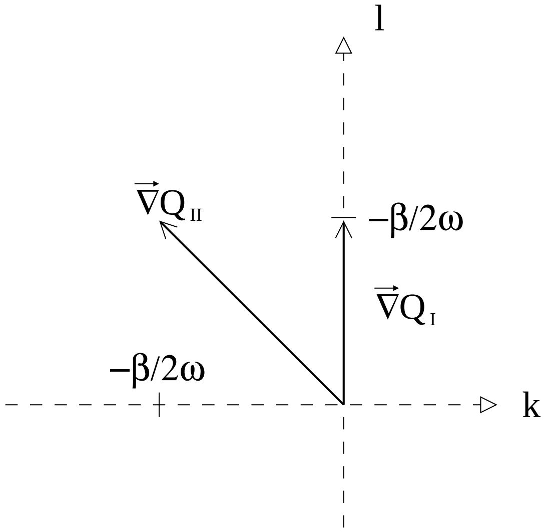


Figure 4: Ambient potential vorticity gradients for a flat bottom region and an x-slope with $f_0 h_x/H_0 = \beta$.

Equation (18) can be integrated trivially in the two open regions $x < 0$ and $x > 0$ because it admits plane wave solutions. The physical situation we want to represent is a plane wave impinging from region I onto region II (figure 3). In the stationary state we must have the motion in region I represented by the sum of two plane waves, one impinging and one reflected. The impinging wave must carry energy towards the boundary and the reflected wave away from it. In region II there should be only a transmitted wave. The kind of solution we seek is therefore:

$$\psi_i + \psi_r = Ae^{j(k_i x + l_i y - w_i t)} + Re^{j(k_r x + l_r y - w_r t)} \quad x > 0 \quad \text{region I} \tag{19}$$

$$\psi_t = Te^{j(k_t x + l_t y - w_t t)} \quad x < 0 \quad \text{region II} \tag{20}$$

where the subscripts $i, r, t,$ stand for impinging, reflected and transmitted respectively and $j = \sqrt{-1}$. $A, R, T$ are the constant amplitudes of the waves.

Only the parameters for the impinging wave are initially given. The others must be found by matching conditions at the boundary ($x = 0$). First of all, we require continuity of pressure and of mass transport at the boundary. If pressure were to be discontinuous, infinite accelerations would ensue. The second condition is conservation of mass since we cannot expect the boundary to be either a source or a sink of this quantity. These two conditions are degenerate in this case, because the flow is barotropic, and both lead to the expressions:

$$\psi_i + \psi_r = \psi_t\,, \quad x = 0 \tag{21}$$
$$A\, e^{j(l_i y - w_i t)} + R\, e^{j(l_r y - w_r t)} = T\, e^{j(l_t y - w_t t)} \tag{22}$$

For this to be true for all times and all $y's$, we must have:

$$l \equiv l_i = l_r = l_t \tag{23}$$
$$\omega \equiv \omega_i = \omega_r = \omega_t \tag{24}$$
$$A + R = T \tag{25}$$

We must also determine $k_r$ and $k_t$ and for this we have the two dispersion relations:

$$\omega = \frac{-\beta k_r}{k_r^2 + l^2}\,, \quad x > 0 \tag{26}$$
$$\omega = \frac{-\beta k_t + \left(\frac{f}{H_0}\right) h_x\, l}{k_t^2 + l^2}\,, \quad x < 0 \tag{27}$$

The relations give quadratic expressions for $k$, so radiation condition must be applied to decide upon the appropriate solution. That is, the value of $k_r$ is chosen so that the reflected wave transmits energy eastward and the value of $k_t$ is chosen so that the transmitted wave transports energy westward.

One more equation is needed to determine $R$ and $T$. We obtain this by integrating (18) across $x = 0$ and taking the interval of integration to zero. In this way we solve the equation on the only point where it can't be trivially integrated. Our equation has a finite discontinuity in one of its, otherwise constant, parameters at $x = 0$. We want to know if this will impose a discontinuity in the across boundary derivative of the stream function, and if so, to quantify this jump.

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial t}\int_{\epsilon}^{-\epsilon}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)dx - \int_{\epsilon}^{-\epsilon} Q_y\psi_x\, dx + \int_{\epsilon}^{-\epsilon} Q_x\psi_y\, dx\right) = 0 \tag{28}$$

(22) and (23) show that all derivatives in the $y$-direction are continuous, so the second term in the first integral is zero and $\psi_y$ can be removed from the third integral. $Q_y$ is also continuous, so it can be removed from the second integral. This leaves us with the equation:

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial t}\int_{\epsilon}^{-\epsilon}\frac{\partial^2\psi}{\partial x^2}\, dx - Q_y\int_{\epsilon}^{-\epsilon}\psi_x\, dx + \psi_y\int_{\epsilon}^{-\epsilon} Q_x\, dx\right) = 0 \tag{29}$$

In our problem, $Q_x$ has a finite discontinuity in $x = 0$ (going from $\beta$ in region I to $\beta + \frac{f}{H_0} h_x$ in region II). Its integral is therefore continuous and the third term is zero. If we suppose a finite discontinuity in $\psi_x$, the second term is also zero and the first term quantifies this discontinuity. Since this is the only term left on the right, the discontinuity must be zero. $\psi_x$ must therefore be continuous in our problem. From this condition we get the expression:

$$\lim_{x \to 0^+} (\psi_i + \psi_r) = \lim_{x \to 0^-} \psi_t \; k_i \psi_i + k_r \psi_r = k_t \psi_t|_{x=0}$$
$$k_i A + k_r R = k_t T \tag{30}$$

Substituting (25) into (30) we get the expression for the reflection and transmission coefficients in terms of the $k$ wave numbers:

$$\frac{R}{A} = \frac{k_t - k_i}{k_r - k_t}, \qquad \frac{T}{A} = 1 + \frac{k_t - k_i}{k_r - k_t}. \tag{31}$$

The problem is now formally solved, for we have as many equations as unknowns. However, even for this simple case where we have fixed the orientation of the slope, the solutions depend on three parameters: $\omega$, $l$, $h_x$, and it is hard to have a qualitative idea of their behavior.

We have, for instance, to worry about the existence of real solutions for $k_t$, given a value of $l$, in (26) and (27). A complex $k_t$ would mean an evanescent wave in $x$, which would transport no energy. In other words, it would mean total reflection. Because the dispersion relations depend on both $\omega$ and $h_x$ the condition for total reflection will depend on these two parameters. Furthermore, the reflection and transmission coefficients depend non-linearly on $l$, $\omega$ and $h_x$ through $k_r$ and $k_t$. Our main goal is to learn about the transmission of energy up the slope, it would be interesting to have some understanding of the role the different parameters play in this.

To gain a more intuitive grasp of the situation we will use the Longuet-Higgins circles. We have two regions with different $\vec{\nabla} Q$ so we will need to consider two circles at the same time. We will not be able to choose the coordinate system in which both dispersion relations have their simplest forms (9), so one of the circles will appear rotated from its familiar position shown in figure 1. Since the along boundary wave number is the same for the three waves, it is convenient to choose the direction of the boundary as the y-axis. In our case the direction of the boundary coincides with that of $\vec{\nabla} Q$ in region I. In figure 5 we see, for this choice of axis and $\frac{f}{H_0} h_x = \beta$, the Longuet-Higgins circles for regions I and II, on the left and right respectively.

As we noted earlier, the centre vectors of the circles are perpendicular to the direction of $\vec{\nabla} Q$ on the anti-clockwise sense, so all wave vectors (and phase velocities) lie to the left of $\vec{\nabla} Q$. The centres of both circles have the same k-coordinate, which is given by the along boundary component of $\vec{\nabla} Q$. The gradient is larger in region II, where there is the planetary $\beta$ and the slope, so the circle is larger.

Figure 6 illustrates our problem. The value of $l$ (determined by the choice of impinging wave) that all three waves must have in common is represented by the horizontal line segment. The wave vectors for the three waves must be the intersections of this segment with the circles. The direction of the group velocities for each wave are shown as the arrows

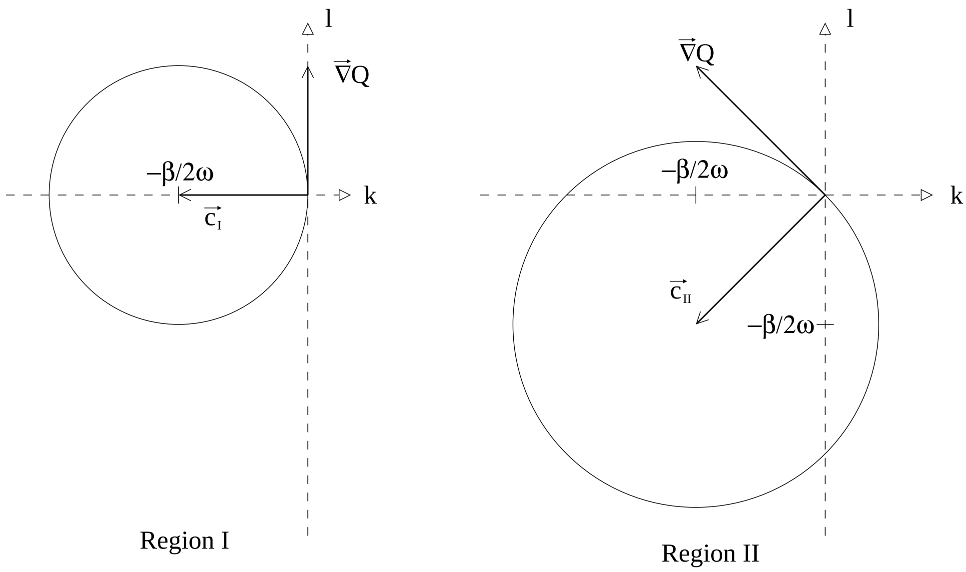


Figure 5: Longuet-Higgins circles for $\vec{\nabla} Q$ (region I) and $\vec{\nabla} Q = (\beta\,\hat{x}, \left(\frac{f}{H}\right) h_x\,\hat{y})$ with $\left(\frac{f}{H}\right) h_x = \beta$ (region II).

from the circumference to the centre. Figure 5 shows how the reflected and transmitted wave vectors are chosen on the basis of the direction of the group velocity. We also notice that, upon crossing the boundary the group velocity veers to the south.

The issue of whether an impinging wave is able to propagate on to the slope is made clear in figure 5, where the two circles are drawn in the same set of axis. For the same frequency, the two media have different ranges of $l$ which correspond to traveling waves. If an impinging wave has a value of $l$ within the interval that is not allowed in region II, the wave will not be able to propagate on and there will be total reflection. All waves within the top shaded area cannot propagate up the slope. Likewise all the waves in the bottom shaded area would not be able to propagate from the slope to the flat bottom region.

The arrows in the figure show group velocities for the critical value of $l$, the value beyond which there is total reflection. We see that for this value the transmitted wave grazes the boundary. Beyond this value of $l$ a transmitted wave would have to veer so much that it wouldn't even enter region II.

The fact that $R$ and $T$ depend on $l$ through $k$ and the importance of veering to this problem suggest that the angle of the group velocity is a more suitable variable than $l$.

A $k$ wave number can be written in terms of this variable as

$$k = k_c - r \sin\gamma \tag{32}$$

where $k_c$, the $k$ coordinate of the center, is $Q_y$, so

$$k = \frac{1}{2\omega}\left(\vec{\nabla} Q \cdot \hat{y} - |\vec{\nabla} Q| \sin\gamma\right). \tag{33}$$

$Q_y$, the along boundary component of $\vec{\nabla} Q$, is the same for both media.

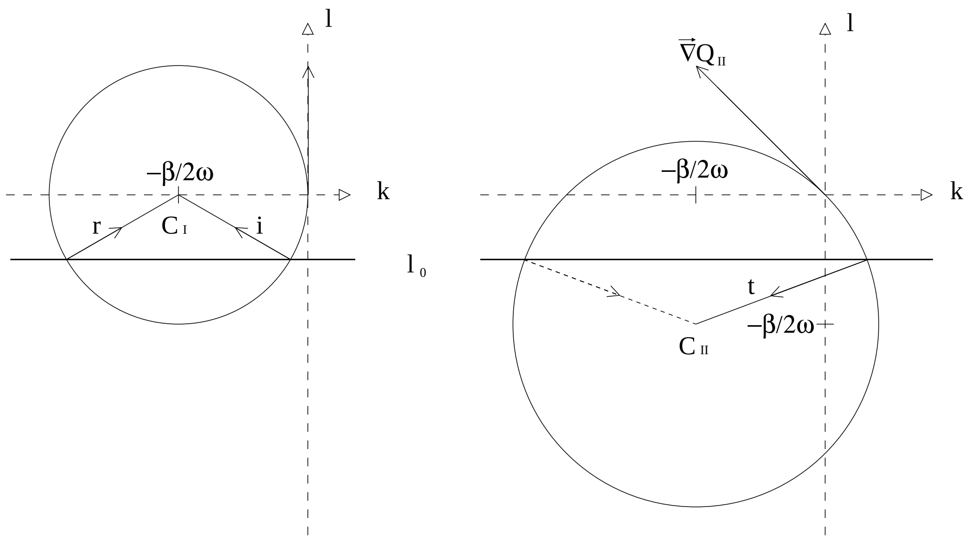


Figure 6: Longuet-Higgins circles for regions I (left) and II (right). The horizontal line segments represent the value of $l$ which is common to all three waves. The intersections of this segment with the circles are the wave vectors that obey the dispersion relation for that value of $l$ in each medium. The arrows from the circumference to the centre represent the group velocities for these vectors. For the impinging and transmitted waves the group velocities have a component to the west.

The coefficients $R$ and $T$, having the form (wave number)/(wave number) are independent of $\omega$. This is a consequence of the rigid lid approximation, which makes the centers of the circles and their radii tangent to the origin and the wave numbers proportional $\omega$.

In these coordinates, $R$ is written:

$$\frac{R}{A} = \frac{-|\vec{\nabla} Q_{II}| \sin \gamma_{II} + |\vec{\nabla} Q_I| \sin \gamma_I}{|\vec{\nabla} Q_I| \sin \gamma_I + |\vec{\nabla} Q_{II}| \sin \gamma_{II}} \tag{34}$$

Having said this and looking again at figure 7 we see that the critical value of $l$ is actually better interpreted as a critical angle, or cutoff angle for transmission ($\gamma_c$). Remembering that the $l$ coordinate of the centre of the circles is given by $\frac{Q_x}{2\omega}$ and the radius by $|\vec{\nabla} Q|$, it is easy to write ($\gamma_c$) in terms of the parameters of region II:

$$|\vec{r}| - \frac{Q_{IIx}}{2\omega} = |\vec{r}| \sin \gamma_c \tag{35}$$

$$\sin \gamma_c = 1 - \frac{Q_{IIx}}{|\vec{\nabla} Q_{II}|} \tag{36}$$

We now calculate $\frac{T(\gamma)}{A}$ and $\frac{R(\gamma)}{A}$ for $\left(\frac{f}{H_0}\right) h_x = \beta$.

The results shown in figure 8 are rather surprising. Not only does the transmission coefficient exceed 1, but it has a cusp at the cutoff angle, where we would expect energy transmission to be zero! All of this makes one worry about conservation of energy.

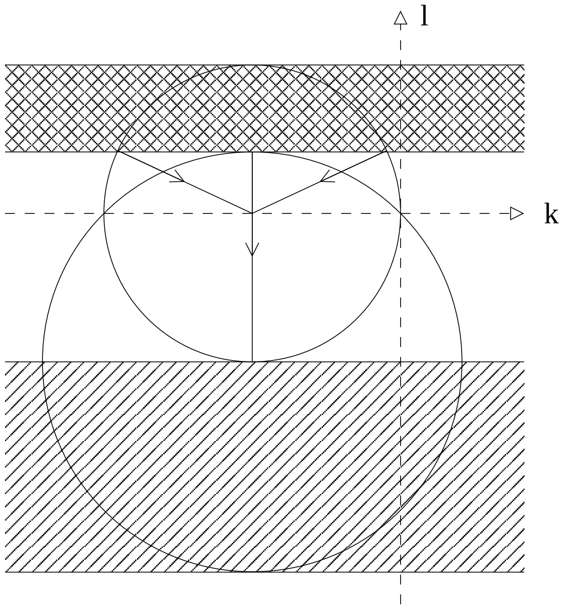


Figure 7: Total reflection: $l$ wave numbers in the shaded areas are only allowed in one of the regions.

However, going back to (14), we see that it is the divergence of the energy flux that plays a part in energy conservation. We have no reason to expect accumulation of energy at the boundary, so we must have zero divergence of $\vec{S}$. As we see from (15) A, T and R give information about the magnitude of $\vec{S}$, but alone they don't contain enough information to account for its divergence.

For a plane wave, all fields are constant on a constant phase line, zero divergence at a point is the same as zero divergence on a finite interval on these lines. The condition for no accumulation of energy at the boundary is:

$$\int_{\Delta l} \vec{S}_I \cdot \hat{n}\, dl - \int_{\Delta l} \vec{S}_{II} \cdot \hat{n}\, dl = 0 \tag{37}$$

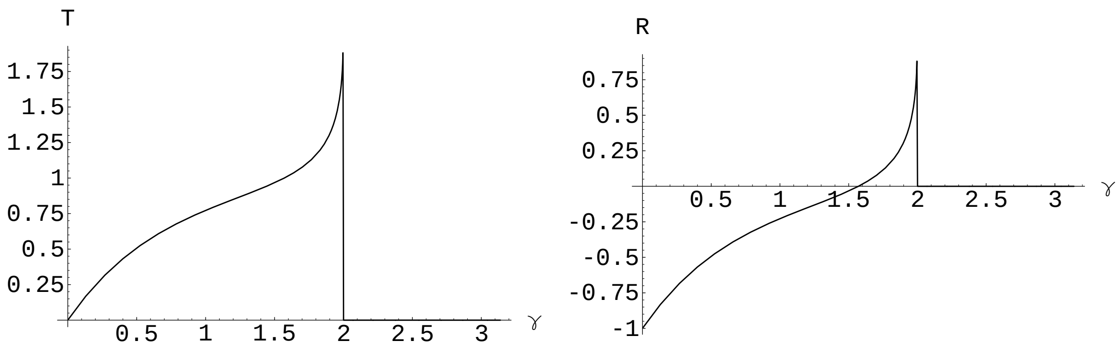


Figure 8: Transmission and reflection coefficients for a an x-slope with $Q_x = \frac{f}{H_0} h_x = \beta$.

We have seen that the group velocity veers when changing regions, so if we were to look at the energy flux vector across the boundary it would be something like figure 9 a).

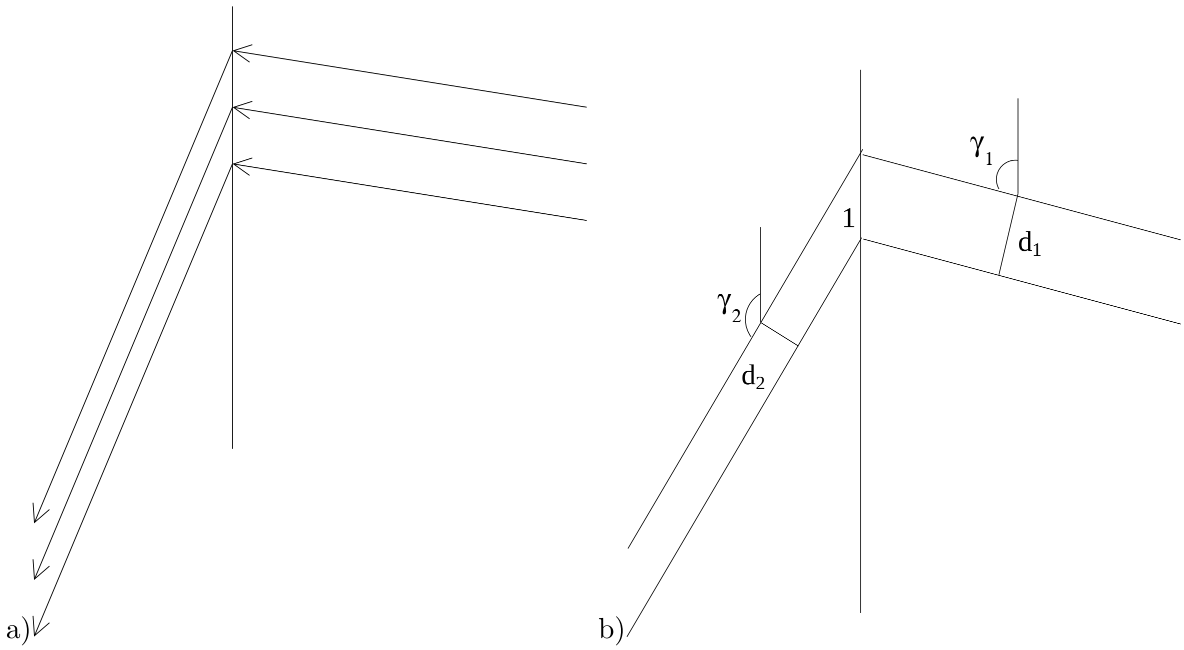


Figure 9: A wave beam, represented here by the energy flux vector, changes its direction and magnitude as it crosses the boundary in such away that its divergence is zero.

$\vec{S}$ (which is proportional to the group velocity) veers when crossing the boundary and the crossectional length of a beam is changed. For (37) to hold, the magnitude of $|\vec{S}|$ must change too. From (16) we see that $|\vec{S}|$ depends on $|\vec{\nabla}Q|$ and on the amplitude coefficient. As the veering depends on the angle of incidence and $|\vec{\nabla}Q|$ doesn't, all the angle dependence of $|\vec{S}|$ must be contained in the amplitude coefficient.

Figure 9 b) shows that the length factor for a beam which makes an angle $\gamma$ with the vertical is $\sin\gamma$. With this last information we arrive at the condition for no accumulation of energy at the boundary:

$$\left(A^2 - R^2\right)|\vec{\nabla}Q_I|\sin\gamma_I = T^2|\vec{\nabla}Q_{II}|\sin\gamma_{II} \tag{38}$$

or

$$\left(\frac{T}{A}\right)^2 \frac{\sin\gamma_I}{\sin\gamma_I}\frac{|\vec{\nabla}Q_I|}{|\vec{\nabla}Q_{II}|} + \left(\frac{R}{A}\right)^2 = 1 \tag{39}$$

We call the terms on the left the transmitted and reflected energy fractions. In figure 10 we show the result of evaluating the LHS of (39) for all angles of incidence. Energy is in fact conserved despite the awkward behavior of the amplitude coefficients seen in figure 8.

We are now ready to consider energy transmission onto the slope. Figure 11 shows the transmitted and reflected energy fractions as a function of the angle of incidence $\gamma_I$. We see that transmission does in fact go to zero at the critical angle (2 for this case, as

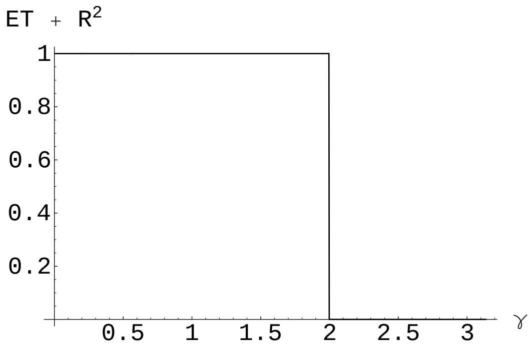


Figure 10: Sum of transmitted and reflected energy fractions calculated according to (39)

shown by the plots and as calculated from (36). We see also that energy transmission is maximum for $\gamma_I = \frac{\pi}{2}$. This corresponds to $\vec{K} = (0,0)$, or a steady current. All energy from a steady current is transmitted onto the slope. The veering is such that the steady current remains perpendicular to $\vec{\nabla}Q$ in the new medium and the length factor for this veering exactly compensates for the change in $|\vec{\nabla}Q|$, in equation (39). Transmission falls off as $|\vec{K}|$ increases towards both sides on the circle.

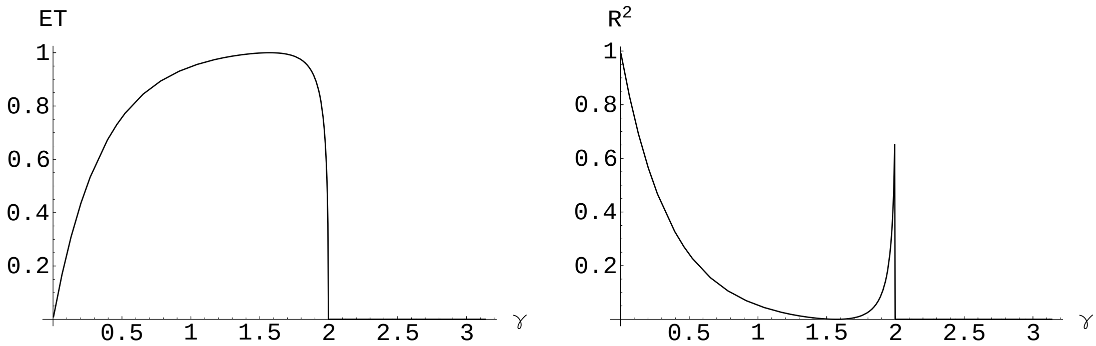


Figure 11: Transmitted and reflected energy fractions for the x-slope with $Q_x = \frac{f}{H_0} h_x = \beta$.

## 2.1 X-slopes of Different Steepness

Although we have up to now worked with a fixed value of $h_x$, changing it brings no surprises. As $h_x$ grows, $\vec{\nabla}Q_{II}$ gets closer to the x-axis and the circle for region II closer to being tangent to the x-axis. This means that only impinging waves with a southerly group velocity component are allowed on the slope (cutoff angle $= \frac{\pi}{2}$). However, large slopes violate the assumption that $\frac{h_x L}{H}$ is small, made to obtain our wave equation and we shouldn't really worry about them. Figure 12 a) shows plots of the transmitted energy fraction for $\left(\frac{f}{H_0}\right) h_x = \beta, 2\beta, ...$ We see that the behavior in all these cases is qualitatively similar but with a cutoff angle approaching $\frac{\pi}{2}$ and the transmitted energy fraction falling off more rapidly with increasing $|\vec{K}|$ for the larger slopes. This is just as one would expect, for it should be harder for energy to proceed up steeper slopes.

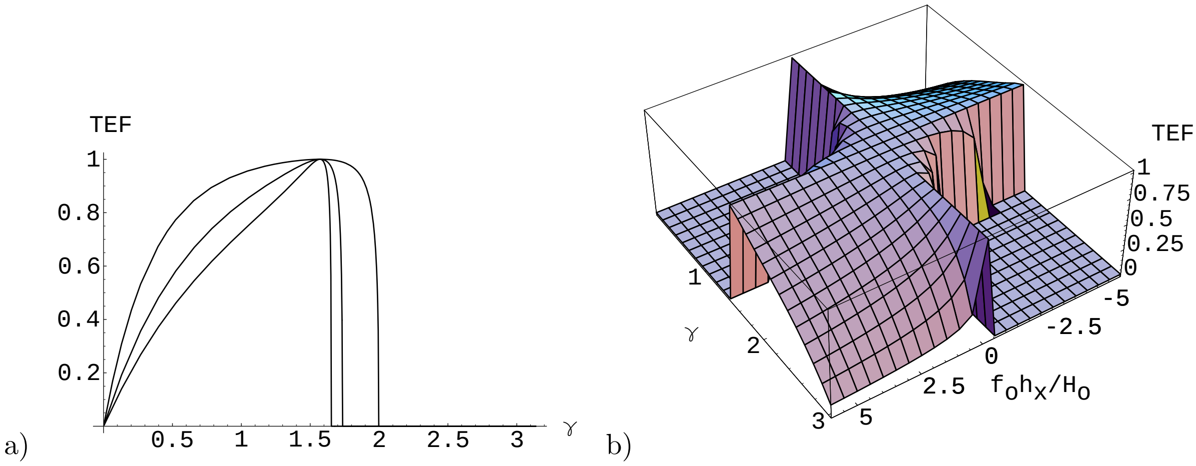


Figure 12: Transmitted energy fraction for a x-slope with $f_0 h_x / H_0$ varying from $-6\beta$ to $6\beta$ and angle of incidence in relation to north ($\gamma$) varying from 0 to $\pi$.

# 3 Slopes with Different Orientations

Rotating the slope (and boundary) around brings more interesting consequences. By rotating the slope we can give the gradients of ambient potential vorticity in the different regions any relative orientation, including parallel and anti-parallel. In figure 13 a) we show the dispersion relations for a flat bottom region and a slope going up towards North. We see that all $l$ wave numbers allowed in region I are also allowed in region II, so some energy can propagate up to the slope for all of wave numbers. As the relative angle between the gradients increases, their dispersion relations move apart in phase space and the two regions share less common $l$ wave numbers. Transmitted energy fractions (TEF) for several of these cases are shown in figure 14 a) as a function of the angle of incidence measured from North ($\gamma + \theta$). We see that as the angle between the gradients increases, the band of waves allowed on the slope narrows, but the transmitted energy fraction is always 1 for $\gamma + \theta = \frac{\pi}{2}$ or incidence from the east, which corresponds to ($k = 0, l = 0$). Figure 14 b) shows the transmitted energy fraction for all relative all orientations of the boundary and angles of incidence.

If a slope that grows towards the south is steep enough to overcompensate for the planetary $\beta$, the gradients of ambient potential vorticity for the two regions are anti-parallel. As can be seen in figure 13 a) , this means that they have no waves in common, except for ($k = 0, l = 0$). The steady current would be the only possibility of Rossby wave energy exchange. However, this movement is East to West and therefore along the boundary. As far as Rossby waves go therefore, the two regions are isolated. Dispersion relations for this case are depicted in figure 13 b).

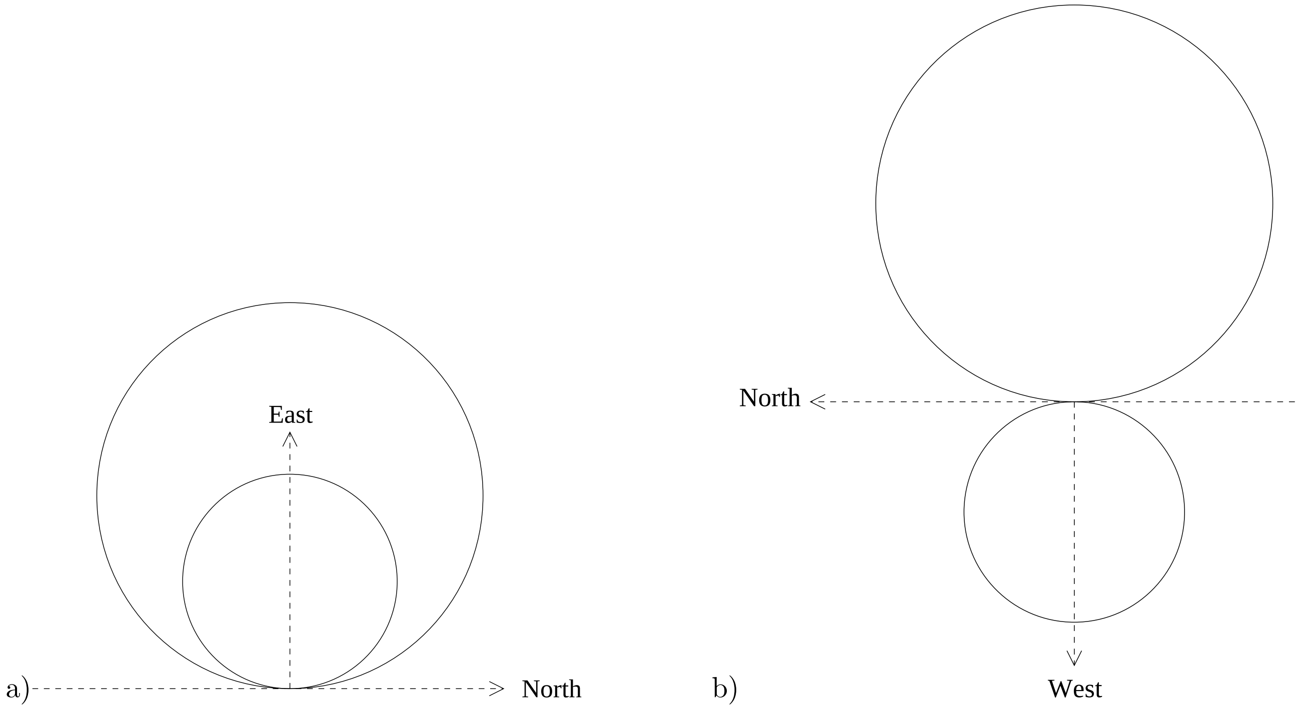


Figure 13: Dispersion relations for a flat bottom region and a) a y-slope growing towards north, b) a y-slope growing towards south.

# 4 Concluding Remarks

This work was intended as a simple first step in the investigation of energy transmission from mesoscale oceanic eddies onto the gentle continental slopes. In using the approximation of small depth perturbations we have confined ourselves to representing behavior not far from the foot of these slopes. In this context, the rigid lid approximation is not an additional restriction for it requires wavelengths to be small compared to the Rossby radius of deformation, which for the flat bottom open ocean is of order 2000Km.

We believe the main weakness of this work is the neglect terms 2 and 4 in equation 4. They are considered small when compared to the other slope terms (5 and 6) because, for the low frequency regime we are working on, $\frac{1}{T} << f_0$. However, terms with corresponding components of $\vec{\nabla} h$ have different spatial scales. For Rossby waves these spatial scales ($X$ and $Y$) may be very different, invalidating the assumption on the relative sizes of the terms. We have realized this after the summer was over and are now working on understanding the effects of keeping the terms.

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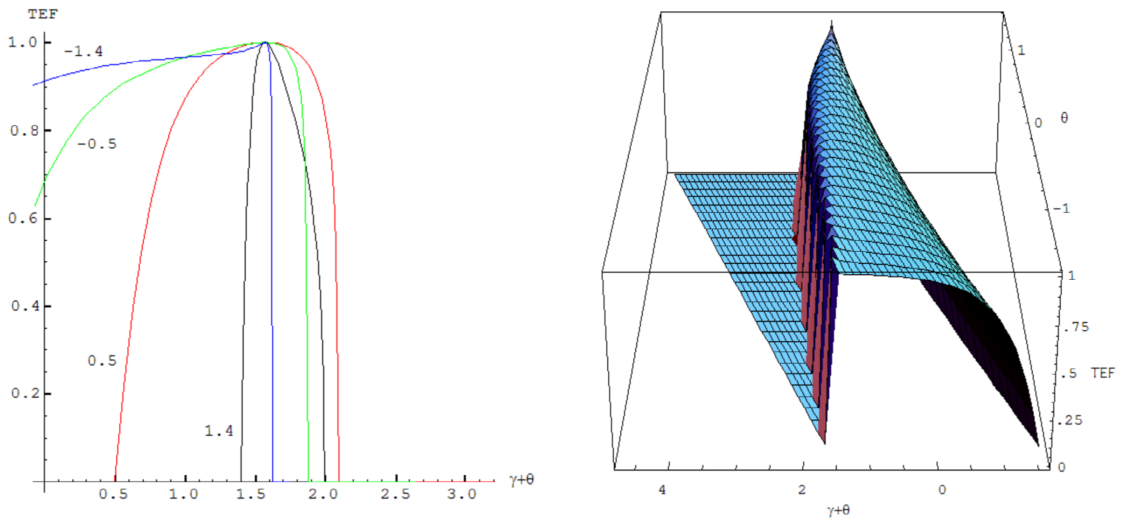


Figure 14: Transmitted energy fraction for $\vec{\nabla} h = -\beta$ for four different orientations of the boundary: 1.4, 0.5, -0.5, 1.4 radians with respect to North.

understanding of oceanography. Neil and Stefan for organizing the program for this year and, together with the other staff members providing such a rich working environment. I would like to thank also all the fellows and Shreyas and Shilpa for the friendship we cultivated during the summer.

# References

[1] M.S. Longuet-Higgins, *Planetary waves on a rotating sphere*, Proc. Ray. Soc. A, **279** (1964) 446-473.

[2] M.S. Longuet-Higgins, *On group velocity and energy flux in planetary wave motions*, Deep-Sea Research, **11** (1964) 35

[3] J. Pedlosky, *Waves in the Ocean and Atmosphere*, Class Notes 12.802, MIT/WHOI Joint Program.