Lecture 1: Introduction to ocean tides

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1 Introduction

The phenomenon of oceanic tides has been observed and studied by humanity for centuries. Success in localized tidal prediction and in the general understanding of tidal propagation in ocean basins led to the belief that this was a well understood phenomenon and no longer of interest for scientific investigation. However, recent decades have seen a renewal of interest for this subject by the scientific community. The goal is now to understand the dissipation of tidal energy in the ocean. Research done in the seventies suggested that rather than being mostly dissipated on continental shelves and shallow seas, tidal energy could excite far traveling internal waves in the ocean. Through interaction with oceanic currents, topographic features or with other waves, these could transfer energy to smaller scales and contribute to oceanic mixing. This has been suggested as a possible driving mechanism for the thermohaline circulation.

This first lecture is introductory and its aim is to review the tidal generating mechanisms and to arrive at a mathematical expression for the tide generating potential.

2 Tide Generating Forces

Tidal oscillations are the response of the ocean and the Earth to the gravitational pull of celestial bodies other than the Earth. Because of their movement relative to the Earth, this gravitational pull changes in time, and because of the finite size of the Earth, it also varies in space over its surface. Fortunately for local tidal prediction, the temporal response of the ocean is very linear, allowing tidal records to be interpreted as the superposition of periodic components with frequencies associated with the movements of the celestial bodies exerting the force. Spatial response is influenced by the presence of continents and bottom topography, and is a less well established matter.

Figure 1 shows a two month tidal record from Port Adelaide, Australia. Even though tidal records vary significantly for different coastal locations, this one in particular can be considered typical in that it clearly shows characteristics of tidal oscillations that can be directly related to astronomical forcings.

Perhaps the first feature to stand out is the semi-diurnal component, two high tides can be seen to occur on each day. A closer look reveals a modulation of the amplitude of the semi-diurnal oscillation, roughly over a one month period. Intervals of high amplitude are known as spring tides while those of lower amplitudes are known as neap tides. As indicated in the figure, the springs-neaps cycle is associated with the phases of the Moon. For a same day, there is often a difference in the amplitude of the two high tides. This is known as the

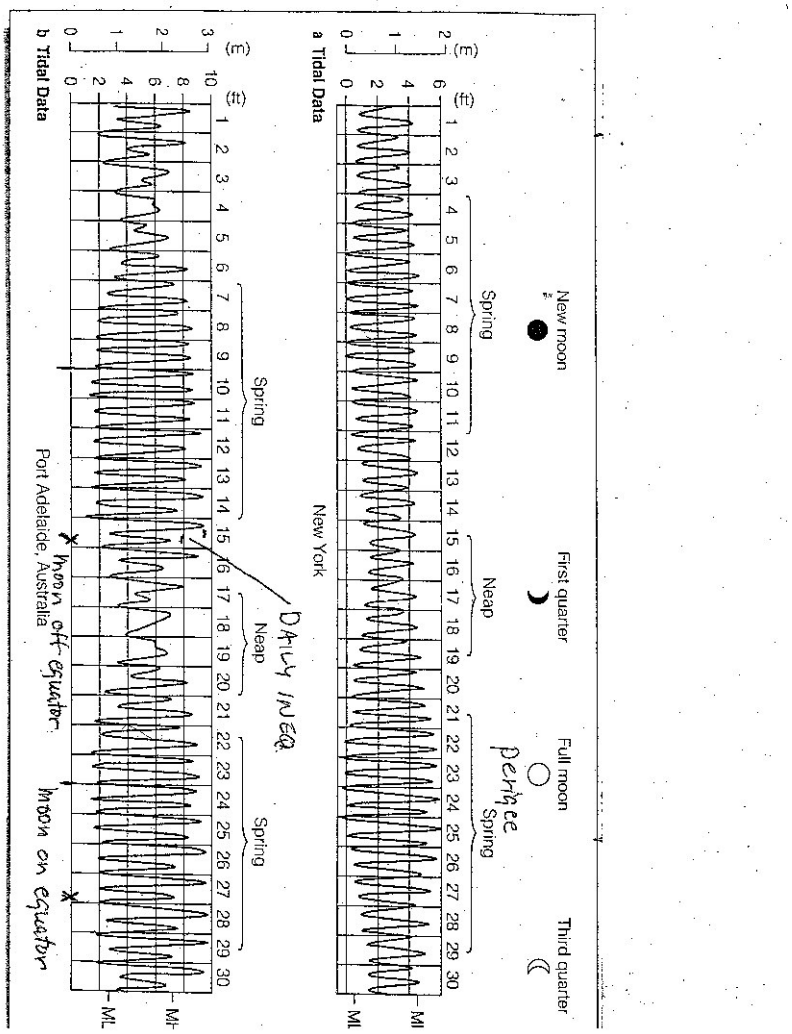

Figure 1: Two months of tidal data for Port Adelaide Australia. We can see in this record some features that can be directly accounted for by the details of the astronomical tidal forcing, such as the springs-neaps cycle, the daily inequality and the absence of daily inequality when the Moon is on the Equator.

"daily inequality" and, as indicated in the figure, disappears when the Moon is over the equator.

We will try to clarify below the basic ideas underlying tidal forcing. We will also try to explain the origin of the forcing terms responsible for causing the tidal record features described above. Finally we will attempt to explain the derivation of the tide generating potential.

3 Tidal Forcing

Even though small compared to the planets and especially to the sun, the Moon is by far the celestial body closest to the Earth. Because gravitational pull decreases linearly with mass and quadratically with distance, the Moon exerts the biggest influence over the Earth, contributing the most to the formation of tides. We will begin by considering its effects.

The centres of mass of the Earth and the Moon orbit around the common centre of mass of the Earth-Moon system. Their movements are such that centrifugal force counterbalances gravitational attraction at the individual centres of mass. The Earth is a rigid body, so every material point in it executes an identical orbit, and is therefore subject to the same centrifugal force, as illustrated in figure 2. Gravitational force however will vary because the distance between these points to the Moon may vary by up to one Earth diameter. Gravitational force will prevail over centrifugal force on the hemisphere closest to the Moon and centrifugal force will prevail on the hemisphere furthest to it. The opposite hemispheres have net forces in opposite directions, causing the ocean to bulge on both sides. As the Earth spins under this configuration, two daily tides are felt.

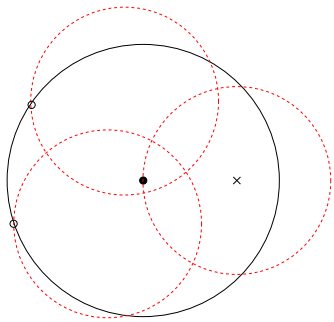

Figure 2: The centre of the Earth, shown as a filled dot, rotates about the centre of mass of the Earth-Moon system, indicated by a 'x' mark. Dashed circles show the orbital movement of the points shown by a 'o' mark on Earth's surface and the center of the Earth.

A common source of confusion regarding the argument above is to suppose that the centrifugal force relevant to the problem is due to the spinning of the Earth around its own axis. This seems reasonable at first sight because this force is constant for every latitude circle, allowing for an imbalance with lunar attraction, which is longitude dependent at any given instant. However, the centrifugal force due to the Earth's spin has permanently deformed the Earth's surface into a spheroid (as opposed to the spherical shape that would ensue from self-gravitation only). That is to say, this centrifugal force is compensated by

the Earth's own gravitational field. A mnemonic phrase to keep in mind is that tides are caused by the action of *other* celestial bodies over the Earth.

Another not entirely uncommon misconception is that tides are partly the result of the variation of centrifugal force over the surface of the Earth. This kind of confusion arises because the distance between the centre of the Earth-Moon system is smaller than one Earth radius and therefore this point lies "within" the Earth. If this were a fixed material point, like the centre of the Earth, around which the planet revolved, there would indeed be a variation of centrifugal force with distance from it. However, the centre of mass of the Earth-Moon system is just a point in space, and as the Earth revolves around it, as indicated in figure 2 none of its material points are fixed.

Even though a constant field, the centrifugal force due to the revolution of the Earth around the system's centre of mass is essential to the semi-diurnality of the tides. We can illustrate this by considering the situation in which the centre of the Earth is fixed. In this case the only force acting upon it is lunar gravitational attraction. Although uneven over the surface, it pulls every point on Earth towards the Moon, causing the water to bulge on the hemisphere closer to it. The spinning of the Earth would therefore make every point on it experience a diurnal tide cycle, instead of a semi-diurnal one. This situation is illustrated in figure 3

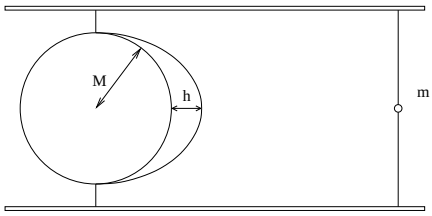

Figure 3: Tidal deformation that would ensue if from lunar attraction if the Earth's centre of mass were fixed.

We can calculate the surface elevation that would result from this forcing. The assumption is that the elevation would be such that net terrestrial gravitational force would exactly compensate for the lunar force at the point.

Net Moon's gravitational force:

$$\frac{GM}{(a+h)^2} - \frac{GM}{a^2} \simeq -\frac{GM}{a^2}\frac{2h}{a}.$$

Net Earth's gravitational force:

$$\frac{GM}{a^2}(\frac{a}{r})^2\frac{m}{M},$$

where M is the mass of the Earth, r is the distance between Earth's and Moon's centres of mass, a is the Earth's radius, h is the sea surface deformation at the sub lunar point and G is the universal gravitational constant. Equating the two forces and isolating h we get:

$$h = \frac{a^3}{2r^2}\frac{m}{M} = 10.7\text{m}. \tag{1}$$

This is an unrealistically high value.

If we now allow the Earth's centre of mass to accelerate, the centrifugal force due to this motion will compensate for the Moon's gravity at that point. We can anticipate that tidal deformation will be smaller, since it will be a response to a smaller resultant force. As explained earlier, predominance of lunar attraction on the hemisphere facing the Moon and of the centrifugal force on the one opposing deforms the surface into an ellipsoid. The water will bulge around the sub-lunar and anti-sub lunar points.

For this case we can also calculate tidal elevation on the sub-lunar point.

$$\frac{Gm}{(r-a)^2} - \frac{Gm}{r^2} \simeq \frac{GM}{r^2}\frac{2a}{r}, \tag{2}$$

where the first term on the right is lunar gravity, the second is the centrifugal force and the term on the right is the net gravitational force of the Earth. Isolating h we get:

$$h = \frac{a^4}{r^3}\frac{m}{M} = 35.8\text{cm} \tag{3}$$

The Moon rotates around the Earth in the same direction as the Earth spins, and the surface deformation must rotate with it. It takes slightly longer than a day for the Moon to be directly over the same point on the Earth's surface, as illustrated in figure 4 this is called a lunar day. Likewise, the period between two high tides is half lunar day. Apart from the semi-diurnal tides, we can expect the presence of the Moon to permanently deform the sea surface. This is an order zero effect called the permanent tide.

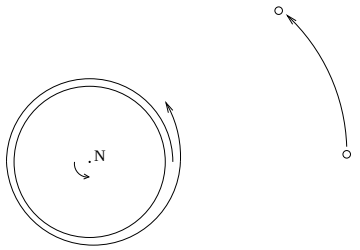

Figure 4: The Earth-Moon system. The position of the Moon with respect to a fixed point on the Earth's surface after one revolution is illustrated.

Up to now, we have considered the orbit of the Moon circular. It is however elliptic, with the Earth-Moon centre of mass being one of the foci. The Moon's gravitational force over the Earth will be modulated over the period of one anomalistic month (figure 5), as will the tidal components it generates.

In summary, tidal forcing by the Moon alone can be represented by the following harmonics:

Lunar Semi-Diurnal Tide (M_2) $2/LD$	12h 25.236 min
Lunar Elliptical (N_2)	$2/LD - 1/perigee$ 12h 3.501 min
Lunar Monthly Elliptical (M_m)	$1/perigee$ 27.5545 days (anomalistic month)

The $2/LD + 1/perigee$ term was left out because it has a small amplitude. Modulation of the amplitude of M_2 is represented by the interaction of M_2 and N_2, which is constructive once each anomalistic month.

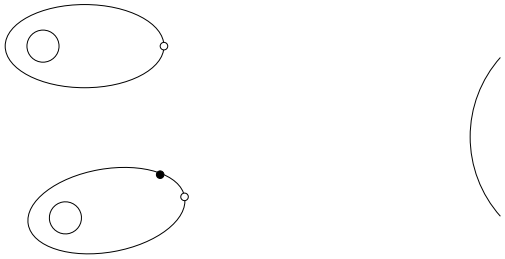

Figure 5: As the Moon rotates around the Earth, the Earth rotates around the Sun. For this reason, after each orbital period (anomalistic month), the Moon is in different position in its orbit with respect to the Sun. In the above picture, we represent on top an initial position of the Earth and Moon, below it their position after one anomalistic month. The black Moon in this case represents the position it would have to be in to exhibit the same phase as in the initial configuration.

All arguments mentioned above are valid for any other celestial body which might be reasonably considered to form a two body system with the Earth, for which the orbits are elliptical. Another such body is the sun, whose tidal effect over the Earth (when the two body system is considered in isolation) can be reduced to the harmonic components below.

Solar Semi-Diurnal Tide (S_2)	$2/SD$ 12h 25.236 min
Solar Elliptical (N_2)	$2/SD - 1/anom.yr.$ 12h 3.501 min
Solar Annual Elliptical ()	$1/perihelion$ 365.25964 days (anomalistic year)

When Moon and Sun are aligned with the Earth, their semi-diurnal components interfere constructively, giving rise to tides of larger amplitude, known as spring tides. When they are in quadrature, the interference is exactly destructive, giving rise to smaller amplitude tidal variations, called neap tides. The relative arrangement of the Earth Sun and Moon is perceived on the Earth as the phases of the Moon, and therefore the springs-neaps cycle has a period of one lunar (synodic) month. A lunar month is the duration required for the Moon to return to a fixed position in its orbit in relation to the Sun as illustrated in figure 5.

Up to now we have assumed that the orbits of the Earth (around the Sun) and Moon are coplanar to the spinning of the Earth at all instants of time. In reality these planes intersect at an angle. The effect this has over the tide is illustrated in figure 6. As the Earth rotates,

it perceives the tidal surface as being "tilted" in relation to latitude. In terms of harmonics, this is represented by a daily component, which gives rise to the daily inequality. When the tide generating bodies intersect the equatorial plane, the daily inequality disappears. In the Port Adelaide record (figure 1), we can see that daily inequality disappears when the Moon is on the Equator.

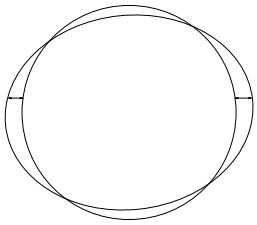

Figure 6: The Moon's orbit is not coplanar with the Equator. The tidal surface is in general "tilted" with respect to the Equator giving rise to the daily inequality.

The amplitude of the declinational components depends on the angle of the bodies orbit to the equatorial plane. Both the lunar orbital plane and the ecliptic precess, modulating the declinational tides. The Moon's orbit precesses over 18.6 years, its angle to the Ecliptic varying between $-5°08'$ and $5°08'$. In relation to the Earth's equatorial plane the variation is between $23°27' - 5°08'$ and $23°27' + 5°08'$.

As mentioned earlier, the usefulness of decomposing the tide generating force into harmonics is due to the linearity of the oceans response to it in time. In fact, we have taken this for granted in the preceding section when we explained the springs-neaps cycle purely as the result of the interference of two forcing terms. This property allows for more precise tidal prediction. Tidal records are not used to determine the important frequencies in their harmonic expansions, these are known from astronomical considerations. Data is used only to determine the amplitudes of local response to these terms.

4 Spatial Structure of the Tides

As the Earth's spinning under the tide generating potential is felt as the propagation of the tidal wave. However, this propagation is obstructed by the presence of continents and bottom topography. Real co-tidal lines therefore look nothing like the constant phase lines of the tide generating potential. As a plane wave enters a basin, it feels the effect of the Earth's rotation and propagates along its borders. The nodal line that would exist in the case with no rotation degenerates into a nodal point, called amphidromic point.The irregularity of the oceanic basins and of bottom topography disrupt the propagation and a precise map of tidal propagation could only be obtained after the advent of satellite altimetry. This data is harmonically analyzed to obtain maps for the different astronomical components. Figure 7 shows a co-tidal map obtained in this manner. Although the amphidromic points are eye catching, the less conspicuous anti-amphidromic points, for which tidal amplitude is maximum and there is almost no phase variation, are probably more useful for testing satellite altimetry.

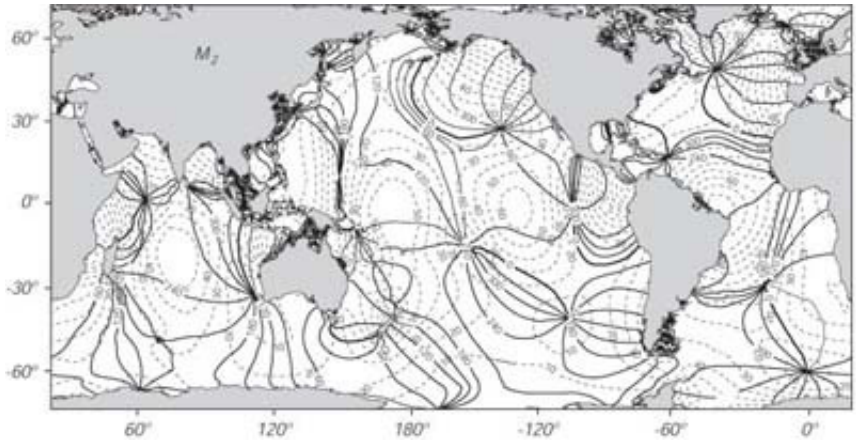

Figure 7: Cotidal map. Amphidromic points are the ones where cotidal lines cross. In these points tidal amplitude is zero and phase speed infinite. Less conspicuous are the anti-amphidromic points, where tidal amplitude is maximum and phase stationary.

In the following section we will formalize the ideas outlined above so as to arrive at an expression for the tide generating potential.

5 The tide generating potential

We want to calculate the tidal force that the Moon or Sun exerts on the Earth, in particular on the oceans. Remember that we are only interested in the effects of *another* body on the Earth, not the effect of the Earth's rotation and gravity on its shape and that of the oceans.

Consider the plane made up of the centre of the Earth, the tide generating body (the Moon or the Sun), assumed to be a point mass, and an observer at P on the surface of the Earth, as shown in figure 8. For the moment, we assume R is constant.

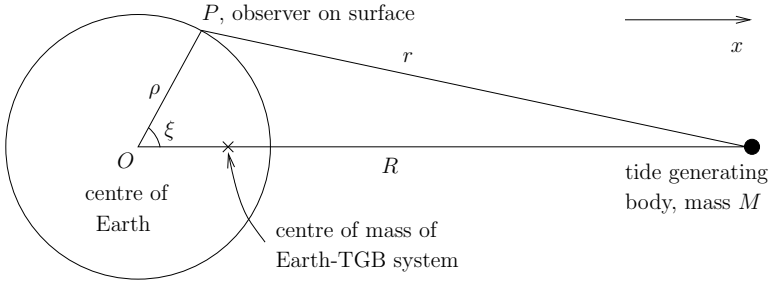

Figure 8: The geometry for calculating the tidal force at P due to the tide generating body.

The tide generating body (TGB) exerts a force, $\mathbf{F}$, on the centre of the Earth

$$\mathbf{F} = \frac{GM}{R^2}\,\hat{\imath},$$

where $\hat{\imath}$ is the unit normal along the x-direction and G is the universal gravitational constant. From this, the potential at the centre of the Earth, O, due to the TGB is

$$V(O) = \frac{GM}{R^2}\, x + \text{const} = \frac{GM}{R^2}\rho\cos\xi + \text{const},$$

where the constant is arbitrary.

Then the resultant force at P is the gravitational force at P, less that at O, with potential

$$V(P) = \frac{GM}{r} - \frac{GM}{R^2}\rho\cos\xi. \tag{4}$$

The sign is chosen as r is measured away from the TGB and the force should be towards it. Note that the force on the Earth as a whole, $\mathbf{F}$, is balanced by the centrifugal force due to the motion of the Earth about the common centre of mass of the Earth–TGB system.

The parameters r, R, ρ and ξ are linked by $r^2 = R^2 - 2\rho R\cos\xi + \rho^2$ from properties of triangles. Hence

$$\frac{1}{r} = \frac{1}{R}\left(1 - \frac{2\rho}{R}\cos\xi + \frac{\rho^2}{R^2}\right)^{-\frac{1}{2}} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n P_n(\cos\xi), \tag{5}$$

where $P_n(z)$ is the nth Legendre polynomial, with $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = (3z^2-1)/2$, ... The final equality in (5) may be found in, for example, Morse and Feshbach [1].

Hence (4) becomes

$$V(P) = \frac{GM}{R}\left[1 + \sum_{n=2}^{\infty}\left(\frac{\rho}{R}\right)^n P_n(\cos\xi)\right].$$

For the Moon, $0.0157 \leq \rho/R \leq 0.0180$ and for the Sun $\rho/R \sim 10^{-4}$. Hence the potential may be truncated at $n = 2$. Forgetting about the constant GM/R, which is unimportant since ultimately we want to find the forces, it becomes

$$V(P) \simeq \frac{GM}{R}\left(\frac{\rho}{R}\right)^2 P_2(\cos\xi). \tag{6}$$

Note that this potential is symmetric in ξ. This is consistent with the discussion in previous sections, where we argued that the tide generating force is symmetrical with respect to the plane that contains the Earth's centre of mass and is orthogonal to the Earth-Moon axis.

5.1 The tide generating potential in geographical coordinates

It is more useful to express the tidal potential in geographical coordinates: actual latitude and longitude of the observer on the Earth and the apparent latitude and longitude of the TGB. This coordinate system is shown in figure 9. We call attention to the fact that the longitudinal angles are measures with respect to the Equator. In this coordinate system the tidal ellipsoid is "tilted", and the tidal potential will therefore have asymmetrical components.

From spherical trigonometry

$$\cos\xi = \sin\theta\sin\delta + \cos\theta\cos\delta\cos H.$$

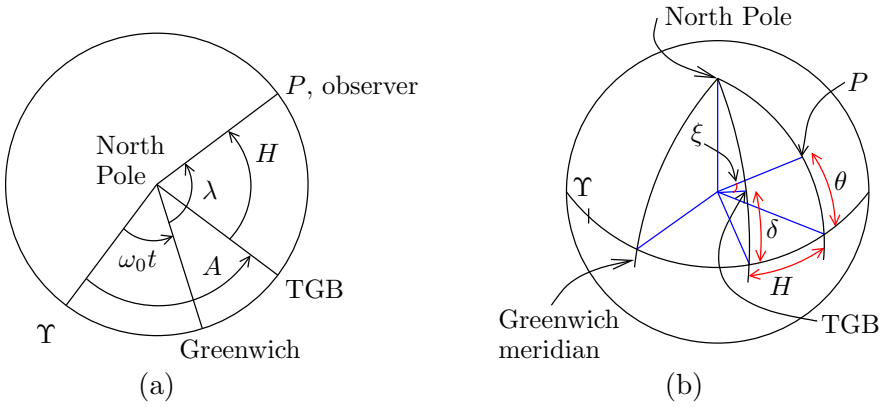

Figure 9: The geographical coordinates: (a) looking down from the North Pole and (b) looking at the spherical Earth. Here λ is the geographic longitude of the observer and θ is the geographic latitude. ξ is the angle between the TGB and the observer as in figure 8. δ is the *declination* of the TGB and A is the *right ascension* of the TGB, or its apparent longitude with respect to an origin Υ. Υ is a celestial reference point from which to measure positions of the TGB: it is the northward crossing of the Sun at equinox, or equivalently the point of intersection of the equatorial plane and the *ecliptic*, the plane of the Earth's orbit about the Sun, as the Sun travels northward. The point Υ is assumed fixed with respect to the fixed stars for the current discussion. However, with respect to the rotating Earth, the point Υ moves. If $t = 0$ is taken to be the time at which Υ lies on the Greenwich meridian, then at time t, the Earth has rotated $\omega_0 t$ giving the angle between Υ and Greenwich, where ω_0 is the frequency associated with the period of rotation of the Earth about its axis so that the TGB appears again in the same Earth–TGB orientation. $H = \omega_0 t + \lambda - A$ is referred to as the hour angle.

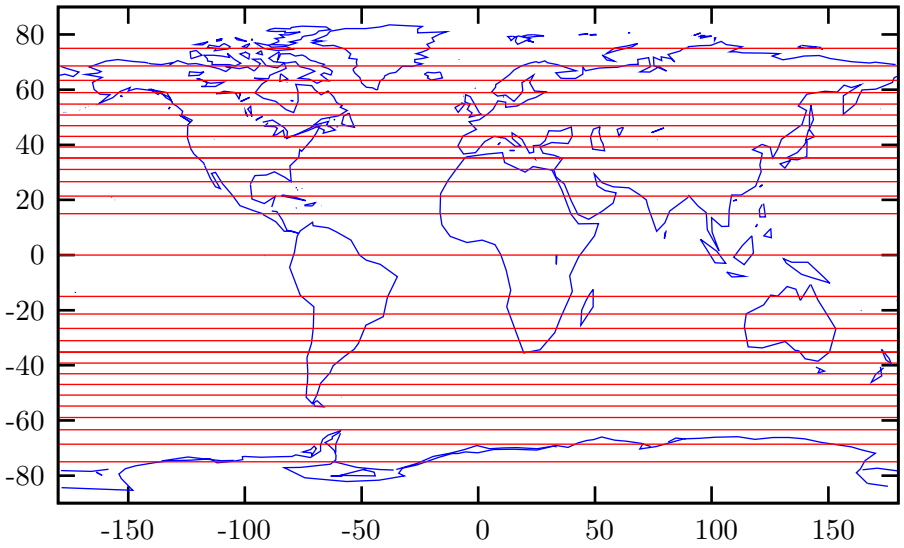

Figure 10: Equipotential lines for the long-period potential.

Substituting this in (6) gives

$$V(\lambda,\theta) \simeq \frac{3GM\rho^2}{4R_0^3}\left(\frac{R_0}{R}\right)^3\left[\frac{4}{3}\left(\frac{1}{2}-\frac{3}{2}\sin^2\theta\right)\left(\frac{1}{2}-\frac{3}{2}\sin^2\delta\right)\right.$$
$$\left.+\sin 2\theta\sin 2\delta\cos H+\cos^2\theta\cos^2\delta\cos 2H\right], \qquad (7)$$

where R_0 is a reference value of the orbital distance R of the TGB. The coefficient $3GM\rho^2/4R_0^3$ is referred to as the *Doodson constant*, D. For the Moon, $D_{\rm Moon}/g = 26.75\,$cm and for the Sun, $D_{\rm sun} = 0.4605 D_{\rm Moon}$.

The first term in the square bracket in (7) has no dependence on the hour angle, H, and gives rise to a long-period[1] potential. The second term gives rise to a diurnal potential and the final term to a semi-diurnal potential. Figures 10-12 show plots of the instantaneous equipotential lines for the three components of (7) and plots of the cotidal[2] and corange[3] lines. The plots shown are in fact representative of the solid Earth tide, since the response time of the Earth is of the order of an hour (deduced from earthquake measurements), much quicker than the time period of the tidal potential and so the solid Earth can adjust to the equipotential surfaces. The oceans have a much longer response time.

In reality, the declination and orbital distance in (7) vary in time. In the following sections, we consider how these variations change the potential.

[1] In the present analysis, infinitely long. However we shall see that each term of the potential is modulated and hence this becomes a long-period potential.

[2] A line passing through points at which high tide occurs at the same number of hours after the Moon transits the Greenwich meridian.

[3] A line passing through points of equal tidal range.

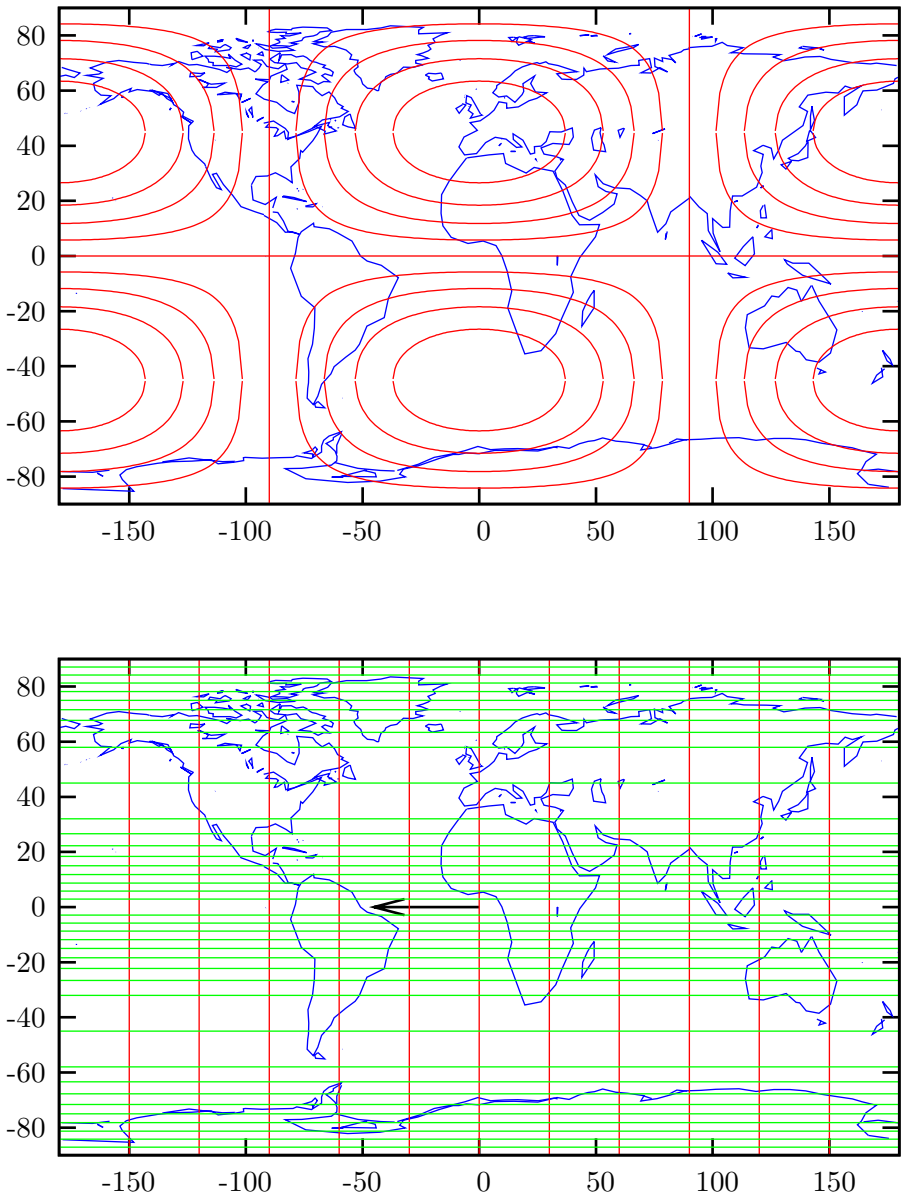

Figure 11: Equipotential lines (top) and cotidal, green, and corange, red, lines (bottom) for the diurnal potential.The arrow indicates the direction of increasing cotidal time.

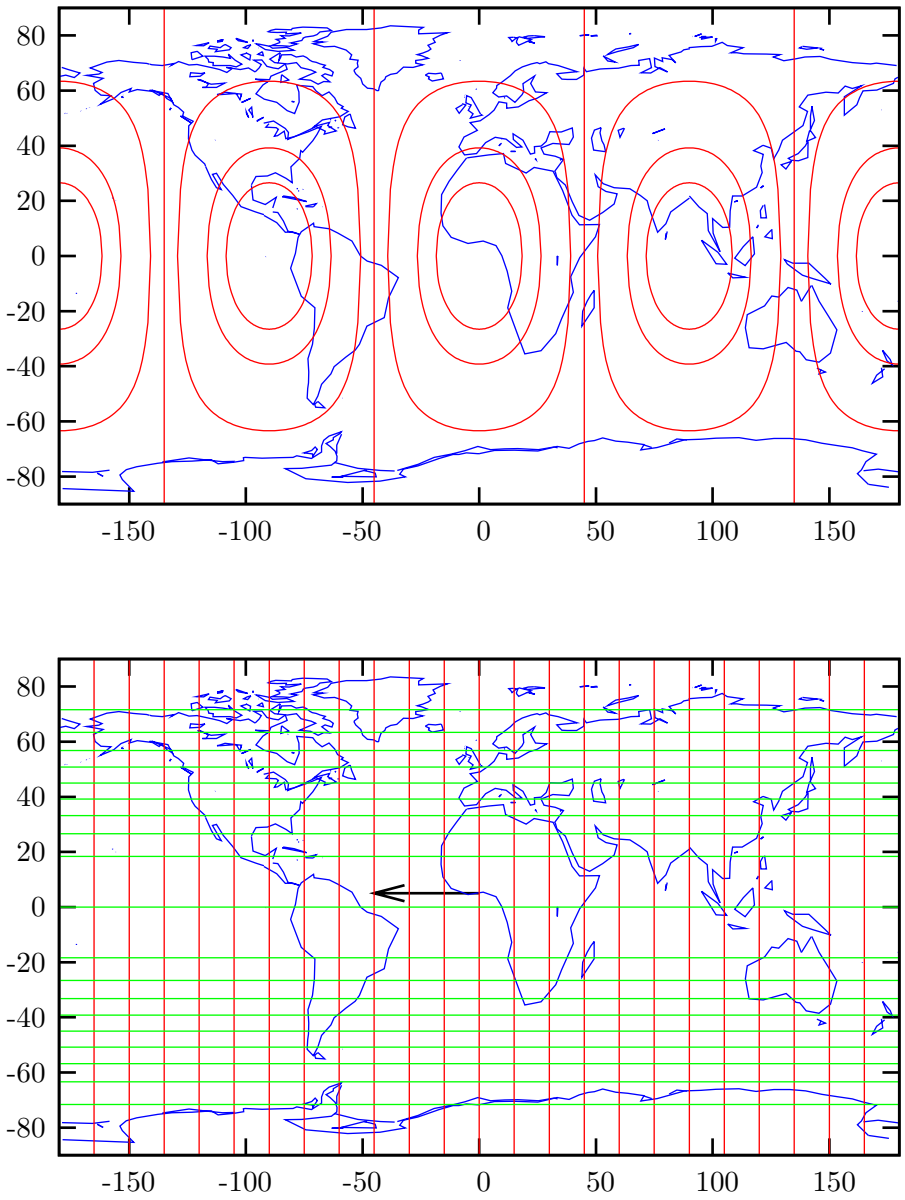

Figure 12: Equipotential lines (top) and cotidal, green, and corange, red, lines (bottom) for the semi-diurnal potential. The arrow indicates the direction of increasing cotidal time.

5.2 Variation of the declination of the tide generating body

The angle δ in figure 9b and (7) varies in time, based on the location of the tide generating body relative to the plane of the equator. This variation has a time period related to the precession of the equinoxes for the Sun and to the precession of the lunar node for the Moon.

5.2.1 Precession of the equinoxes

The Earth's rotational axis is tilted, at present at $24°27'$, to the ecliptic.[4] As the Earth rotates about the Sun, this means that the declination of the Sun varies, as shown in figure 13a. Hence only after a 'year' is the declination expected to return to its initial value.

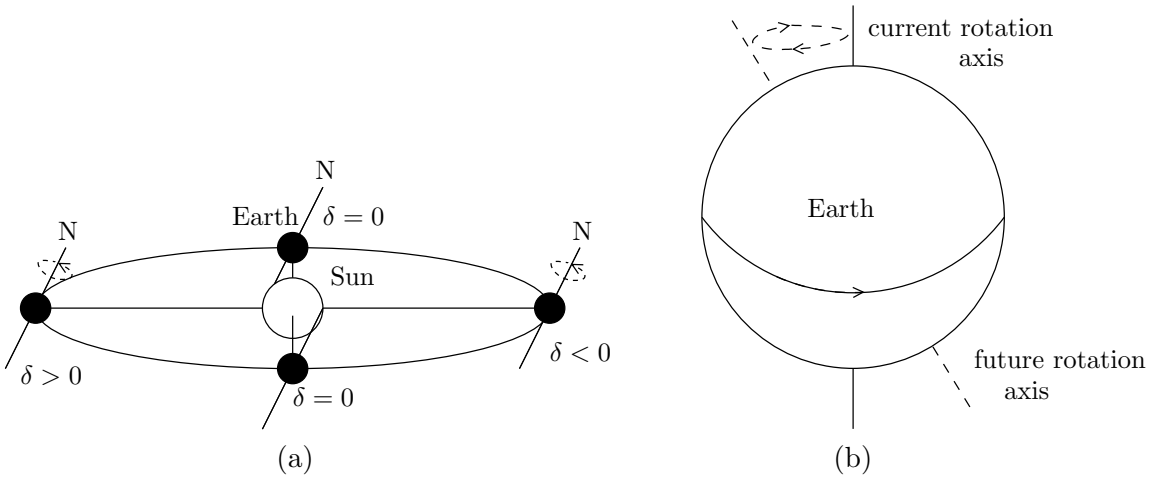

Figure 13: (a) The variation in declination due to the rotation of the Earth about the Sun. (b) The precession of the equinoxes.

Lunar and solar gravity act on the oblate Earth, making it spin like a top, with its rotation axis precessing as depicted in figure 13b. The celestial point where the Sun crosses the plane of the equator, moving from south to north is known as the *Point of Aries* or the vernal equinox and is the point Υ in figure 9 at which $\delta = 0$. Due to precession, it moves eastward relative to the fixed stars.[5] The period of precession is 25570 years. The time period of the revolution of the Earth about the Sun from vernal equinox to vernal equinox is the tropical year and is 365.242199 mean solar days. This is the 'year' we are interested in for the declination returning to its original value.

5.2.2 Precession of the lunar node

The plane of the Moon's orbit around the Earth is inclined at $5°08'$ with respect to the ecliptic. It precesses with a period of 18613 years due to the Earth's gravity.

[4] The plane of the Earth's orbit around the Sun.

[5] In antiquity, it *was* in the constellation Aries. Now it is in Pisces.

The intersection of the plane of the Moon's orbit with the equatorial plane as the Moon goes from south to north is the *ascending lunar node*, Ω. The mean time period separating adjacent passages through Ω is the tropical month of 27.321582 mean solar days. This is the time period before the same declination of the Moon is again achieved.

5.3 Variation of the orbital distance of the TGB

In reality, R is not fixed in (7), since the Moon and Sun are not a constant distance from the Earth. Here we consider how the orbital distance of the Moon varies in time. The same analysis also applied for the Sun.

5.3.1 Kepler's laws

The orbit of the Moon about the Earth is an ellipse (Kepler's first law), as we now show. Consider the geometry shown in figure 14. From Newton's laws,

$$m\ddot{\mathbf{x}} = -GmM\mathbf{x}/R^3,$$

where $\mathbf{x}$ is the position vector of the Moon relative to the Earth.

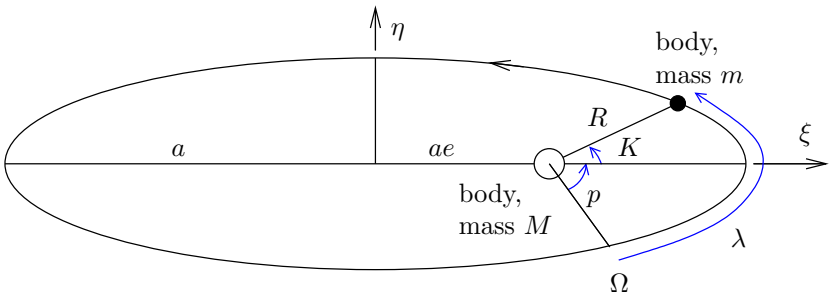

Figure 14: The Moon of mass m in orbit about the Earth of mass M. Assume that the Earth is fixed in space.

Rewriting this equation in polar coordinates, with $x = R\cos\lambda_e$ and $y = R\sin\lambda_e$ gives

$$\ddot{R} - R\dot{\lambda_e}^2 = -GM/R^2, \qquad R\ddot{\lambda_e} + 2\dot{R}\dot{\lambda_e} = 0.$$

Integrating the second equation gives Kepler's second law

$$R^2\dot{\lambda_e} = \text{constant} = h, \tag{8}$$

which says that the line joining the orbiting Moon and the Earth sweeps out equal areas in equal intervals of time. Integrating the first equation, by setting $u = 1/R$, gives

$$\frac{1}{R} = A'\cos K + \frac{GM}{h^2}, \tag{9}$$

where A' is a constant of integration and we have used the fact that $1/R$ is symmetric about the line $\eta = 0$. This is the equation of an ellipse with semi-major axis a and eccentricity e satisfying $A' = e/a(1-e^2)$ and $gM/h^2 = 1/a(1-e^2)$.

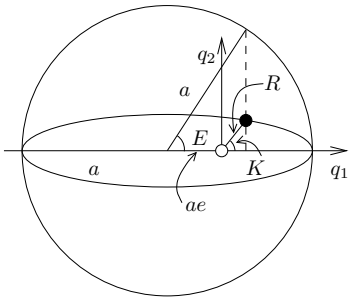

Figure 15: The definition of the eccentric anomaly E.

We now want to see how this modifies the potential given in (7). Instead of using the *true anomaly*, K, it is more useful to write (9) in terms of the *eccentric anomaly*, E, defined as shown in figure 15. Then it is possible to write

$$R = a(1 - e\cos E) \tag{10}$$

and the motion of the Moon in its orbit is given by *Kepler's equation*

$$E - e\sin E = \omega_k t, \tag{11}$$

in which $t = 0$ at perigee[6] and $\omega_k = \sqrt{GM/a^3}$, the *Kepler frequency*. The *mean anomaly*, E_0 is related to E by

$$E_0 = E - \sin E$$

and increases uniformly in time: $E_0 = \omega_k t$ from (11).

In (7) we then have, assuming the eccentricity e is small,

$$R_0/R \equiv a/R = (1 - e\cos E)^{-1} \sim 1 + e\cos E_0 + e^2\cos 2E_0 + \ldots,$$

from (10) and

$$R_0/R = a/R = (1 + e\cos K)/(1 - e^2),$$

from (9). Hence $K \simeq E_0 + 2e\sin E_0 + \ldots$.

Remembering from figure 14 that the *ecliptic longitude*, $\lambda_e = p + K$ and setting the *mean longitude* to be $h = p + E_0$ we obtain

$$\begin{aligned} R/R_0 &= 1 + e\cos(h - p) + e^2\cos 2(h - p) + \ldots, \\ \lambda_e &= h + 2e\sin(h - p) + \ldots. \end{aligned}$$

This can be simply translated into geographical coordinates for the Sun as the tide generating body. For the Sun

$$\sin\delta = \sin\lambda_e \sin\epsilon, \qquad A = \lambda_e - \tan^2(\epsilon/2)\sin\lambda_e,$$

[6]The point in the orbit of the Moon nearest the Earth.

where δ is the declination, λ_e is the ecliptic longitude, A is the right ascension of the Sun and ϵ is the angle between the ecliptic and the equatorial plane.

For the Earth–Moon system it is actually more complex to write the solution in terms of the geographical coordinates as the lunar node does not coincide with the point of Aries and the Moon's orbit is not in the ecliptic. Furthermore, there is a strong solar perturbation to its orbit.

5.4 Tidal harmonics

The effect of the variations of declination and distances to the tide generating bodies is to alter the coefficients for terms in (7). These variations may be Fourier decomposed and result in modulations of the basic tidal frequencies: the long-period, diurnal and semi-diurnal. Then the potential given in (7) may be written as

$$V(\lambda,\theta) = V_0(\lambda,\theta) + V_1(\lambda,\theta) + V_2(\lambda,\theta), \tag{12}$$

where

$$V_s(\lambda,\theta) = DG_s \sum_j C_j \cos(\sigma_j t + s\lambda + \theta_j)$$

with $G_0 = (1 - 3\sin^2\theta)/2$, $G_1 = \sin 2\theta$ and $G_2 = \cos^2\theta$; D the Doodson constant and C_j the amplitude of the component. The harmonic frequency σ_j is a linear combination of the angular velocity of the Earth's rotation ω [already seen in (7) in the hour angle] and the sum and the difference of angular velocities ω_k with $k = 1,\ldots,5$ which are the five fundamental astronomical frequencies, having the largest effect modifying the potentials (it is possible to include many more). These five frequencies are given in table 1. Hence

$$\sigma_j = s\omega + \sum_{k=1}^{5} m_k^j \omega_k,$$

where $s = 0, 1, 2$ for the long-period, diurnal and semi-diurnal respectively; $m_k^j = 0, \pm1, \pm2, \ldots$ and ω is either taken to be $\omega_0 - \omega_1$ for the Moon as the TGB, or $\omega_0 - \omega_2$ for the Sun with ω_0 the sidereal[7] frequency. λ is the longitude of the observer.

All tidal harmonics with amplitudes $C > 0.05$ are given in table 2.

5.4.1 Doodson numbers

For convenience, the frequencies σ_j may be written as

$$\text{Doodson number} = s\, m_1^j\, m_2^j\, m_3^j\, m_4^j\, m_5^j + 0\,5\,5\,5\,5\,5,$$

where the addition of 055555 is simply so that the Doodson numbers are all positive (since in general the m_i^j lie in the range $-5 \le m_i^j < 5$). The Doodson numbers are also given in table 2.

[7]The length of time between consecutive passes of a given 'fixed' star in the sky over the Greenwich meridian. The sidereal day is 23 hr56 min, slightly shorter than the 'normal' or solar day because the Earth's orbital motion about the Sun means the Earth has to rotate slightly more than one turn with respect to the 'fixed' stars in order to reach the same Earth–Sun orientation.

Period	Nomenclature
$360^\circ/\omega_1 = 27.321582\,\text{days}$	period of lunar declination
$360^\circ/\omega_2 = 365.242199\,\text{days}$	period of solar declination
$360^\circ/\omega_3 = 8.847\,\text{years}$	period of lunar perigee rotation
$360^\circ/\omega_4 = 18.613\,\text{years}$	period of lunar node rotation
$360^\circ/\omega_5 = 20940\,\text{years}$	period of perihelion rotation

Table 1: The fundamental periods of the Earth's and the Moon's orbital motion. *From Bartels* [2].

5.4.2 Spectra of the tides

Figure 16 shows a plot of the spectrum of equilibrium tides with frequencies near twice per day (the semi-diurnal tides). The spectrum is split into groups separated by a cycle per month (0.55°hr^{-1}). Each of these is further split into groups separated by a cycle per year (0.04°hr^{-1}). The finest splitting in the figure is at a cycle per 8.847 years ($0.0046^\circ\text{hr}^{-1}$).

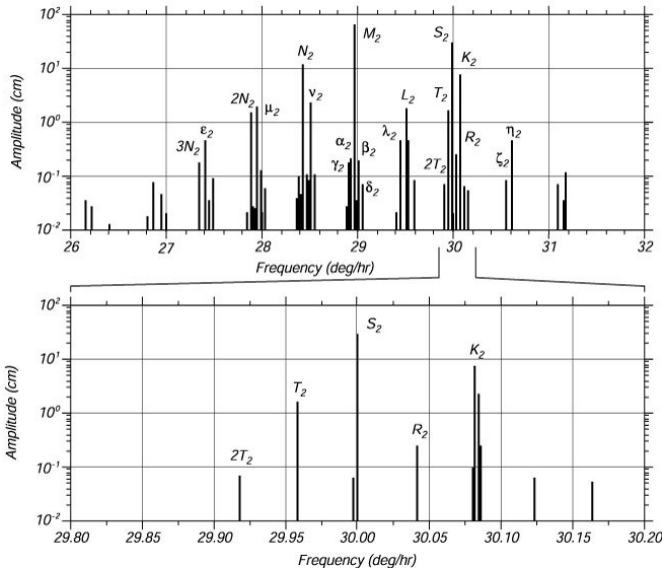

Figure 16: A spectrum for equilibrium tides. *From* `oceanworld.tamu.edu/resources/`.

Notes by Josefina Arraut and Anja Slim.

References

[1] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[2] J. Bartels, *Handbuch der Physik* (Springer, Berlin, 1967), Vol. 48.

Ampl., C	Frequency, σ $^\circ\,\mathrm{hr}^{-1}$	Period $360^\circ/\sigma$	Doodson number	Notation
		Long period tides		
0.2341	0		055555	S_0 (solar constant)
0.5046	0		055555	M_0 (lunar constant)
0.0655	$\omega_4 = 0.00221$	18.613 years	055565	– (nodal M_0)
0.0729	$2\omega_2 = 0.08214$	182.621 days	057555	S_{sa} (declinational S_0)
0.0825	$\omega_1 - \omega_3 = 0.54437$	27.555 days	065455	M_m (elliptical M_0)
0.1564	$2\omega_1 = 1.09803$	13.661 days	075555	M_f (declinational M_0)
0.0648	$2\omega_1 + \omega_4 = 1.10024$	13.663 days	075565	– (nodal M_0)
		Diurnal tides		
0.0722	$\omega_(- 2\omega_1 + \omega_3 = 13.39866$	26.868	135655	Q_1 (elliptical O_1)
0.0710	$-\omega_(+ \omega_1 - \omega_4 = 13.94083$	25.823	145565	– (nodal O_1)
0.3769	$\omega_(- \omega_1 = 13.94304$	25.819	145555	O_1 (basic lunar)
0.1755	$\omega_\odot - \omega_2 = 14.95893$	24.066	163555	P_1 (basic solar)
0.1682	$\omega_\odot + \omega_2 = 15.04107$	23.934	165555	K_1^S (declinational P_1)
0.3623	$\omega_(+ \omega_1 = 15.04107$	23.934	165555	K_1^M (declinational O_1)
0.0718	$\omega_(- \omega_1 + \omega_4 = 15.04328$	23.931	145565	– (nodal K_1^M)
		Semi-diurnal tides		
0.1739	$2\omega_(- \omega_1 + \omega_3 = 28.43973$	12.658	245655	N_2 (elliptical M_2)
0.9081	$2\omega_(= 28.98410$	12.421	255555	M_2 (basic lunar)
0.4229	$2\omega_\odot = 30.00000$	12.000	273555	S_2 (basic solar)
0.0365	$2\omega_\odot + 2\omega_2 = 30.08214$	11.967	275555	K_2^S (declinational S_2)
0.0786	$2\omega_(+ 1\omega_2 = 30.08214$	11.967	275555	K_2^M (declinational M_2)
		Combined tides		
0.5305	$\omega_0 = 15.04107$	23.934		K_1 (lunar-solar declinational)
0.1151	$2\omega_0 = 30.08214$	11.967		K_2 (lunar-solar declinational)

Table 2: The tidal harmonics with amplitude coefficients $C > 0.05$. $\omega_\odot$ is $\omega_0 - \omega_2$, associated with the Sun and $\omega_($ is $\omega_0 - \omega_1$, associated with the Moon. In the table, for the Doodson numbers, $\omega = \omega_($ and hence $\omega_\odot = \omega_(+ \omega_1 - \omega_2$. Note that the table also includes the K_2^S harmonic, even though it's amplitude is less than 0.05. Its frequency coincides with that of K_2^M and they are completely indistinguishable. They are combined into the single lunar-solar semi-diurnal wave, K_2. *From Bartels* [2].