Vibrating pendulum and stratified fluids

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1 Abstract

The problem posed is the stabilization of the inverted state of a simple pendulum induced by high–frequency vertical oscillations of the pivot point. The stability conditions are derived by means of the multiscale perturbation leading to the averaged dynamics as well as by linearization. Then the concept and methods are applied to the study of an incompressible, inviscid, stratified fluid under the Boussinesq approximation. The mechanism of the stabilization of the fluid system was found to be analogous to that of pendulum provided that the density disturbance has the form of a wave or the sum of waves. However, the analogy in case of a general density disturbance is not obvious.

2 Introduction

A simple pendulum has only one stable state, the straight–down position. However, if its support vibrates in the vertical or, equivalently, when the gravity is modulated at a frequency much greater than the natural frequency of the pendulum, then it is also possible for the inverted (upside–down) position to be a stable state. The problem dates back to 1908 when Stephenson showed that it is indeed possible to stabilize an inverted pendulum by subjecting the pivot to small vertical oscillations of suitably high frequency ([17], [18], [19]). However, it was the work of Piotr Kapitza ([10]) that drew broader attention and commenced a series of studies concerned with this interesting phenomenon, called sometimes for that reason "Kapitza pendulum". Similar behavior of parametrically forced systems in this parameter regime was found in other problems, like particle trapping and even evolution of market prices (e.g see [6], [7]).

The purpose of this work is to investigate the stabilization of the inverted pendulum and to apply the concept and the methods developed to fluid dynamics. The pendulum system is treated by means of the multiscale perturbation which leads to the averaged dynamics, as well as by linearization, which reduces the problem to Mathieu equation. As a simple fluid analog we choose incompressible, stratified fluid under the Boussinesq approximation in periodic domain, subjected to a rapidly varying gravitational field. We focus on the multiscale technique and averaged dynamics to find the stabilization mechanism equivalent to that obtained for the Kapitza pendulum.

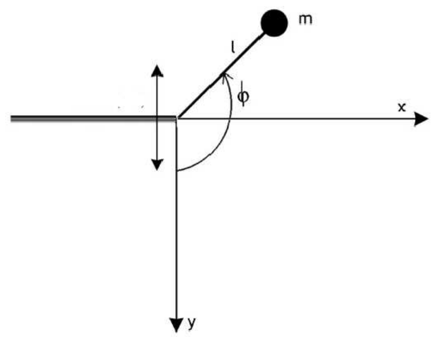

Figure 1: Kapitza pendulum.

3 Vibrating inverted pendulum

3.1 Problem formulation. Equation of motion

We consider a simple, nonlinear pendulum of mass m and length l, moving on a vertical plane in the uniform gravitational field and subjected to a vertical, rapid vibration of the pivot point. By rapid vibration we mean oscillation of high frequency and small amplitude of the pivot motion, given a form:

$$\zeta(t) = a\cos(\gamma t). \tag{1}$$

The parameters of the external forcing, the amplitude of the vertical motion and the frequency, obey

$$a \sim \mathcal{O}(\epsilon), \quad \gamma \sim \mathcal{O}(\frac{1}{\epsilon}),$$

where ϵ is a small number. Following the classical work of [11], we choose the coordinate system depicted in figure 1 and the following transformation:

$$\begin{aligned} x &= l\sin\phi \\ y &= l\cos\phi + a\cos(\gamma t), \end{aligned} \tag{2}$$

where $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ is the angle that the pendulum forms with the downward vertical. From the Lagrangian of the system (for the derivation, see Appendix A), we obtain the equation of motion of the vibrationally forced pendulum:

$$ml^2\ddot{\phi} + mg_o l\sin\phi + mal\gamma^2\cos(\gamma t)\sin\phi = 0, \tag{3}$$

where g_o is the gravitational constant. It is worth noting that by a simple rearrengement leading to:

$$l\ddot{\phi} + \big(g_o - a\gamma^2\cos(\gamma t)\big)\sin\phi = 0,$$

we can regard the system of the pendulum with the vertically oscillating support as a immotile pendulum in the field of a modulated, rapidly varying modified gravity of the form:

$$g = g_o + \frac{1}{\epsilon}g_{-1}(\frac{t}{\epsilon}).$$

Defing the natural frequency ω_o of the pendulum by $\omega_o^2 = \frac{g_o}{l}$, we get a more concise form of the equation of motion:

$$\ddot{\phi} + (\omega_o^2 + \frac{a}{l}\gamma^2 \cos(\gamma t)) \sin\phi = 0. \tag{4}$$

It is more convenient to operate with non–dimensional parameters. Without the loss of generality, we let $\omega_o^2 = 1$ and divide (4) by it. The nondimensional time is set to be $t^* = \omega_o t$. We define the ratio of forcing and natural frequencies Ω, and the relative amplitude of the forcing β as, respectively:

$$\Omega = \frac{\gamma}{\omega_o} = \gamma, \qquad \beta = \frac{a}{l},$$

where $\Omega \sim \mathcal{O}(\frac{1}{\epsilon})$ and $\beta \sim \mathcal{O}(\epsilon)$. Dropping asterics, the equation of motion of the so–called "normalized pendulum" becomes:

$$\ddot{\phi} + (1 + \beta\Omega^2 \cos(\Omega\, t)) \sin\phi = 0. \tag{5}$$

This is a nonlinear equation with periodic coefficients, and is nonintegrable. Note that it describes a forced motion in a uniform field of gravitational potential U_o:

$$\frac{\partial^2\phi}{\partial t^2} = -\frac{\partial U_o}{\partial \phi} - \beta\Omega^2 \cos(\Omega\, t)\sin\phi, \quad \text{where} \quad U_o = -\cos\phi \tag{6}$$

The stability properties of this system have been studied either by means of averaging (e.g. [10], [11]), i.e. effective potential method, or by linearization around fixed points which leads to Mathieu equation (e. g. [18], [8]). The information obtained by neither of the two approaches is complete, however it is in some sense complementary; therefore we found worthwile to apply both of them. We obtain the averaged dynamics by an alternative technique, the multiscale perturbation, and then zoom into the neighborhood of the equilibra by means of linearization.

3.2 $\frac{1}{\epsilon}$ problem and multiscale perturbation

We note that there are two well separated time scales in our problem, corresponding to the **slow** motion of the pendulum and **fast** oscillation of the pivot point. We can therefore attempt to find an asymptotic solution valid on long time scales of the pendulum motion. We will define a perturbation parameter and its relation with the forcing parameters by:

$$\epsilon = \frac{1}{\Omega}, \quad |\epsilon| \ll 1, \quad \beta = \frac{a}{l} = \epsilon\tilde{\beta}. \tag{7}$$

The independent time scales in our problem are:

$$\begin{aligned} &\text{slow time:} \quad t, \quad t \sim \mathcal{O}(1) \\ &\text{fast time:} \quad \tau = \frac{t}{\epsilon}, \quad \tau \sim \mathcal{O}(\frac{1}{\epsilon}), \end{aligned}$$

and so the equation of motion (4) can be expressed as:

$$\ddot{\phi} + (1 + \frac{\tilde{\beta}}{\epsilon}\cos\tau) \sin\phi = 0. \tag{8}$$

The first and the second time derivatives become, respectively:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}, \qquad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial \tau \partial t} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \tau^2}.$$

Expanding the variable ϕ in power series of ϵ yields:

$$\phi(t,\tau) = \phi_o(t,\tau) + \epsilon\phi_1(t,\tau) + \epsilon^2\phi_2(t,\tau) + \dots . \tag{9}$$

Inserting the perturbation series (9) into (8) and assembling powers of ϵ yields a set of equations for the subsequent orders in ϵ. Manipulation of these equations results in a group of terms that give unbounded, linear growth of the solution in fast time τ which obviously destroys the solution on long time scales. Such terms are called secular terms and a standard procedure in multiscale perturbation technique is to remove them by making them equal to zero and vanish ([9]). For the justification and more detailed treatment of the problem, see Appendix B. The condition for the solution to be valid uniformly on t gives the equation of motion of the leading order quantity:

$$\frac{\partial^2\phi_o}{\partial t^2} = -(1 + \frac{(\beta\Omega)^2}{2}\cos\phi_o)\sin\phi_o. \tag{10}$$

It is worth to note that the same result would be obtained if the **average** of the $\mathcal{O}(1)$ equation over the fast time has been taken, defined as following:

$$\epsilon \ll T \ll 1, \quad \overline{\psi}(t) \equiv \frac{1}{T}\int_{\tau}^{\tau+T} \psi(t,\tau')d\tau', \tag{11}$$

(see Appendix B). Using the fact that $\overline{\cos^2\tau} = \frac{1}{2}$, we get:

$$\frac{\partial^2\overline{\phi}}{\partial t^2} = -(1 + \frac{(\beta\Omega)^2}{2}\cos\overline{\phi})\sin\overline{\phi}. \tag{12}$$

Therefore, we can consider (10), governing the dynamics of the $\mathcal{O}(1)$ quantity, to be equivalent to the dynamics of a variable averaged over the fast oscillations (12).

3.3 Averaged dynamics. Effective potential.

We can note the averaged dynamics of the vibrationally forced pendulum governed by (12), can be perceived as a motion in the field of effective potential $\mathcal{U}$:

$$\frac{\partial^2\overline{\phi}}{\partial t^2} = -\frac{\partial\mathcal{U}}{\partial\phi}, \quad \text{where} \quad \mathcal{U} = -\cos\overline{\phi} + \frac{\beta^2\Omega^2}{4}\sin^2\overline{\phi}. \tag{13}$$

Interestingly, there is no explicit time dependence in the governing equation, as the effective potential is dependent only on the mean state of the system and parameters of the forcing (compare with (6)). This observation allows for the following physical explanation of the process, given by [10]. Vibrational gravitational field (induced by the oscillatory motion of the pivot) leads to the production the vibrational torgue which on average manifests as an ordinary force. This force tends to set the rod of the pendulum in the direction of the axis of the oscillations. Provided the vibrational force is balanced by the gravity, the inverted state exhibits the dynamical equilibrum and becomes stable. We can now express the stability condition for the upper equilibrum in terms of the parameters of the forced system. Let's take a closer look into the stability of the system.

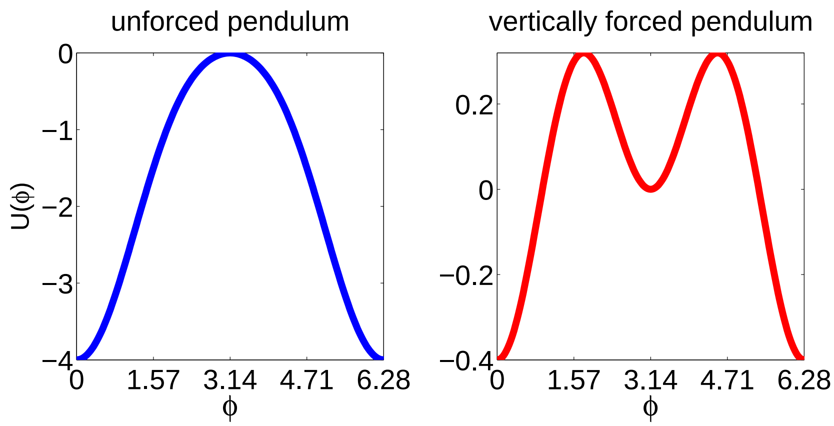

Figure 2: Left: Potential as a function of ϕ for the unforced, nonlinear pendulum. Right: effective potential for vibrationally forced pendulum. Note that scales of the two figures are set different show the shape of the functions.

3.4 Stability of equilibria of averaged dynamics

The stability of equilibra can be most easily determined from (13). By setting the right hand side to zero, we find positions of equilbra, i.e the extrema of $\mathcal{U}$. There are four such points, in contrary to the case of the unforced nonlinear pendulum, when there are only two (fig. 2). The nature of the extremum is determined by the sign of the second derivative of $\mathcal{U}$, $\frac{\partial^2\mathcal{U}}{\partial\phi^2} < 0$ indicates maximum (i.e. an unstable equilibrium), $\frac{\partial^2\mathcal{U}}{\partial\phi^2} > 0$, minimum (a stable equilibrum called a potential well):

Equilibrum	Stability
$\phi_1^* = 0$ (noninverted)	stable
$\phi_2^* = \arccos(-\frac{(\beta\Omega)^2}{2})$	unstable
$\phi_3^* = \arccos(-\frac{(\beta\Omega)^2}{2})$	unstable
$\phi_4^* = \pi$ (inverted)	conditionally stable

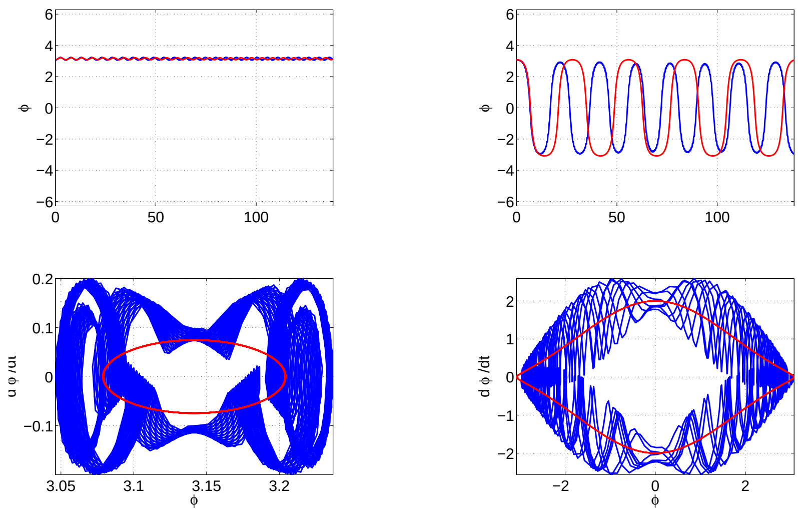

Figure 3: Time series and phase plots of the stabilized inverted pendulum. Blue color indicates the nonaveraged and red – averaged dynamics. Left: stable configuration $\Omega = 11,\ \beta = 0.2$. Right: unstable configuration (rotational mode) $\Omega = 11,\ \beta = 0.1$. Note that the scales of the phase space plots are set different to show the shape of trajectories.

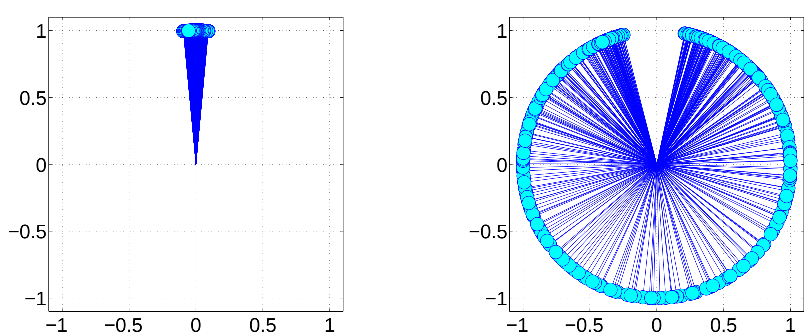

Figure 4: Time trace of the vibrationally forced pendulum in physical space. Left: stable configuration $\Omega = 11,\ \beta = 0.2$. Right: unstable configuration $\Omega = 11,\ \beta = 0.1$.

From (13), the condition for the stable upper equalibrum ϕ_4^* is:

$$\frac{\beta^2\Omega^2}{2} > 1. \tag{14}$$

The stability of the inverted position depends solely on the parameters of the system and requires a forcing of suitably high frequency. Note also that an angle $\phi_{2,3}^* = \arccos(-\frac{(\beta\Omega)^2}{2})$ can be interpreted as a width of the potential well, namely maximum initial displacement that allows for the stabilization of the upper equilibrum for given properties of the forcing. Exemplary time series and phaseplots of the pendulum in a stable and unstable regime is presented on figure 3, for both nonaveraged and averaged dynamics. Corresponding behavior of the pendulum in the physical space is shown in figure 4.

There is another point of view of the averaged dynamics. The problem of the invertible pendulum may be considered in the $1\frac{1}{2}$–degree–of–freedom Hamiltonian setting. The nonlinear dynamics is then described by the Poicaré map or equivalently by its integrable approximation, a planar Hamiltonian, obtained from the normal form theory: successive transformations leading to the removal of the explicit time dependence. As the effective potential corresponds to the potential energy, the planar Hamiltonian is equal to the total energy of the averaged system. In this framework, by defining a parameter $\lambda = \frac{2}{\beta^2\Omega^2}$, the transition of the inverted equilibrum from a minimum for $\lambda < 1$ to a saddle point for $\lambda > 1$ may be considered a subcritical Hamiltonian pitchfork bifurcation ([5]), see figure 5.

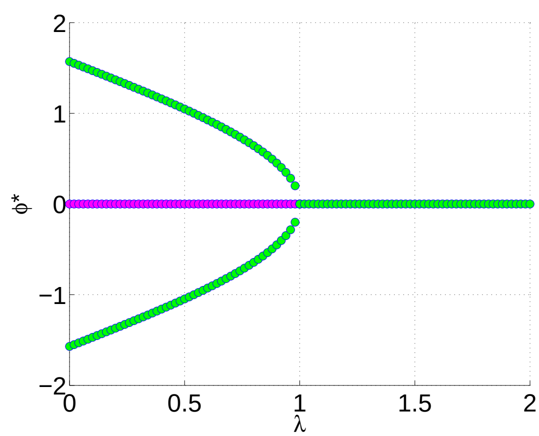

Figure 5: Bifurcation diagram in the (λ, ϕ^*) - plane for the inverted pendulum, where $\lambda = \frac{2}{\beta^2\Omega^2}$ and ϕ^* is a fixed point. Green dots indicate instability, magenta dots – stability.

3.5 Mathieu equation

Another approach to the study of the inverted pendulum is the linearization of the dynamics near an equilibrum. We define $\alpha = \frac{1}{\Omega^2}$ and look again into the dynamics of (5), but now as evolving in the fast time $\tau = \gamma t$:

$$\phi_{\tau\tau} + (\alpha + \beta \cos\tau)\sin\phi = 0. \tag{15}$$

Now we zoom into the dynamics near π, so it is convenient to define the complementary angular displacement φ with respect to ϕ, $\varphi = \pi - \phi$. This is the angular displacement from the upper equilibrum position. Then $\sin\varphi = \sin(\pi - \phi) = +\sin\phi$ and $\varphi_{\tau\tau} = -\phi_{\tau\tau}$. The governing equation becomes:

$$\varphi_{\tau\tau} - (\alpha + \beta \cos\tau)\sin\varphi = 0. \tag{16}$$

The sign of β is not important as it corresponds to the instantenous amplitude of the pivot motion around the center of the coordinate system, which can be postive or negative. The sign of α, however, matters. We define the variable ψ as

$$\psi = \begin{cases} \phi : & \text{angular displacement near the lower equilibrum} \\ \varphi : & \text{angular displacement near the upper equilibrum} \end{cases}$$

and linearize $\sin\psi \sim \psi$, obtaining the canonical form of **Mathieu equation**:

$$\frac{\partial^2\psi}{\partial\tau^2} + (\alpha + \beta\cos\tau)\psi = 0. \tag{17}$$

Thus equation (17) describes linearized dynamics around the either fixed point, depending on the sign of α:

$$\alpha > 0 : \quad \text{lower equilibrum} \quad \alpha < 0 : \quad \text{upper equilibrum} \tag{18}$$

This specific formulation allows the investigation of the linear stability near the either equilibrum based on the general results for the Mathieu equation. It is also a manifestation of the fact that we can tackle the stability of the inverted state by changing the sign of the gravity g. The stability of the periodic solutions of the Mathieu equation, given by the Floquet theory, can be determined from the diagram in (β, α) – parameter space ([9]). The so–called transition curves separate regions of values α and β corresponding to unstable and stable solutions. Kapitza pendulum regime, with high forcing frequency (small α) and small amplitude of the pivot motion (small β) lies in the region marked by the red circle in fig (6). The most surprising is the fact that, contrary to the averaged dynamics, linearized approach gives the possibility for destabilization of the lower equilibrum! We can obtain quite exact values for the parameter regions corresponding to stable inverted and noninverted equilibra:

Equilibrum	Stability condition
inverted	$\frac{\sqrt{2}}{\Omega} < \beta < 0.45 + \frac{1.799}{\Omega^2}$
noninverted	$0 < \beta < 0.45 - \frac{1.799}{\Omega^2}$

It is worth noting, that the linear stability condition for inverted pendulum based on linearization was generalized to an inverted N-pendulum and proven by [1]. In that case the system can be reduces to N uncoupled Mathieu equations and the stabilty condition yields

$$\frac{\sqrt{2}g}{\gamma\omega_{min}} < \beta < \frac{0.450g}{\omega_{max}^2}, \tag{19}$$

where ω_{min} and ω_{max} are the lowest and the highest of the natural frequencies of any single pendulum member of the full configuration.

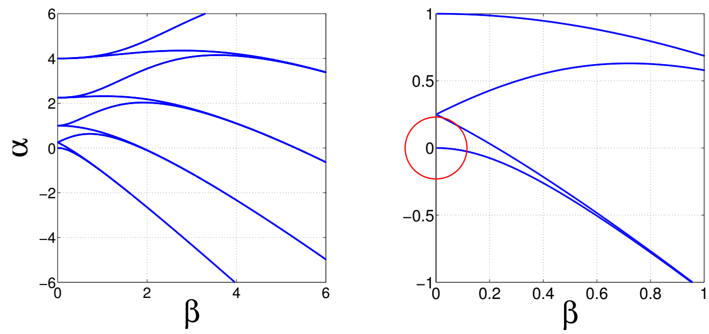

Figure 6: Left: Stability diagram for Mathieu equation. Right: the Kapitza pendulum regime.

3.6 Averaged dynamics vs Mathieu equation

Both approaches to the study of the dynamics of the pendulum system under the influence of a rapidly oscillating forcing are not completely adequate, i.e the relevance of either of them is limited. Here we summarize the key points concerning the applicability of the two methods.

Averaged dynamics:

- approximates long time asymptotic behavior,
- concerns global dynamics, i.e arbitrary displacements from the equilibrum position. Hence, apart from the very proximity of the inverted state, allows the study of the rotating state of the pendulum,
- gives only lower stability bound in the parameter space for the inverted equilibrium,
- gives the maximum angular displacement that permit stabilization of the inverted state for a given set of parameter values,

- predicts the stability of the lower equilibrum for all the values of the parameters,
- gives physical explanation for the dynamical stabilization phenomenon in terms of the effective potential.

Certainly some subtle details in the pendulum dynamics are lost in the approximate average analysis, which refers only to the slow component of the motion. In this method one introduces an approximation with no control on the relevance of the discarded dynamics, except of the estimate of their magnitude in terms of ϵ. There is always a region in parameter space where the averaging fails predicting unstable configuration – the region over the upper bound in case of the inverted state and the whole unstable region for the lower equilibrum, which can be found from the Mathieu equation.

Linearized theory:

- reduces the problem to the well-known Mathieu equation,
- applies only to the vicinity of the fixed points, i.e is valid only for small angular displacements from the equilibrum position,
- gives more precisely defined stability regions: lower and upper bounds on the parameter values,
- admits the possibility for destabilization of the lower equilibrum and gives the range of parameter values for which it should be observed,
- does not provide with any physical explanation of the phenomenon.

In general, the applicability of the Mathieu equation is limited as in general the linear stability cannot be extended immediately to the full system. However, it gives correct results for the zoomed view into the dynamics near the equilibra.

4 Kapitza pendulum in fluid systems

4.1 Kapitza pendulum vs fluid systems

As already mentioned, we can regard a pendulum with a vertically oscillating support as equivalent to a pendulum with a stationary support in a periodically varying gravity field. This observation allows to attempt at finding an analogy between the Kapitza pendulum and fluid systems acted upon by vibrating gravitational forces, and thus study the dynamics of the latter by methods developed in the preceding sections. A similar analogy was already mentioned by Lord Rayleigh ([15]), who considered a phenomenon observed in the famous Faraday experiment: a surface wave instability in a vertically vibrated container filled with fluid, with the frequency close to resonance with the natural frequency of the system. From that time on the problem has been studied widely both analytically (e.g. [2], [12], [13]) and experimentally (e.g. [4]). The "Kapitza regime" discussed here is different as the forcing frequency is much higher then the natural one and it would not be trivial to think of an experiment similar to that of Faraday. One can rather search for an idealized physical situation. We have chosen a simple fluid system described by Boussinesq equations, characterized by density stratification and related buoyancy force, which action, combined with that of gravity, provides the mechanism for inertial oscillations with bounded natural frequency.

4.2 Boussinesq equations. Problem formulation

We will study the dynamics of a incompressible, inviscid, stratified, hydrostatic, non–rotating fluid, governed by nonlinear Boussinesq equations (the momentum equations, continuity and conservation of density), see e.g. [16]):

$$\frac{Du}{Dt} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} = 0 \tag{20}$$

$$\frac{Dw}{Dt} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} = -\frac{g\rho}{\overline{\rho}} = b \tag{21}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{22}$$

$$\frac{D\rho}{Dt} = 0 \tag{23}$$

Here t is time and u and w are the components of the nondivergent velocity field in horizontal (x) and vertical (z) directions in a Cartesian frame of reference. The positive z–direction is antiparallel to gravity. We will work in the (x, z) – plane, which implies that all characteristics of the system are uniform in y.

We consider a hydrostically balanced reference state upon which perturbations are to be imposed and make use of the Boussinesq approximation, which means that the density field, represented by $\rho(x, z, t) = \overline{\rho} + \rho_o(z) + \Delta\rho(x, z, t) = \overline{\rho}\big(1 - \mathcal{S}\big(\frac{z}{H}\big)\big) + \Delta\rho(x, z, t)$, satisfies $\overline{\rho} >> \rho_o(z) >> \Delta\rho$. The mean fluid density $\overline{\rho}$ is a uniform costant. The density stratification is assumed to be linear and $\mathcal{S}$ is a stability parameter defined as:

$$\mathcal{S} = \begin{cases} +1 & \text{stable stratification} \\ -1 & \text{unstable stratification} \end{cases}$$

The vertical extent of the domain is small compared to a density depth scale $H = |\frac{1}{\overline{\rho}}\frac{d\rho_0}{dz}|^{-1}$. Pressure field p is assumed to be hydrostatically related. The joint effect of gravity and density stratification leads to a buoyancy force b in the vertical. The natural frequency of the system, so–called the Brunt–Väisälä frequency, is defined through $N^2(z) = -(\frac{g}{\overline{\rho}}\frac{d\rho_o}{dz})$, which is constant in case of linear stratification, and with notation used here $N = \frac{g\mathcal{S}}{H}$. We will constraint us to the flows with periodic boundary conditions.

Nondivergence of the velocity field allows to introduce a stream function ψ, such that $u = \psi_z$ and $w = -\psi_x$. In an infinite medium the Boussinesq equations are satisfied by planar internal waves of the form $\psi = \Psi\cos(kx + mz)\, e^{-i\omega t}$ (with horizontal and vertical wave numbers k and m and $\kappa = \sqrt{k^2 + m^2}$), which obey the dispersion relation $\omega = \pm N\frac{k}{|\kappa|}$. In polar coordinates chosen so that $k = \kappa\cos\alpha$ and $m = \kappa\sin\alpha$ we have $\omega = \pm N\cos\alpha$. The frequency is therefore a function of the angle that the wave vector makes with the vertical. The condition for progressive waves is therefore $0 \leq \omega \leq N$ so N acts as the upper bound of internal wave frequencies, corresponding to entirely vertical flow (buoyancy oscillation) ([3]).

Now we will embed the system in a modified gravity field, varying in vertical as $g(t) = g_o + a\gamma^2 \cos(\gamma t)$, where g_o is a gravitational costant, the amplitude of oscillatory motion satisfies $a << g_o$, and the frequency of the oscillations γ is much higher then the Brunt–Väisälä frequency, that is $\gamma >> N$.

We will nondimensionalize the system as follows:

$$(x^*, z^*) = \big(\frac{x}{H}, \frac{z}{H}\big), \qquad t^* = \frac{t}{\sqrt{\frac{H}{g_o}}},$$

which results in the nondimensional components of the velocity:

$$(u^*, w^*) = \big(\frac{u}{\sqrt{g_o H}}, \frac{w}{\sqrt{g_o H}}\big).$$

The nondimensional density and pressure fields are:

$$\rho^*(x,z,t) = \big(1 - \mathcal{S}\big(\frac{z}{H}\big)\big) + \Delta\rho^*(x,z,t), \quad \text{where} \quad \Delta\rho^* = \frac{\Delta\rho}{\bar{\rho}}, \quad \text{and} \quad p^* = \frac{p}{\bar{\rho} g_o H},$$

which gives the nondimensional Brunt–Väisälä frequency $N^{*2} = \mathcal{S}$. By introducing the nondimensional parameters:

$$\Omega = \frac{\gamma}{\frac{H}{g_o}}, \quad \beta = \frac{a}{H},$$

the ratio of the forcing and natural frequencies and the relative magnitude of the forcing, respectively, we get

$$g^*(t) = \frac{g(t)}{g_o} = 1 + \beta\Omega^2 \cos(\Omega t^*). \tag{24}$$

By dropping asterics, the nondimensional Boussinesq equations are:

$$\begin{aligned}
\frac{Du}{Dt} + \frac{\partial p}{\partial x} \qquad\qquad &= 0 && (25)\\
\frac{Dw}{Dt} + \frac{\partial p}{\partial z} \qquad\qquad &= -\Delta\rho(1 + \beta\Omega^2\cos(\Omega t)\big) && (26)\\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \qquad\qquad &= 0 && (27)\\
\frac{\partial \Delta\rho}{\partial t} + u\frac{\partial \Delta\rho}{\partial x} + w\frac{\partial \Delta\rho}{\partial z} - w\mathcal{S} &= 0 && (28)
\end{aligned}$$

We can eliminate the pressure by focusing on the vorticity equation, which is:

$$\frac{Dq}{Dt} = \frac{D}{Dt}\big(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\big) = \frac{\partial \Delta\rho}{\partial x}\big(1 + \beta\Omega^2\cos(\Omega t)\big). \tag{29}$$

It is evident that the vibrating gravity field modifies the basic mechanism of the baroclinic generation of vorticity. We will now investigate this phenomenon in detail, using the methods derived for the stabilized inverted pendulum.

4.3 Multiscale expansion

We can observe that, similarly to the case of the Kapitza pendulum discussed above, in our Boussinesq problem we have two well separated time scales: the slow gravity oscillations and the fast oscillation of the gravity field. Thus, an attempt to find an asymptotic solution valid on long time scales is justified. Analogously as for the pendulum, we can define a perturbation parameter by $\epsilon = \frac{1}{\Omega}$, and we have $\beta \sim \mathcal{O}(\epsilon)$. The slow time is then $t \sim \mathcal{O}(1)$, and the fast time $\tau = \Omega t$, $\tau \sim \mathcal{O}(\frac{1}{\epsilon})$, compare with (7). The oscillating gravity field (24) can be expressed as $g(t) = 1 + \frac{1}{\epsilon}\tilde{g}_{-1}(\tau)$. Unlike in the pendulum case, system variables depend now not only on time, but also on space, therefore we have for the first, the second and material time derivatives, respectively:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}, \qquad \frac{\partial^2}{\partial t^2} \to \frac{\partial^2}{\partial t^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial \tau \partial t} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \tau^2}, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.$$

With the averaging operator defined as (11), the components of the velocity perturbation are expanded in perturbation series as:

$$\begin{aligned} u(x,z,t,\tau) &= \overline{u_o}(x,z,t) + u_o'(x,z,t,\tau) + ... \\ w(x,z,t,\tau) &= \overline{w_o}(x,z,t) + w_o'(x,z,t,\tau) + ... , \end{aligned} \tag{30}$$

where $(\overline{u_o}, \overline{w_o})$ refer to the mean perturbation velocity, while (u_o', w_o') correspond to the disturbance due to modified acceleration of the order $\mathcal{O}(\frac{1}{\epsilon})$; after time integration, we can expect them to be of the order $\mathcal{O}(1)$, but as they came from the oscillatory motion, they will vanish when averaged over the fast time. We will define the mean substantial derivative:

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u_o}\frac{\partial}{\partial x} + \overline{w_o}\frac{\partial}{\partial z}.$$

The density perturbation is expanded in ϵ as:

$$\Delta\rho(x,z,t,\tau) = \Delta\rho_o(x,z,t,\tau) + \epsilon\Delta\rho_1(x,z,t,\tau) + ... \tag{31}$$

Inserting the series (31) and (30) into the density equation (28), we get:

$$\begin{aligned} &\frac{\partial\Delta\rho_o}{\partial t} + \frac{1}{\epsilon}\frac{\partial\Delta\rho_o}{\partial\tau} + \epsilon\frac{\partial\Delta\rho_1}{\partial t} + \frac{\partial\Delta\rho_1}{\partial\tau} + \overline{u_o}\frac{\partial\Delta\rho_o}{\partial x} + u_o'\frac{\partial\Delta\rho_o}{\partial x} + \overline{u_o}\frac{\partial\Delta\rho_1}{\partial x} + u_o'\frac{\partial\Delta\rho_1}{\partial x} + \\ &+ \overline{w_o}\frac{\partial\Delta\rho_o}{\partial x} + w_o'\frac{\partial\Delta\rho_o}{\partial x} + \overline{w_o}\frac{\partial\Delta\rho_1}{\partial x} + w_o'\frac{\partial\Delta\rho_1}{\partial x} - \overline{w_o}\mathcal{S} - w_o'\mathcal{S} = 0, \end{aligned} \tag{32}$$

Gathering terms of the order $\mathcal{O}(\frac{1}{\epsilon})$ we conclude that $\Delta\rho_o = \overline{\Delta\rho}_o(x,z,t)$. In the order of $\mathcal{O}(1)$, we will first average out the terms containing perturbation quantities $\Delta\rho_1$ and (u_o', w_o'), so that the remaining terms give an evolution equation for the mean density perurbation:

$$\frac{\overline{D}\,\overline{\Delta\rho}}{Dt} - \overline{w_o}\mathcal{S} = 0. \tag{33}$$

Subtraction of (33) from (32) results in an evolution equation for $\Delta\rho_1$:

$$\frac{\partial\Delta\rho_1}{\partial\tau} + u_o'\frac{\partial\overline{\Delta\rho}_o}{\partial x} + w_o'\frac{\partial\overline{\Delta\rho}_o}{\partial z} - w_o'\mathcal{S} = 0. \tag{34}$$

Inserting the perturbation series (30) into (29), we get:

$$\frac{\partial}{\partial t}\big(\frac{\partial \overline{u_o}}{\partial z}-\frac{\partial \overline{w_o}}{\partial x}\big)+\frac{1}{\epsilon}\frac{\partial}{\partial \tau}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)+\overline{u_o}\frac{\partial}{\partial x}\big(\frac{\partial \overline{u_o}}{\partial z}-\frac{\partial \overline{w_o}}{\partial x}\big)+\overline{w_o}\frac{\partial}{\partial z}\big(\frac{\partial \overline{u_o}}{\partial z}-\frac{\partial \overline{w_o}}{\partial x}\big)+$$
$$+\overline{u_o}\frac{\partial}{\partial x}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)+\overline{w_o}\frac{\partial}{\partial z}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)+u_o'\frac{\partial}{\partial x}\big(\frac{\partial \overline{u_o}}{\partial z}-\frac{\partial \overline{w_o}}{\partial x}\big)+w_o'\frac{\partial}{\partial z}\big(\frac{\partial \overline{u_o}}{\partial z}-\frac{\partial \overline{w_o}}{\partial x}\big)+$$
$$+u_o'\frac{\partial}{\partial x}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)+w_o'\frac{\partial}{\partial z}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)=\frac{\partial \overline{\Delta\rho}_o}{\partial x}+\frac{1}{\epsilon}\tilde{g}_{-1}(\tau)\frac{\partial \overline{\Delta\rho}_o}{\partial x}+\tilde{g}_{-1}(\tau)\frac{\partial \Delta\rho_1}{\partial x}+\epsilon\frac{\partial \Delta\rho_1}{\partial x}.$$

Terms of the order $\mathcal{O}(\frac{1}{\epsilon})$ gather in the evolution equation for q_o':

$$\frac{\partial q_o'}{\partial \tau}=\frac{\partial}{\partial \tau}\big(\frac{\partial u_o'}{\partial z}-\frac{\partial w_o'}{\partial x}\big)=\tilde{g}_{-1}(\tau)\frac{\partial \overline{\Delta\rho}_o}{\partial x}. \tag{35}$$

By applying the averaging operation to the $\mathcal{O}(1)$ ensamble, terms containing products of the leading order and prime quantities are eliminated and the evolution equation for the mean perturbation vorticity follows:

$$\frac{\overline{D}\overline{q}}{Dt}=\frac{\partial \overline{\Delta\rho}_o}{\partial x}+\overline{\big(\frac{\partial \Delta\rho_1}{\partial x}\tilde{g}_{-1}(\tau)\big)}-\overline{\big(u_o'\frac{\partial q_o'}{\partial x}+w_o'\frac{\partial q_o'}{\partial z}\big)}. \tag{36}$$

We will use the fact that averaging is a linear operator so that from the continuity equation (28) to obtain:

$$\frac{\partial \overline{u_o}}{\partial x}+\frac{\partial \overline{w_o}}{\partial z}=0, \quad \text{and} \quad \frac{\partial u_o'}{\partial x}+\frac{\partial w_o'}{\partial z}=0. \tag{37}$$

To summarize, we have arrive at two separated sets of equations with no explicit time dependence: for averaged perturbation variables and those generated by the forcing of our system. By analogy to the Kapitza pendulum, a stable stratification corresponds to the stable equilibrium of the pendulum, while the unstable stratification – to the inverted state. Now we will analyze the response of the system to the instantenous density disturbance to determine how the vertical oscillations of the gravity field influence stability properties of the system as a whole.

4.4 Stability of the vibrating Boussinesq system

4.4.1 Monochromatic wave disturbance in x direction

We will start with a perturbation of the mean state of the form:

$$\overline{\Delta\rho}=\varrho_o\cos(kx). \tag{38}$$

Assuming a perturbation streamfunction of the form $\psi_o'=\Psi_o'\cos(kx)$, from (35) we have $u_o'=0$ and $w_o'=w_o'(x)=\frac{(\Delta\rho_o)_{xx}}{k^2}\int\tilde{g}_{-1}(\tau')d\tau'$. Therefore in the (35), the advection terms cancel out yielding $\frac{d\Delta\rho_1}{d\tau}=\mathcal{S}w_o'$ and they vanish in the equations for the evolution of the mean density perturbation (33) the mean vorticity perturbation (36) to give:

$$\frac{\overline{D}\overline{q}}{Dt}=\frac{\partial \overline{\Delta\rho}_o}{\partial x}\big(1+\frac{\mathcal{S}(\beta\Omega)^2}{2}\big), \qquad \frac{\overline{D}\overline{\Delta\rho}_o}{Dt}=\mathcal{S}\overline{w_o}. \tag{39}$$

Manipulation of these expresions yields the evolution equation for the mean density perturbation:

$$\frac{\overline{D}^2\overline{\Delta\rho}}{Dt^2} + S_*^2\overline{\Delta\rho} = 0, \quad \text{where} \quad S_*^2 = \mathcal{S} + \frac{\beta^2\Omega^2}{2}, \tag{40}$$

is the new modified nondimensional frequency, expressed by means of the stability parameter (note that $\mathcal{S}^2 = 1$ irrespective of the initial stability). Thus we conclude that in case of a stable initial stratification ($\mathcal{S} = 1$), the stability is augmented, while in case of unstable initial stratification there is a stabilizing effect of the vertically oscillating gravity field. The stability condition in the initially unstable case is:

$$\frac{\beta^2\Omega^2}{2} > 1, \tag{41}$$

which is identical to the analogous condition for the inverted Kapitza pendulum (14). In fact, the equation (40) itself may be perceived as a linear analog of (12) – unintentionally linearized by the dynamics itself. We can thus expect all the results obtained from the Mathieu equation for the inverted pendulum to be valid in the problem discussed here.

Assuming the plane wave solution of the form $\overline{\psi} = \overline{\psi}_o \cos(kx - \omega t)$, we obtain the dispersion relation for the internal waves supported by our system:

$$\omega^2 = S_*^2,$$

which is equivalent to the dispersion relation of the buoyancy oscillation typical to the unforced Boussinesq system, with N replaced by S^*, i.e. the system is stabilized and the frequency of the vertical oscillation is higher then the maximum frequency in the unforced case.

4.4.2 Monochromatic plane wave disturbance

In case the initial density perturbation has a form of the planar wave:

$$\overline{\Delta\rho} = \varrho_o \cos(kx + mz), \tag{42}$$

the procedure is conducted analogously as in the previous case. Although $u_o' \neq 0$, from the continuity equation (37) we have $\overline{u_o}\frac{\partial}{\partial x} = \overline{w_o}\frac{\partial}{\partial z} = u_o'\frac{\partial}{\partial x} = w_o'\frac{\partial}{\partial x} = 0$ leading to the cancellation of the advection terms. Consequently, the multiscale technique and averaging leads to the following stability condition for the averaged system:

$$\frac{\overline{D}^2}{Dt^2}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)\overline{\Delta\rho} + S_*^2\,\Big(\frac{\partial^2}{\partial x^2}\Big)\,\overline{\Delta\rho} = 0, \quad \text{where} \quad S_*^2 = \mathcal{S} + \Big(\frac{k^2}{k^2+m^2}\Big)\frac{\beta^2\Omega^2}{2}.$$

As before, we conclude that in case of a stable initial stratification the stability is amplified, while in case of the unstable initial stratification, the vibrating gravity results in the stabilizing effect whenever:

$$\Big(\frac{k^2}{k^2+m^2}\Big)\frac{\beta^2\Omega^2}{2} > 1, \tag{43}$$

which is equivalent to (41), except that now the orientation of the perturbation in space (i.e. $\left(\frac{k^2}{k^2+m^2}\right)$), matters. The dispersion relation in this case is:

$$\omega^2 = S_*^2 \frac{k^2}{k^2+m^2}. \tag{44}$$

Again, the form of the dispersion relation is equivalent to the dispersion relation of the internal gravity wave solutions to the unforced case, with N replaced by S^*. The effect of the vibrational gravity forcing depends not only on the values of the parameters of the forcing, but also on the angle of the wavevector $\mathbf{k}$: for a given forcing properties, vertical disturbances are more difficult to suppress.

4.4.3 Perturbation of an arbitrary form

If we allow the instantenous density disturbance to have an arbitrary form $\overline{\Delta\rho}(x,z)$, the advective terms in (33, 34) and (36) do not cancel out and the mean vorticity equation takes a form of an integro–differential equation:

$$\frac{\overline{D}\overline{q}}{Dt} = \frac{\partial\overline{\Delta\rho}}{\partial x} + \overline{\left(\frac{\partial\Delta\rho_1}{\partial x}\tilde{g}_{-1}\right)} - \mathcal{J}(\psi', \nabla^2\psi'). \tag{45}$$

Without further assumptions, it is difficult to construct any meaningful stability condition. However, if we could represent $\overline{\Delta\rho}$ in Fourier series and linearize in u_o', w_o' and $\Delta\rho$, we can get rid of the advective terms and obtain the following form of the mean vorticity equation:

$$\frac{\overline{D}\overline{q}}{Dt} = \sum_{n=1}^{\infty}\left(\frac{\partial\overline{\Delta\rho}_o}{\partial x}\right)_n\left(1 + \left(\frac{k_n^2}{k_n^2+m_n^2}\right)\frac{\beta^2\Omega^2}{2}\right), \tag{46}$$

which shows the additive effect of any single perturbation wave component of the series to the baroclinic generation of the mean vorticity, modified by the vibrating gravity in similar way as in the previous simpler cases. The corresponding stability condition for an unstable initial stratification is:

$$\frac{\beta^2\Omega^2}{2}\sum_{n=1}^{\infty}\left(\frac{k_n^2}{k_n^2+m_n^2}\right) > 1, \tag{47}$$

so again we can see that provided the instantenous density perturbation can be given a form of the sum of waves, there is an analogy between the effect of the vibrational forcing on the stability of the system in case of the Kapitza pendulum and dynamics of a fluid described by ideal Boussinesq equations, with a modification due to nonlocality of the problem: for a given values of forcing parameters the stability is strongly affected by the direction of the propagation of the density disturbance.

5 Summary and conclusions

In this work we considered the inverted pendulum with the vibrating support. The application of multiscale perturbation, leading to the averaged dynamics, as well as linearization, allowed us to study the stabilization phenomenon. Then we used the multiscale technique and averaging to an incompressible, inviscid, linearly stratified, nonlinear Boussinesq system in a periodic domain, subjected to rapidly oscillating gravity field. We have shown that, provided the instantenous density perturbation can be given a form of a wave or sum of waves, the stabilization mechanism induced by the vibrational forcing is analogous to that exhibited by the Kapitza pendulum. However, the dynamics of the Boussinesq system is more complicated, as it evolves not only in time, but also in space. The resulting stability condition for the initially unstable configuration is modified: it requires not only suitably high frequency and small amplitude of the vibrating motion – the direction of propagation of the density perturbation in the space also plays a role. In case of a disturbance of a general form it is difficult to draw the conclusions about the system stability without further assumptions.

The work presented here is not just an idealized, educative example that contributes to the understanding of the instability phenomena. There are indeed real physical situations that permits the use of Boussinesq approximation with the forcing regime as prescribed here, though certainly requiring adequate boundary conditions and generally more complex analysis. As an example, we give convective phenomena in radially pulsating stars, treated in ([14]) by means of linearization. The appeal of the multiscale perturbation and averaging methods discussed here is that they provide the description of the global behavior of the averaged variable, expected to be the one related to any observed quantity. This observation strongly encourages the application of these methods in any future investigation of the stability mechanisms in more realistic and complex fluid systems forced parametrically.

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7 Appendix A

In the coordinate system (2), the kinetic energy of the system is expressed as:

$$T = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) = \frac{1}{2}m(l^2\dot{\phi} + a^2\gamma^2\sin^2(\gamma t) + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}) =$$
$$= \frac{1}{2}ml^2\dot{\phi} + mal\gamma^2\cos(\gamma t)\cos\phi + \frac{1}{2}ma^2\gamma^2\sin^2(\gamma t) - \frac{d}{dt}\big[mal\gamma\sin(\gamma t)\cos(\phi)\big],$$

and the potential is $U = -mg_o y = -mg_o l\cos\phi$. The Lagrangian of the system is therefore:

$$L = T - U = \frac{1}{2}ml^2\dot{\phi} + mal\gamma^2\cos(\gamma t)\cos\phi + mg_o l\cos\phi + \frac{d}{dt}\big[mal\gamma\sin(\gamma t)\cos(\phi) + \frac{1}{2}ma\gamma\sin^2(\gamma t)\big].$$

The complete time derivative on RHS does not enter the action, so from the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0,$$

we obtain the equation of motion of the vibrationally forced pendulum (3):

$$ml^2\ddot{\phi} + mg_o l \sin\phi + mal\gamma^2 \cos(\gamma t)\sin\phi = 0.$$

8 Appendix B

Inserting the perturbation series (9) into (8) gives:

$$\frac{d^2\phi_o}{dt^2} = -\sin\phi_o - \epsilon\phi_1\cos\phi_o\epsilon^2\phi_2\cos\phi_o - \frac{\tilde{\beta}}{\epsilon}\cos\tau\sin\phi_o - \tilde{\beta}\phi_1\cos\tau\cos\phi_o - \tilde{\beta}\epsilon\phi_2\cos\tau\cos\phi_o$$

A typical initial condition we can think about is the initial displacement from the inverted position with zero angular velocity:

$$\phi_o|_{\substack{t=0\\ \tau=0}} = A_o, \qquad \frac{d\phi_o}{dt}|_{\substack{t=0\\ \tau=0}} + \frac{1}{\epsilon}\frac{d\phi_o}{d\tau}|_{\substack{t=0\\ \tau=0}} = 0. \tag{48}$$

And the absence of the perturbed quantities (no fast initialization):

$$\phi_n|_{\substack{t=0\\ \tau=0}} = 1, \qquad \frac{d\phi_n}{dt}|_{\substack{t=0\\ \tau=0}} + \frac{1}{\epsilon}\frac{d\phi_n}{d\tau}|_{\substack{t=0\\ \tau=0}} = 0, \quad n > 0. \tag{49}$$

By assembling powers of ϵ one obtains equations for the subsequent orders:

$$\begin{aligned}
\mathcal{O}(\frac{1}{\epsilon^2}): &\quad \frac{\partial^2\phi_o}{\partial\tau^2} = 0,\\
\mathcal{O}(\frac{1}{\epsilon}): &\quad \frac{\partial^2\phi_1}{\partial\tau^2} = -\tilde{\beta}\cos\tau\sin\phi_o(t),\\
\mathcal{O}(1): &\quad \frac{\partial^2\phi_2}{\partial\tau^2} = -\frac{\partial^2\phi_o}{\partial t^2} - 2\frac{\partial^2\phi_1}{\partial\tau\partial t} - \sin\phi_o - \tilde{\beta}\phi_1\cos\tau\cos\phi_o(t).
\end{aligned}$$

Let's look at them in detail. The integral of the first of them is:

$$\mathcal{O}(\frac{1}{\epsilon^2}): \quad \frac{\partial^2\phi_o}{\partial\tau^2} = 0, \quad \phi_o = F_o(t)\tau + G_o(t).$$

This gives unbounded, linear growth of the solution in fast time τ which obviously destroys the solution on long time scales. Such terms are called secular terms and a standard procedure in multiscale perturbation technique is to remove them by making them equal to zero and vanish ([9]). This argument can be justified by using the initial conditions (48), from which we get $F_o(t) = 0$ and $G_o(t) = A_o$. Therefore, the leading order solution is equal to $A_o(t)$, a function of long time t but a constant with respect to τ, $\phi_o = \phi_o(t)$. Using analogous argument with the initial conditions (49) incorporated, we get for the next order term:

$$\mathcal{O}(\frac{1}{\epsilon}): \quad \phi_1 = \tilde{\beta}(\cos\tau - 1)\sin\phi_o(t). \tag{50}$$

The equation for the order $\mathcal{O}(1)$ becomes:

$$\begin{aligned}
\mathcal{O}(1): \quad \frac{\partial^2\phi_2}{\partial\tau^2} &= 2\tilde{\beta}\sin\tau\cos\phi_o\dot{\phi}_o(t) + \tilde{\beta}^2\cos\tau\sin\phi_o\cos\phi_o - \frac{\tilde{\beta}^2}{2}\cos 2\tau\sin\phi_o\cos\phi_o -\\
&- \frac{\partial^2\phi_o}{\partial t^2} - \sin\phi_o - \frac{1}{2}\tilde{\beta}^2\sin\phi_o(t)\cos\phi_o(t).
\end{aligned}$$

Terms that are constants with respect to the fast time τ are a potential source for a secular growth in our solution. The condition for the solution to be valid uniformly on t gives the equation of motion of the leading order quantity (10). Note also that by taking the average defined by (11) of the equation (50), one gets (12).