Simple Models with Cascade of Energy and Anomalous Dissipation

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1 Introduction

The phenomenon of turbulence remains one of modern physics greatest unresolved challenges. Turbulent fluid flow exhibits an extraodinarily complex structure which manifests itself in a wide range of length and time scales, posing a significant problem in its analytical study as well as in numerical simulation. While many aspects of turbulence are quite controversial, it is generally accepted that turbulence is characterized by a nonlinear transfer of energy from large length scales to smaller and smaller ones, wherein energy is dissipated at the length scale of the molecular viscosity ν [5, 6, 8]. What can then be said about energy dissipation in the regime of fully-developed turbulence—that is, as $Re \to \infty$, or equivalently, as $\nu \to 0$?

In his 1949 paper, Onsager [7] made the surprising conjecture that turbulent flow can remain dissipative *even in the inviscid limit*. By transferring energy to ever smaller scales and gradually dividing it amongst infinitely many degrees of freedom, the driving mechanism behind such "anomalous dissipation" is the energy cascade itself! Onsager thus suggested that the role of viscosity in energy dissipation is secondary to that of the cascade process. The purpose of this paper is to present simple exactly solvable models which exhibit these very features of a cascade of energy and anomalous dissipation, and to demonstrate that Onsager's conjecture is indeed realizable within this elementary framework.

1.1 Energy Balance and Onsager's Conjecture

We begin our discussion in the context of the 3D incompressible Navier-Stokes equations with viscosity $\nu > 0$, forcing $\mathbf{f}$, and periodic boundary conditions

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}, & \mathbf{x} \in \Omega = [0, L_f]^3 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1}$$

where $\mathbf{f}(\mathbf{x}, t)$ is a stationary, homogeneous forcing acting on large scales:

$$\mathbb{E}\mathbf{f}(\mathbf{x}, t) = 0,\ \mathbb{E}\mathbf{f}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}', t') = F\left(\frac{\mathbf{x} - \mathbf{x}'}{L_f}\right)\delta(t - t'). \tag{2}$$

For our present discussion, let

$$\mathbf{f}(\mathbf{x},t) = \sigma \sum_{i=1}^{3} \left(\dot{W}_{x_i}(t) \sin \frac{2\pi x_i}{L} + \dot{W}'_{x_i}(t) \cos \frac{2\pi x_i}{L} \right) \hat{\mathbf{e}}_i \tag{3}$$

which satisfies (2) with $F(\mathbf{x}) = \frac{1}{2}\sigma^2 \sum_{i=1}^{3} \cos(2\pi|\mathbf{x}\cdot\mathbf{e}_i|)$. Consider now the energy density of the system

$$E(t) \doteq \frac{1}{|\Omega|} \int_\Omega \frac{1}{2} \mathbb{E}\mathbf{u}^2(\mathbf{x},t) d\mathbf{x} = \frac{1}{|\Omega|} \int_\Omega \frac{1}{2} \left(\lim_{T\to\infty} \frac{1}{t+T} \int_{-T}^{t} \mathbf{u}^2(\mathbf{x},t') dt' \right). \tag{4}$$

where the last equality holds by ergodicity. Multiplying (1) by $\mathbf{u}$ and integrating by parts gives

$$\dot{E}(t) = \frac{d}{dt} \frac{1}{|\Omega|} \int_\Omega \frac{1}{2} \mathbb{E}|\mathbf{u}|^2 d\mathbf{x} = -\nu \frac{1}{|\Omega|} \int_\Omega \mathbb{E}|\nabla \mathbf{u}|^2 d\mathbf{x} + \varepsilon \tag{5}$$

where $\varepsilon = \frac{3}{2}\sigma^2$, the density of the energy flux into the system through forcing, appears by Îto's formula. Assuming the system is in a statistical steady state ($\dot{E}(t) = 0$) then there exists a global energy balance between forcing and dissipation through viscosity:

$$\nu \frac{1}{|\Omega|} \int_\Omega \mathbb{E}|\nabla \mathbf{u}|^2 d\mathbf{x} = \varepsilon \tag{6}$$

To arrive at a local energy balance, consider a dimensional argument with $L =$ *length* and $T =$ *time*. Since

$$[L_f] = L \quad [\varepsilon] = \frac{L^2}{T^3} \quad [\nu] = \frac{L^2}{T}, \tag{7}$$

the only length scale which can be derived from ν and ε is the viscous length scale

$$l_\nu = C \epsilon^{-\frac{1}{4}} \nu^{\frac{3}{4}} \tag{8}$$

which vanishes in the inviscid limit. From a local perspective, energy that is pumped into the system at the forcing length scale L_f cascades to smaller and smaller scales and is subsequently removed from the system at the length scale of the viscosity l_ν (see Figure 1).

The cascade picture is more readily observed in the Fourier space setting. The Fourier representation of (1) is

$$\frac{d}{dt} \hat{u}_{\mathbf{k}} = -i \mathbf{P}_{\mathbf{k}^\perp} \sum_{\mathbf{q}} (\mathbf{q} \cdot \hat{u}_{\mathbf{k}-\mathbf{q}}) \hat{u}_{\mathbf{q}} - \nu |\mathbf{k}|^2 \hat{u}_{\mathbf{k}} + \hat{f}_{\mathbf{k}} \tag{9}$$

where $\mathbf{P}_{\mathbf{k}^\perp} = \mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}$ is the projection on the space of divergence-free velocity fields and

$$\hat{u}_{\mathbf{k}}(t) = \frac{1}{L^{\frac{3}{2}}} \int_{[0,L]^3} \mathbf{u}(\mathbf{x},t) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$$

$$\hat{f}_{\mathbf{k}}(t) = \frac{1}{L^{\frac{3}{2}}} \int_{[0,L]^3} \mathbf{f}(\mathbf{x},t) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}.$$

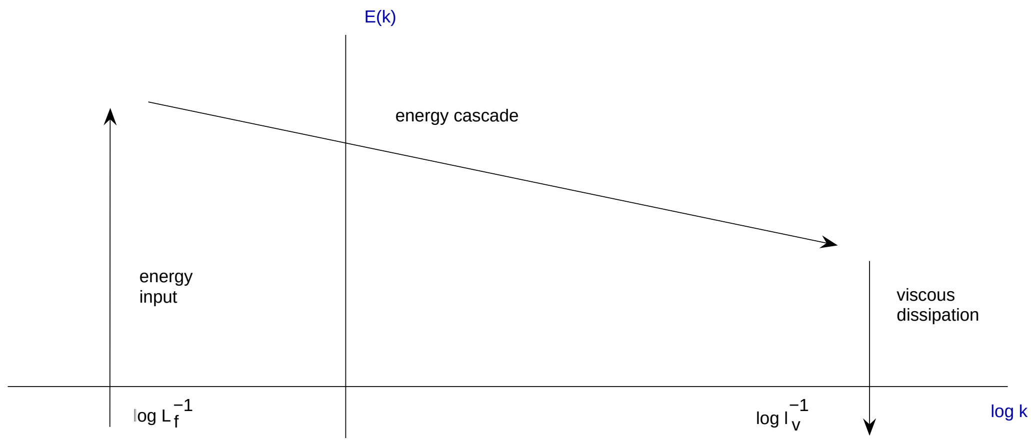

Figure 1: Cascade of energy from forcing length scale L_f to viscous length scale l_ν.

The first, second, and third terms on the right-hand side of (9) correspond to the mechanism of energy transfer between modes, dissipation at l_ν^{-1}, and energy input at L_f^{-1}, respectively. By Parseval's identity, the energy equation is then

$$\begin{aligned} \dot{E}(t) &= \frac{d}{dt}\sum_{\mathbf{k}} \frac{1}{2}\mathbb{E}|\hat{u}_{\mathbf{k}}|^2 \\ &= i\sum_{\mathbf{k}} \mathbf{P}_{\mathbf{k}^\perp}\mathbb{E}\sum_{\mathbf{q}}(\mathbf{q}\cdot\hat{u}_{\mathbf{k}-\mathbf{q}})(\hat{u}^*_{\mathbf{k}}\cdot\hat{u}_{\mathbf{q}}) + c.c. - \nu\sum_{\mathbf{k}}|\mathbf{k}|^2\mathbb{E}|\hat{u}_{\mathbf{k}}|^2 + \varepsilon \end{aligned} \tag{10}$$

with $c.c.$ denoting the complex conjugate of the first term. The summands within the energy transfer terms are commonly known as "triad interactions" due to the appearance of coupling between the modes $\mathbf{k}, \mathbf{q},$ and $\mathbf{k}-\mathbf{q}$, consequently resulting in a nonlinear transfer of energy. A formal rearrangement of the sum gives that

$$i\sum_{\mathbf{k}} \mathbf{P}_{\mathbf{k}^\perp}\mathbb{E}\sum_{\mathbf{q}}(\mathbf{q}\cdot\hat{u}_{\mathbf{k}-\mathbf{q}})(\hat{u}^*_{\mathbf{k}}\cdot\hat{u}_{\mathbf{q}}) + c.c. = 0, \tag{11}$$

implying a global energy balance between forcing and viscous dissipation analogous to (6) for statistical steady state solutions to 3D Navier-Stokes:

$$\nu\sum_{\mathbf{k}}|\mathbf{k}|^2\mathbb{E}|\hat{u}_{\mathbf{k}}|^2 = \varepsilon. \tag{12}$$

Is this formal rearrangement actually valid? If we presume the existence of a steady state solution to (1) in the inviscid limit (Euler equation with forcing) then (11) is strikingly false! For steady state solutions to Euler, the energy transfer and forcing terms balance:

$$i\sum_{\mathbf{k}} \mathbf{P}_{\mathbf{k}^\perp}\mathbb{E}\sum_{\mathbf{q}}(\mathbf{q}\cdot\hat{u}_{\mathbf{k}-\mathbf{q}})(\hat{u}^*_{\mathbf{k}}\cdot\hat{u}_{\mathbf{q}}) + c.c. = \varepsilon. \tag{13}$$

The fact that the sum in (13) does not vanish provides some insight into the lack of regularity of solutions to the forced Euler equation [3, 4]. In particular, since the Fourier coefficients of $\mathbf{u}$ do not decay rapidly enough to allow absolute convergence of the sum we have that such solutions maintain shocks, which allow for the anomalous dissipation of energy. This is the heart of Onsager's conjecture: In the regime of fully-developed turbulence, steady state solutions correspond to the *most regular* weak solutions of the 3D Euler equation that allow for anomalous dissipation. In addition, Onsager proposed that weak solutions of Euler conserve energy if they are Hölder continuous with exponent n greater than 1/3 [5]. In Fourier space, the Hölder condition is

$$\sum_{\mathbf{k}} |\mathbf{k}|^n |\hat{u}_{\mathbf{k}}| < \infty \tag{14}$$

so if the previous sum is absolutely convergent with $n > 1/3$ then the conjecture gives that the formal rearrangement in (11) is valid and energy is conserved. The sufficiency of this condition was proved in 1994 by Constantin *et al.* [2] but necessity still remains an open question.

The loss of regularity of steady solutions to forced 3D Euler can be observed through a dimensional analysis argument. Define the second-order structure function

$$E(\mathbf{x} - \mathbf{x}', t) \doteq \mathbb{E}|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}', t)|^2, \tag{15}$$

where the homogeneity of solutions $\mathbf{u}$ to (1) has been used. Then

$$[E(\mathbf{x}, t)] = \frac{L^2}{T^2} \quad [\mathbf{x}] = L. \tag{16}$$

and there exists a function $\mathcal{F}$ such that

$$\mathcal{F}\left(E^{-\frac{3}{2}}\varepsilon\mathbf{x}, \frac{\mathbf{x}}{L_f}, \frac{\mathbf{x}}{l_\nu}\right) = 0. \tag{17}$$

Assuming isotropy,

$$E = C'\varepsilon^{\frac{2}{3}}|\mathbf{x}|^{\frac{2}{3}} g\left(\frac{|\mathbf{x}|}{L_f}, \frac{|\mathbf{x}|}{l_\nu}\right) \tag{18}$$

for some g. Since we are interested in the inertial range $l_\nu \ll |\mathbf{x}| \ll L_f$ with L_f fixed, we first let $l_\nu \to 0$ (that is, let $\nu \to 0$) and then take $|\mathbf{x}| \to 0$ to arrive at the celebrated Kolmogorov two-thirds law

$$E = C\varepsilon^{\frac{2}{3}}|\mathbf{x}|^{\frac{2}{3}}, \tag{19}$$

where we have made the assumption that $\lim_{\xi\to 0}\lim_{\eta\to\infty} g(\xi, \eta)$ exists and is finite. In Fourier space, the previous display is equivalent to the five-thirds law

$$E_{\mathbf{k}} \sim \varepsilon^{\frac{2}{3}}|\mathbf{k}|^{-\frac{5}{3}} \tag{20}$$

with $E_{\mathbf{k}} = \mathbb{E}|\hat{u}_{\mathbf{k}}|^2$. Using (19) and Hölder's inequality, one finds that the velocity field is Hölder continuous with exponent 1/3:

$$\mathbb{E}|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{x}',t)| \le \sqrt{E(\mathbf{x}-\mathbf{x}',t)} = O(|\mathbf{x}-\mathbf{x}'|^{\frac{1}{3}}). \tag{21}$$

Formally,

$$\mathbb{E}|\nabla\mathbf{u}(\mathbf{x})|^2 = \lim_{|\mathbf{y}|\to 0}\frac{1}{|\mathbf{y}|^2}\mathbb{E}|\mathbf{u}(\mathbf{x}+\mathbf{y})-\mathbf{u}(\mathbf{x})|^2 \simeq |\mathbf{y}|^{-\frac{4}{3}} \simeq |l_\nu|^{-\frac{4}{3}} \sim \nu^{-1} \tag{22}$$

where we have used the two-thirds law and concerned ourselves with the regularity of $\mathbf{u}$ at the level of the viscous length scale. It can then be seen that

$$\nu\frac{1}{|\Omega|}\int_\Omega \mathbb{E}|\nabla\mathbf{u}|^2 d\mathbf{x} \sim O(1).$$

Alternatively, since (6) is valid for all $\nu > 0$, we have that $\lim_{\nu\to 0}\nu\frac{1}{|\Omega|}\int_\Omega \mathbb{E}|\nabla\mathbf{u}|^2 d\mathbf{x} = \varepsilon$. Even in the inviscid limit energy is *still* removed by loss of regularity of solutions to Euler's equation!

1.2 Anomalous Dissipation in Burgers' Equation

In our discussion to present we have made several significant assumptions, such as that of the existence of a unique steady state solution to 3D Navier-Stokes with random forcing (which is in fact a reasonable assumption, see [1]). While Onsager's conjecture is somewhat speculative for the 3D or 2D Navier-Stokes and Euler equations, it is realizable and easily illustrated within the framework of Burgers' equation with forcing and periodic boundary conditions:

$$u_t + \frac{1}{2}(u^2)_x = \nu u_{xx} - \frac{\pi}{2}\sin(2\pi x),\ \ x\in[0,1]. \tag{23}$$

For $\nu > 0$, (23) admits smooth solutions with "shock layers" of size $O(\nu)$; however, if $\nu \to 0$, solutions develop discontinuities which allow for anomalous dissipation of energy. In Fourier space, the solution of (23) with $\nu = 0$ is

$$u(x,t) = \frac{1}{2}\sum_{n\in\mathbb{Z}} b_n(t)\sin(2n\pi x) \tag{24}$$

where $b_n(t) = \mathrm{Im}\ \hat{u}_n(t) = 2\int_0^1 u(x,t)\sin(2n\pi x)dx$ and satisfies

$$\begin{aligned}\dot{b}_n &= 2\int_0^1 u_t\sin(2n\pi x)dx \\ &= 2\int_0^1 [-uu_x + f(x)]\sin(2n\pi x)dx \\ &= -2n\pi\sum_{m\in\mathbb{Z}} b_m b_{m-n} - \frac{\pi}{2}(\mathbb{1}_{n=1} - \mathbb{1}_{n=-1}).\end{aligned} \tag{25}$$

By Parseval's identity and the previous display, the unique steady state solution must satisfy

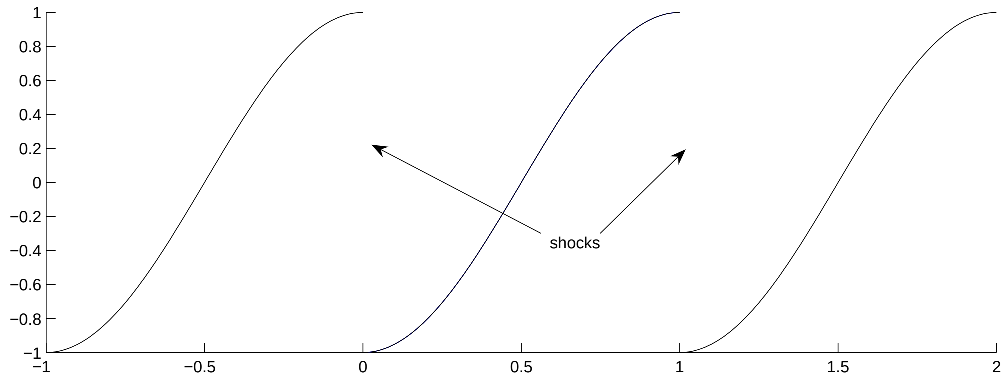

Figure 2: Dissipative solution to forced inviscid Burgers' equation.

$$\dot{E}(t) = \frac{d}{dt} \sum_{n \in \mathbb{Z}} \frac{1}{2} b_n^2 = - \sum_{n,m \in \mathbb{Z}} 2n\pi b_n b_m b_{n-m} + \pi b_1 = 0. \tag{26}$$

If the sum in the previous display is formally reorganized,

$$\begin{aligned} \sum_{n,m \in \mathbb{Z}} 2n\pi b_n b_m b_{n-m} &= \sum_{n,m \in \mathbb{Z}} 2n\pi b_{-n} b_m b_{-n-m} \\ &= - \sum_{n+m+p=0} 2n\pi b_n b_m b_p \\ &= -\frac{1}{3} \sum_{n+m+p=0} 2(n+m+p)\pi b_n b_m b_p \\ &= 0. \end{aligned} \tag{27}$$

We are then led to believe that the system has no steady state solution (with $b_1 \neq 0$). Yet, it is simple to show that

$$w(x) = -\cos \pi x \tag{28}$$

is a solution of (23) in the inviscid limit! How can this be?

The answer lies in the fact that w is a *weak* solution and rearrangement of the sum is invalid because the coefficients b_n do not decay fast enough. In order to compensate for the lack of a viscous dissipation mechanism, w has lost regularity and developed shocks (see Figure 2). One has that

$$b_n = -\frac{8n}{\pi(4n^2 - 1)} \sim -\frac{2}{\pi n} \text{ as } |n| \to \infty \tag{29}$$

so $b_n^2 \sim O(|n|^{-2})$. In this case, the energy of the steady state $E = \sum_{n \in \mathbb{Z}} \frac{1}{2} b_n^2 < \infty$ and dissipation is nonzero:

$$\sum_{n,m\in\mathbb{Z}} 2n\pi b_n b_m b_{n-m} = \pi b_1 = -\frac{8}{3} < 0. \tag{30}$$

The above example illustrates that there exist steady state solutions with finite energy that dissipate through shocks. In the next section, we develop yet simpler models that exhibit an energy cascade and anomalous dissipation.

2 Simple Models

Consider the following infinite dimensional dynamical system:

$$\begin{cases} \dot{a}_n = \alpha\{(n-1)^p a_{n-1} - n^p a_{n+1}\} + f(t)\mathbb{1}_{n=1}, \ n \in \mathbb{N} \\ a_0 = 0 \end{cases} \tag{31}$$

where $\alpha \in \mathbb{R}$, $p = 0, 1, 2$ and $f(t)\mathbb{1}_{n=1}$ is a time-dependent forcing term on the first mode. The system (31) describes a linear shell model with nearest-neighbor coupling and (as we shall see) the feature that it allows for anomalous dissipation. We will focus here on the case $p = 1$ with forcing $f(t) = \sqrt{2\varepsilon}\dot{W}(t)$ and will speculate on cases with $p \neq 1$. In the case of white noise forcing—which has the advantage of being uncorrelated with the modes a_n—an energy balance relation analogous to (5) can be derived:

$$\dot{E}(t) = \frac{d}{dt}\sum_{n\in\mathbb{N}} \frac{1}{2}\mathbb{E}a_n^2 = \alpha \sum_{n\in\mathbb{N}} \mathbb{E}\{(n-1)^p a_n a_{n-1} - n^p a_n a_{n+1}\} + \varepsilon \tag{32}$$

and we have that anomalous dissipation is possible if

$$-\alpha \sum_{n<N} \mathbb{E}\{(n-1)^p a_n a_{n-1} - n^p a_n a_{n+1}\} = \alpha N^p \mathbb{E} a_N a_{N+1} \underset{N\to\infty}{\longrightarrow} \varepsilon. \tag{33}$$

This requires that for steady state solutions

$$a_n \sim n^{-p/2} \text{ as } n \to \infty \tag{34}$$

since if a_n scales with any other exponent, solutions will have either zero or infinite dissipation. This is consistent with Onsager's conjecture since steady states of the model correspond to the most rapidly decaying $\{a_n\}$ which allow for anomalous dissipation! By a simple scaling argument, one has that steady state solutions must lie on the boundary of the weighted l_2 spaces

$$l_{2,p} \doteq \left\{ \{a_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} n^{p-1} a_n^2 < \infty \right\}, \tag{35}$$

with no dissipation in the interior of $l_{2,p}$ and infinite dissipation in the exterior of $l_{2,p}$ (see Figure 3). The representation formula for the unique steady state with $p = 1$, to be derived in a subsequent section, agrees with this picture.

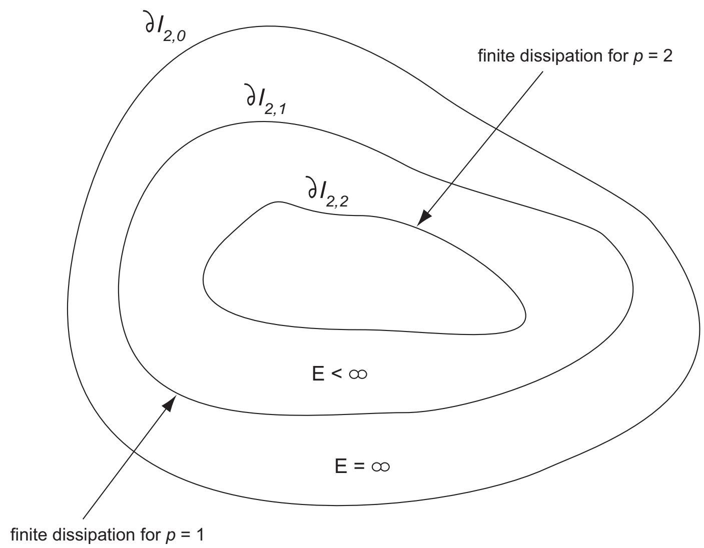

Figure 3: Spaces $l_{2,p}$ and corresponding rates of dissipation.

2.1 The case $p = 1$

We now show how to solve the simple model in the case $p = 1$ with forcing $f(t) = \sqrt{2\varepsilon}\dot{W}(t)$. Some remarks on the case $p = 2$ wil be made in Section 3.

Consider the set of Laguerre polynomials $L_n(x)$ which satisfy

$$\begin{aligned} &\text{(i)} && L_0(x) = 1 \\ &\text{(ii)} && (n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x),\ n \in \mathbb{N} \\ &\text{(iii)} && xL_n'(x) = nL_n(x) - nL_{n-1}(x) \\ &\text{(iv)} && \int_0^\infty L_m(x)L_n(x)e^{-x}dx = \delta_{nm} \end{aligned} \tag{36}$$

Let $f_0 = 0, f_n(z) = L_{n-1}(z)$ for $n \in \mathbb{N}$, and define

$$g(z,t) = \sum_{n\in\mathbb{N}} a_n(t)f_n(z). \tag{37}$$

Then one has that

$$\begin{aligned} \dot{g}(z,t) &= \sum_{n\in\mathbb{N}} \dot{a}_n(t)f_n(z) \\ &= \sum_{n\in\mathbb{N}} \alpha\{(n-1)a_{n-1} - na_{n+1}\}f_n(z) + \sqrt{2\varepsilon}\dot{W}(t) \\ &= \sum_{n\in\mathbb{N}} \alpha a_n\{2zf_n'(z) + (1-z)f_n(z)\} + \sqrt{2\varepsilon}\dot{W}(t) \\ &= 2\alpha z\frac{\partial g}{\partial z}(z,t) + \alpha(1-z)g(z,t) + \sqrt{2\varepsilon}\dot{W}(t) \end{aligned} \tag{38}$$

by shifting indices and by properties (36) of the Laguerre polynomials. Solving the stochastic partial differential equation in the previous display, one can use the orthonormality of $\{f_n(z)\}_{n\in\mathbb{N}}$ to find a_n:

$$a_n(t) = \int_0^\infty g(z,t) f_n(z) e^{-z} dz. \tag{39}$$

Furthermore,

$$\begin{aligned} E(t) &= \sum_{n\in\mathbb{N}} \frac{1}{2}\mathbb{E}a_n^2 = \int_0^\infty \frac{1}{2}\mathbb{E}g^2(z,t)e^{-z}dz \\ \dot{E}(t) &= \sum_{n\in\mathbb{N}} \mathbb{E}a_n\dot{a}_n = \int_0^\infty \mathbb{E}g(z,t)\dot{g}(z,t)e^{-z}dz \end{aligned} \tag{40}$$

where equality follows from Parseval's identity. Note that from (38) one can also derive the conservation form

$$\begin{aligned} \frac{1}{2}(g^2e^{-z})^{\cdot} &= g\dot{g}e^{-z} \\ &= g\left[2\alpha z\frac{\partial g}{\partial z} + \alpha(1-z)g + \sqrt{2\varepsilon}\dot{W}\right]e^{-z} \\ &= \frac{\partial}{\partial z}(\alpha z g^2 e^{-z}) + \sqrt{2\varepsilon}\dot{W}ge^{-z}. \end{aligned} \tag{41}$$

We are now equipped with the tools necessary to determine properties of explicitly determined solutions.

2.2 Representation Formulas for the Steady State and IVP

We now derive the steady state solution of (31) for $p = 1$ with white-noise forcing and examine the initial value problem (IVP) without forcing. To simplify discussion, take $\alpha = 1$. Solving (38) with initial conditions $g_0(z) \doteq g(z,0) = \sum_{n\in\mathbb{N}} a_n(0)f_n(z)$, one has that

$$g(z,t) = e^{(\frac{z}{2}+t)}\left[e^{(-\frac{z}{2}e^{2t})}g_0(ze^{2t}) + \sqrt{2\varepsilon}\int_0^t e^{(-\frac{z}{2}e^{2(t-s)})}dW(s)\right]. \tag{42}$$

The explicit representation for the unique statistical steady state solution is then

$$a_n(t) = \sqrt{2\varepsilon}\int_{-\infty}^t dW(s)\int_0^\infty dz L_{n-1}(z)e^{\left[t-s-\frac{z}{2}(e^{2(t-s)}+1)\right]} \tag{43}$$

so $a_n(t)$ is a Gaussian field with mean 0 and covariance

$$\begin{aligned} \mathbb{E}(a_n(t)a_m(t)) &= 2\varepsilon\int_0^\infty ds\int_0^\infty dz_1\int_0^\infty dz_2 L_{n-1}(z_1)L_{m-1}(z_2)e^{\left[2s-\frac{z_1+z_2}{2}(e^{2s}+1)\right]} \\ &= \frac{2\varepsilon}{n+m-1}. \end{aligned} \tag{44}$$

In particular,

$$-\lim_{N\to\infty}\sum_{n\le N}\{(n-1)\mathbb{E}(a_n a_{n-1}) - n\mathbb{E}(a_n a_{n+1})\} = \lim_{N\to\infty} N\mathbb{E}a_N a_{N+1} \equiv \varepsilon$$

and we have anomalous dissipation.

A simple consequence of the conservation form (41) is that energy is conserved if one begins with finite energy:

$$\sum_{n\in\mathbb{N}}\mathbb{E}a_n^2(0) < \infty \text{ implies } \sum_{n\in\mathbb{N}}\mathbb{E}a_n^2(t) = \sum_{n\in\mathbb{N}}\mathbb{E}a_n^2(0) \text{ for } t>0. \tag{45}$$

There also exist dissipative solutions with $\sum_{n\in\mathbb{N}}\mathbb{E}a_n^2(0) = \infty$; that is, solutions such that

$$\mathbb{E}|a_n| \ge Cn^{-1/2} \text{ as } n\to\infty. \tag{46}$$

Finally, as we have shown above, with forcing there exists a unique statistical steady state with equilibrium distribution supported on the the most regular dissipative solutions, i.e., such that

$$\mathbb{E}a_n^2 = O(n^{-1}) \qquad \text{as } n\to\infty.$$

3 PDE Approximation to Simple Models

When scaled properly, the system (31) very closely resembles a finite difference scheme for the wave equation $u_t = cu_x$. Keeping this in mind, for $\alpha > 0$ one can find PDEs whose solutions mimic those of the simple model for the IVP.

We begin by setting $\alpha = 1$ and by recalling that with no forcing,

$$\dot{a}_n = (n-1)^p a_{n-1} - n^p a_{n+1}.$$

Let $a_n(t) = A(nh, th^{-p+1})$ and send $h\to 0$ with $nh\to x$ and $th^{-p+1}\to\tau$ to obtain

$$\frac{\partial A}{\partial\tau} = -px^{p-1}A - 2x^p\frac{\partial A}{\partial x}. \tag{47}$$

The previous equation can also be written in the conservation form

$$\frac{1}{2}\frac{\partial A^2}{\partial\tau} = -\frac{\partial}{\partial x}(x^p A^2). \tag{48}$$

Thus, one has

$$\frac{1}{2}\frac{d}{d\tau}\int_0^L A^2(x,\tau)dx = -L^pA^2(L,\tau) \underset{L\to\infty}{\not\to} 0 \qquad \text{if } A^2(L,\tau) > CL^{-p} \tag{49}$$

which is exactly the phenomenon of anomalous dissipation!
Now consider the characteristic equation and solution of (47):

$$\frac{dX}{d\tau} = -2X^p, \qquad X(0) = x \tag{50}$$

$$A(x,\tau) = A_0(X(\tau)) \exp\left(-p\int_0^\tau X^{p-1}(\tau')d\tau'\right). \tag{51}$$

3.1 The case $p = 1$

If $p = 1$, then $X(\tau) = xe^{-2\tau}$ and there is no anomalous dissipation if $\int_0^\infty A_0^2(x)dx < \infty$ since energy is conserved:

$$\int_0^\infty A^2(x,\tau)dx = \int_0^\infty A_0^2(xe^{-2\tau})e^{-2\tau}dx = \int_0^\infty A_0^2(\eta)d\eta. \tag{52}$$

Anomalous dissipation only occurs in this case if $A_0^2(x) > Cx^{-1}$ as $x \to \infty$, and in particular, $A^2(x,\tau) = Cx^{-1}$ if $A_0^2(x) = Cx^{-1}$.

3.2 The case $p = 2$

If $p = 2$, then $X(\tau) = \frac{x}{1+2\tau x}$ and anomalous dissipation occurs even if $\int_0^\infty A_0^2(x)dx < \infty$ since

$$\begin{aligned}\int_0^\infty A^2(x,\tau)dx &= \int_0^\infty A_0^2\left(\frac{x}{1+2\tau x}\right)\frac{dx}{(1+2x\tau)^2} \\ &= \int_0^{1/2\tau} A_0^2(\eta)d\eta \\ &\leq \int_0^\infty A_0^2(\eta)d\eta.\end{aligned} \tag{53}$$

Notice also that

$$A^2(x,\tau) = A_0^2\left(\frac{x}{1+2\tau x}\right)\frac{1}{(1+2x\tau)^2} \sim A_0^2(1/2\tau)(2x\tau)^{-2} \quad \text{as} \quad x \to \infty \tag{54}$$

so that one has anomalous dissipation for $\tau \geq \tau_\star$, where

$$\tau_\star = \min\{\tau : A_0^2(1/2\tau) \neq 0\}. \tag{55}$$

3.3 Properties of the Solutions

It can easily be seen that the approximating PDE is consistent with the simple model for $p = 1$ since the aforementioned properties exactly mirror those given in the previous section. We can speculate that this is true for all other values of p as well. Moreover, since the behavior of the characteristics completely determine the properties of the IVP solution, by (50) one has that solutions with $p < 1$ behave like the $p = 1$ solution, and solutions with $p > 2$ behave like the $p = 2$ solution with regards to anomalous dissipation.

There is an interesting analog between solutions of the approximating PDE for $p = 2$ and $p = 1$ and those of the 3D and 2D Euler equations, respectively. As discussed in [3], solutions of 3D Euler with finite energy are expected to dissipate, as in the case $p = 2$. In contrast, it has been proved [4] that solutions to 2D Euler (in which the *enstrophy*

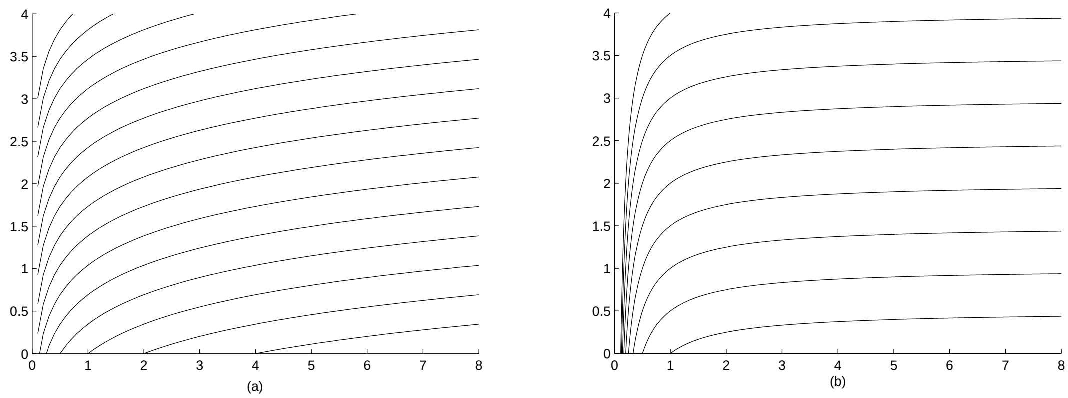

Figure 4: Characteristics of the approximating PDE for (a) $p = 1$ and (b) $p = 2$.

$\mathcal{E} = \frac{1}{2}|\omega|^2 = \frac{1}{2}|\nabla \times \mathbf{u}|^2$, rather than the energy, cascades to small scales) conserve enstrophy whenever the enstrophy itself is finite, just as in the case $p = 1$! Furthermore, if we define the "turnover time" $\tau(k)$ in the context of 3D Euler as the only time scale which can be derived from the wavenumber magnitude $k = |\mathbf{k}|$ and energy flux ε (equivalently, the enstrophy flux $\hat{\varepsilon}$ in 2D Euler), dimensional analysis gives that

$$\tau(k) = C\varepsilon^{-1/3}k^{-2/3} \text{ (3D Euler)}, \qquad \hat{\tau}(k) = C\hat{\varepsilon}^{-2/3} \text{ (2D Euler)}. \tag{56}$$

Since the turnover time describes the time required for energy (enstrophy) to pass through wavenumbers of magnitude k and $\mu(k) = \ln(k)$ is the natural scale-invariant measure associated with k, the total time for energy (enstrophy) to move from $k = 1/L_f$ to $k = \infty$ is

$$\int_{1/L_f}^{\infty} \tau(k) d\mu(k) < \infty \text{ (3D Euler)}, \qquad \int_{1/L_f}^{\infty} \hat{\tau}(k) d\mu(k) = \infty \text{ (2D Euler)}. \tag{57}$$

Analogously, characteristic lines in the case $p = 2$ go to infinity in finite time, while those for $p = 1$ go to infinity in infinite time (see Figure 4)!

It is worth noting that in the derivation of the PDE (47) we have made some smoothness assumptions on solutions of the IVP in the simple model. These assumptions can be justified if $\alpha > 0$ but are quite wrong if $\alpha < 0$. To see this, let $\alpha = -1$ and consider the IVP $a_n(0) = \mathbb{1}_{n=1}$ for $p = 0$. Then for $0 < t \ll 1$ one has that $a_n \gg a_m$ for $n < m$, and so

$$\dot{a}_n(t) \simeq -a_{n-1}(t) \text{ for } n \in \mathbb{N}, \text{ with } a_0 = 0.$$

This implies that for small values of t,

$$a_n(t) \simeq (-1)^{n-1} t^n.$$

The solution of the IVP is thus initially highly oscillatory and cannot be approximated by a smooth function in any strong sense.

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