Lecture 9: The use of variational methods for high-contrast conductivity problems

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1 Introduction

In this lecture we will consider the conductivity in a high-contrast medium. Besides its physical importance, the model under consideration will serve as an illustration of the use of variational principles. This will provide a good introduction to variational principles before using them in a more difficult form in the next lecture, where convection-diffusion problems at high Péclet numbers (strong convection versus weak diffusion) are considered.

2 General formulation

Consider a smooth region $\Omega \subset \mathbb{R}^2$ with outward unit normal $\mathbf{n}(\mathbf{x})$ and with given non-negative *conductivity* $\sigma(\mathbf{x})$[1]. The governing equation for the *potential* Φ is

$$\nabla \cdot [\sigma(\mathbf{x})\nabla\Phi] = 0, \qquad \mathbf{x} \in \Omega, \tag{1}$$

with Neumann boundary condition[2]

$$\sigma(\mathbf{x})\frac{\partial\Phi}{\partial n} = I(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega. \tag{2}$$

The *outgoing current* $I(\mathbf{x})$ is assumed to be equilibrated, hence

$$\int_{\partial\Omega} I dS = 0. \tag{3}$$

Next, let us introduce $s(\mathbf{x})$ by assuming that the conductivity has the form

$$\sigma(\mathbf{x}) = \sigma_0 e^{-s(\mathbf{x})/\epsilon}, \tag{4}$$

where we are interested in the *high-contrast* limit characterised by $\epsilon \downarrow 0$.

Plugging (4) into (1) leads to

$$\epsilon\Delta\Phi - \nabla s \cdot \nabla\Phi = 0. \tag{5}$$

[1]The analysis will not be affected qualitatively in 3 dimensions, but for simplicity we will consider $\mathbb{R}^2$.
[2]Dirichlet boundary condition $\Phi|_{\partial\Omega} = \Psi$ can be analysed as well, without qualitative changes.

Notice that the operator on Φ in (5) is self-adjoint, as opposed to the similar equation for a divergence-free fluid, which will be discussed in more detail in the next lecture. In fact, (5) is difficult to solve in general, hence in the following section the classical variational principles will be introduced to help us estimate the solution without solving the equation itself.

3 Variational principles

To introduce the classical variational principles, we first need to define *Dirichlet-to-Neumann (DtN)* and *Neumann-to-Dirichlet (NtD)* maps.

The DtN map Λ takes Dirichlet boundary data $\Phi|_{\partial\Omega} = \Psi$ to the outgoing current $I = \sigma\frac{\partial\Phi}{\partial n}$, hence $\Lambda\Psi = I$. Furthermore, given (3) the NtD map can be defined as the inverse of the DtN map, namely, $\Psi = \Lambda^{-1}I$. Without going into details, let us note that after determining these two maps, one has almost all the information about the problem that can be observed at the boundary.

Λ is a self-adjoint, positive semidefinite map with respect to the standard inner product. Indeed,

$$\begin{aligned}
(\Lambda\Psi, \Psi) &= \int_{\partial\Omega} \Lambda\Psi(\mathbf{x})\Psi(\mathbf{x})\, dS = \int_{\partial\Omega} I(\mathbf{x})\Psi(\mathbf{x})\, dS = \text{(using the boundary conditions)} \\
&= \int_{\partial\Omega} \sigma(\mathbf{x})\frac{\partial\Phi(\mathbf{x})}{\partial n}\Phi(\mathbf{x})\, dS = \int_{\partial\Omega} \Phi(\mathbf{x})\sigma(\mathbf{x})\nabla\Phi\cdot\mathbf{n}\, dS = \text{(by the divergence theorem)} \\
&= \int_{\Omega} \nabla\cdot(\Phi(\mathbf{x})\sigma(\mathbf{x})\nabla\Phi)\, dV = \int_{\Omega} \sigma(\mathbf{x})\nabla\Phi\cdot\nabla\Phi\, dV \geq 0. \qquad (6)
\end{aligned}$$

which demonstrates that Λ is positive semidefinite. In the last step we integrated by parts and used (1). Now let Ψ_1 and Ψ_2 be two different sets of Dirichlet boundary data. Using (6) we see that

$$\begin{aligned}
(\Lambda\Psi_1, \Psi_2) &= \int_{\partial\Omega} \Lambda\Psi_1(\mathbf{x})\Psi_2(\mathbf{x})\, dS = \int_{\Omega} \sigma(\mathbf{x})\nabla\Phi_1\cdot\nabla\Phi_2\, dV = \text{(by symmetry)} \\
&= \int_{\partial\Omega} \Lambda\Psi_2(\mathbf{x})\Psi_1(\mathbf{x})\, dS = (\Psi_1, \Lambda\Psi_2) \qquad (7)
\end{aligned}$$

and thus the map Λ is self adjoint.

Now we are ready to introduce the *Dirichlet variational principle(DVP):*

$$(\Lambda\Psi, \Psi) = \min_{\tilde{\Phi}} \left\{ \int_{\Omega} \sigma\nabla\tilde{\Phi}\cdot\nabla\tilde{\Phi}\, dV \mid \nabla\tilde{\Phi} \text{ is square-integrable and } \tilde{\Phi}|_{\partial\Omega} = \Psi \right\}. \qquad (8)$$

To prove the DVP (8), we consider the Euler-Lagrange equations for the variational problem on the right hand side. If an integral K is of the form

$$K = \int_{\Omega} f\left(\tilde{\Phi}, \tilde{\Phi}_{x_i}\right)\, dV, \qquad \text{where } \tilde{\Phi}_{x_i} = \frac{\partial\tilde{\Phi}}{\partial x_i}, \qquad (9)$$

then the corresponding Euler-Lagrange equations for solving $\delta K = 0$ by varying Φ are, using the summation convention,

$$\frac{\partial f}{\partial \tilde{\Phi}} - \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial \tilde{\Phi}_{x_i}}\right) = 0. \tag{10}$$

For equation (8) $f(\tilde{\Phi}, \tilde{\Phi}_{x_i}) = \sigma \tilde{\Phi}_{x_i} \tilde{\Phi}_{x_i}$ and thus the Euler-Lagrange equations become

$$\frac{\partial}{\partial x_i}\left(\sigma \tilde{\Phi}_{x_i}\right) = 0. \tag{11}$$

This is simply our original conductivity equation (1), and thus the integral in the DVP (8) is minimised when $\tilde{\Phi} = \Phi$ where Φ solves (1). The integral in the DVP is called the *Dirichlet integral* and measures the rate of energy dissipation.

The DVP can be written in another form, namely,

$$(\Lambda\Psi, \Psi) = \min_{\tilde{\mathbf{E}}}\left\{\int_{\Omega} \sigma \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}\, dV \mid \tilde{\mathbf{E}} = \nabla\tilde{\Phi} \text{ is a curl-free field and } \tilde{\Phi}|_{\partial\Omega} = \Psi\right\}. \tag{12}$$

This form of the DVP helps to illustrate better the *duality* of DVP with the *Kelvin variational principle(KVP):*

$$(I, \Lambda^{-1} I) = \min_{\tilde{\mathbf{j}}}\left\{\int_{\Omega} \sigma^{-1} \tilde{\mathbf{j}} \cdot \tilde{\mathbf{j}}\, dV \mid \nabla \cdot \tilde{\mathbf{j}} = 0 \text{ and } \tilde{\mathbf{j}} \cdot \mathbf{n}|_{\partial\Omega} = I\right\}, \tag{13}$$

where $\tilde{\mathbf{j}} = \sigma\nabla\tilde{\Phi}$ is the divergence-free current. A similar calculation shows that the minimum is realized by $\mathbf{j} = \sigma\nabla\Phi$, where Φ is again the solution of (1).

Notice that while we cannot solve the conductivity equation (1) in general, we know *beforehand* that the solution must be the minimiser of the functionals appearing in both the DVP and KVP. This important feature allows us to bound both Λ and Λ^{-1} from above by taking appropriately well-constructed test functions $\tilde{\Phi}$ and $\tilde{\mathbf{j}}$. It can be shown that this is equivalent to finding both upper and lower bounds for the map Λ. In some problems these bounds coincide, giving rise to the *exact* solution. This method is particularly well-suited for problems in the high-contrast limit, where we try to find an asymptotic form for the solution or at least bound it from both above and below.

4 The high-contrast conductivity problem

In this part of the lecture, we consider a particular problem which will serve as a benchmark to illustrate the application of the variational principles in the conductivity-related problems.

Consider the conductivity equation (1) with $\Phi(x, y) = \chi(x, y) + x$,

$$\nabla \cdot [\sigma(\nabla\chi + \mathbf{e_1})] = 0, \tag{14}$$

where for simplicity we have dropped the arguments (x, y). Here $\mathbf{e_1}$ is the unit vector along the x-axis. Let the domain be the square region $D = [-1/2, 1/2] \times [-1/2, 1/2]$. Consider solutions with periodic potential $\chi(x, y)$, which must be unique up to an additive constant. We will be interested in the quantity

$$\sigma^*(\mathbf{e_1}) = \langle \sigma(\nabla\chi + \mathbf{e_1}) \cdot \mathbf{e_1} \rangle, \tag{15}$$

where the average $\langle\ \rangle$ is taken over D. $\sigma^*(\mathbf{e_1})$ can be interpreted as the *average flux* per unit average gradient in the direction $\mathbf{e_1}$.

Suppose $\sigma(x, y)$ satisfies $\sigma(x, y) = \sigma_0 e^{-s(x,y)/\epsilon}$ with the high-contrast assumption $0 < \epsilon \ll 1$. Then we can write $\sigma^*(\mathbf{e_1})$ in the following form (justified by an integration by parts and (15)):

$$\sigma^*(\mathbf{e_1}) = \langle \sigma(\nabla\chi + \mathbf{e_1}) \cdot (\nabla\chi + \mathbf{e_1}) \rangle = \int_D \sigma(\nabla\chi + \mathbf{e_1}) \cdot (\nabla\chi + \mathbf{e_1})\, dx\, dy. \tag{16}$$

Furthermore, using the Dirichlet variational principle,

$$\sigma^*(\mathbf{e_1}) = \min_{\tilde{\chi}} \int_D \sigma_0 e^{-s/\epsilon} (\nabla\tilde{\chi} + \mathbf{e_1}) \cdot (\nabla\tilde{\chi} + \mathbf{e_1})\, dx\, dy. \tag{17}$$

We will now consider the case where there is a single saddle point in our domain (Figure 1). The integral in (17) cannot be tackled by Laplace's method, since χ itself depends on the infinitesimal parameter ϵ. In fact, the major contribution in the integral comes from the neighbourhood of the saddle point, which, without loss of generality, can be assumed to be at the origin with principal axes aligned with the coordinate axes. Since the gradient of the function at a saddle point vanishes, we have the following Taylor expansion of $s(x, y)$ up to second order:

$$s(x, y) \approx s_0 - \frac{k_1}{2} x^2 + \frac{k_2}{2} y^2, \tag{18}$$

where k_1 and k_2 are the principal curvatures of the level curves of s intersecting at the saddle point.

Next, we pass into an approximate inequality by shrinking the integration region to $\Delta = [-\delta, \delta] \times [-\delta, \delta]$ and plugging the truncated expansion (18) into (17), as well as by minimising only among the functions $\chi(x, y) = \chi(x)$:

$$\begin{aligned} \sigma^*(\mathbf{e_1}) &\lesssim \min_{\tilde{\chi}(x)} \int_\Delta \sigma_0 \exp\left(-\frac{1}{\epsilon} \big[s_0 - \frac{k_1}{2} x^2 + \frac{k_2}{2} y^2 \big] \right) (\tilde{\chi}_x + 1)^2\, dx\, dy \\ &\approx \sigma_0 e^{-s_0/\epsilon} \sqrt{\frac{2\pi\epsilon}{k_2}} \min_{\tilde{\chi}(x)} \int_{-\delta}^{\delta} e^{\frac{k_1}{2\epsilon} x^2} (\tilde{\chi}_x + 1)^2\, dx. \end{aligned} \tag{19}$$

By DVP, the solution of (14) is the minimiser of the functional above, hence we will look for $\chi(x)$ satisfying the equation

$$(e^{\frac{k_1}{2\epsilon} x^2} (\tilde{\chi}_x + 1))_x = 0, \tag{20}$$

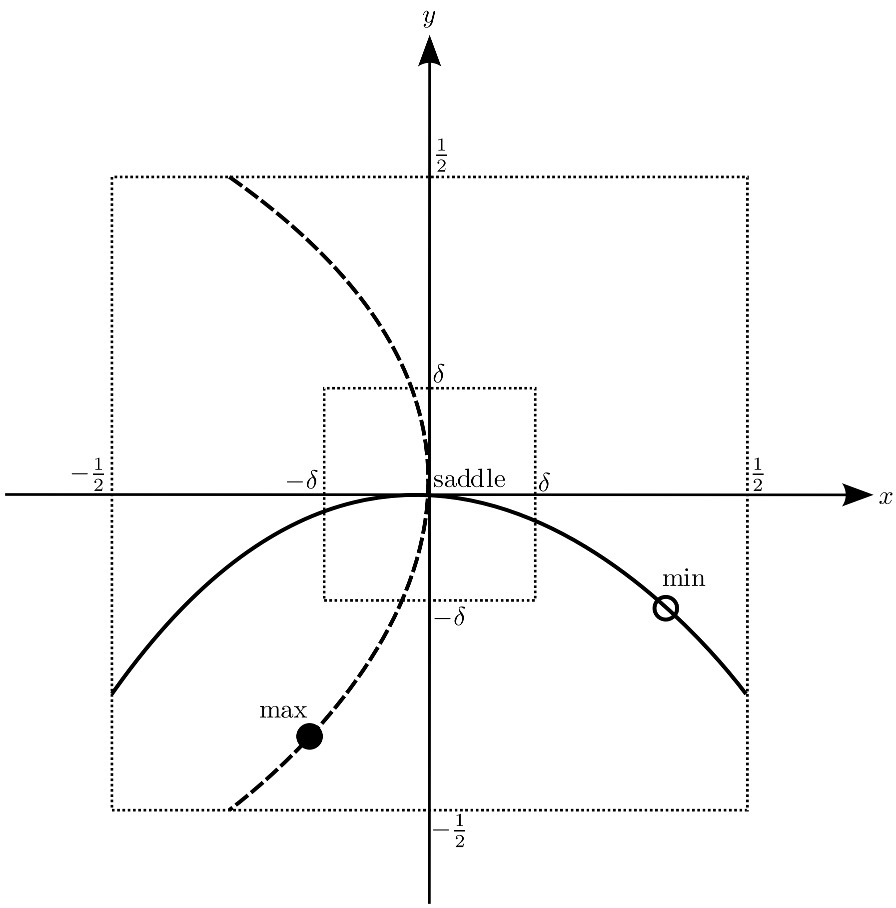

Figure 1: Topography of $s(x, y)$ in the square region D. At the origin there is a saddle point. The filled line through the origin is a valley, the dashed line a ridge. The local maximum is a filled circle, the local minima a hollow circle. The analysis focuses on a small region Δ near the saddle point.

as well as the periodicity condition for χ, which implies

$$\langle \tilde{\chi}_x \rangle = 0. \tag{21}$$

Since ϵ is very small, the average in (21) can be taken over the neighbourhood of the saddle point $[-\delta, \delta]$ as well as over the whole interval $[-1/2, 1/2]$ without changing the leading order asymptotic term.

Solving (20), one finds

$$\tilde{\chi}_x + 1 = Ce^{-\frac{k_1}{2\epsilon}x^2}, \tag{22}$$

while the constant C can be found from (21):

$$1 = \langle \tilde{\chi}_x + 1 \rangle = \langle Ce^{-\frac{k_1}{2\epsilon}x^2} \rangle \sim C\sqrt{2\pi\frac{\epsilon}{k_1}}\,, \tag{23}$$

Hence the optimising function satisfies the approximate equation, which is asymptotic to the solution of (14) as $\epsilon \to 0$

$$\tilde{\chi}_x + 1 \sim \frac{1}{\sqrt{2\pi \frac{\epsilon}{k_1}}} e^{-\frac{k_1}{2\epsilon} x^2} \,. \tag{24}$$

Plugging into the integral (19) leads to

$$\sigma^*(\mathbf{e_1}) \lesssim \sigma_0 e^{-s_0/\epsilon} \sqrt{2\pi \frac{\epsilon}{k_2}} \int_{-\delta}^{\delta} \frac{e^{-\frac{k_1}{2\epsilon} x^2}}{2\pi \frac{\epsilon}{k_1}} \quad \sim \sigma_0 e^{-s_0/\epsilon} \sqrt{\frac{k_1}{k_2}} \,, \tag{25}$$

which is the conductivity at the saddle point $\sigma_0 e^{-s_0/\epsilon}$, multiplied by the factor $\sqrt{\frac{k_1}{k_2}}$ determined by the curvatures of the level sets passing through that saddle point. For instance, small k_2 corresponds to a narrow saddle point, where the conductivity is large.

Using KVP for the backward NtD map, one can find a lower asymptotic bound for $\sigma^*(\mathbf{e_1})$ which turns out to be exactly the same as in (25)! This leads to the *exact* asymptotic expression for the average resulting flux in the x-direction

$$\sigma^*(\mathbf{e_1}) \sim \sigma_0 e^{-s_0/\epsilon} \sqrt{\frac{k_1}{k_2}} \,, \qquad \text{as } \epsilon \to 0. \tag{26}$$

The corresponding resistance $\rho^* = 1/\sigma^*$ is given by

$$\rho^*(\mathbf{e_1}) \sim \frac{1}{\sigma_0} e^{s_0/\epsilon} \sqrt{\frac{k_2}{k_1}} \,, \qquad \text{as } \epsilon \to 0. \tag{27}$$

5 Complicated topography

We now consider the situation where we have multiple saddle points in our domain. Figure 2 gives an example of such a situation.

To understand how current flows through the domain in Figure 2 it is useful to make an analogy with the flow of water. Consider the case where current flows into the domain over a. It will flow directly to the nearest point of maximum conductivity, node 1. There current will "pool" before escaping through the "channels" (saddle points) to the adjacent nodes. From these nodes current will then flow either to other nodes via the channels, or out of the boundary. Hence intuitively the domain can be thought of as behaving as a network of channels.

More formally, we have that the dominant contribution to the DtN map Λ as $\epsilon \downarrow 0$ is determined by the saddle points of $s(x, y)$. At each saddle point we can calculate the resistance of the saddle using the result for a single saddle (27). Denote the resistance of the saddle point between node i and node j as R_{ij}. Note that R_{ij} is symmetric: $R_{ij} = R_{ji}$. Since each saddle can be considered as a single resistor, we can reduce the problem to a simple resistor network (Figure 3).

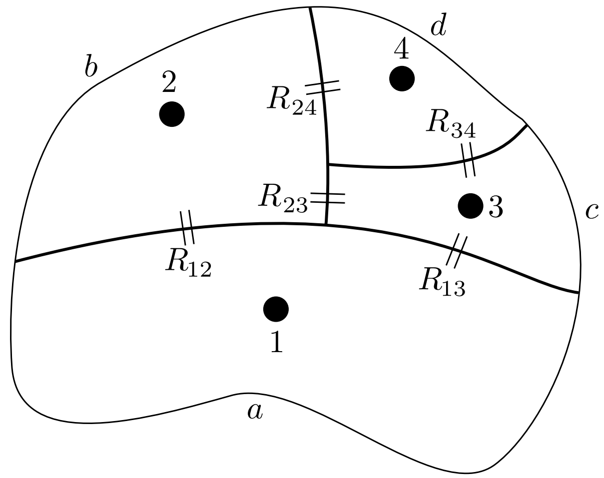

Figure 2: An example domain with multiple saddle points. The topography of $s(x, y)$ is shown. The filled circles are local maxima of conductivity, and are numbered 1, 2, 3, 4 (we will refer to these as the *nodes*). The lines dividing the domain are the valleys. The corresponding divided parts of the boundary are labelled a, b, c, d. The saddle points are shown as two short parallel lines, resembling channels. The resistance R_{ij} has been labelled by each saddle (see later discussion).

The DtN map Λ of the full problem is asymptotic as $\epsilon \downarrow 0$ to the DtN map of the resistor network. Consider the Dirichlet problem where the potential Φ is specified on the boundary, $\Phi|_{\partial\Omega} = \Psi$. Then equation (8) becomes

$$(\Lambda\Psi, \Psi) = \min_{\tilde{\Phi}, \tilde{\Phi}|_{\partial\Omega}=\Psi} \int_\Omega \sigma \nabla\tilde{\Phi} \cdot \nabla\tilde{\Phi}\, dV \lesssim \min_{\tilde{\Phi}_k, \tilde{\Phi}_k|_{\partial\Omega}=\Psi_k} \sum_{j \in \text{nodes}} \sum_{k \in \nu_j} \frac{1}{R_{jk}} \left(\tilde{\Phi}_j - \tilde{\Phi}_k\right)^2. \qquad (28)$$

The above expression specifies an asymptotic upper bound for the DtN map Λ. Here the set ν_j is the set of nodes adjacent to the node j, and $\tilde{\Phi}_j$ is the potential at node j. Ψ_k are the integrated potentials specified on the sections of boundaries k. The boundary condition is now that the potentials $\tilde{\Phi}_k$ of nodes adjacent to the boundary are equal to the potentials Ψ_k on the boundaries. For the example domain, the boundary condition becomes $\tilde{\Phi}_1 = \Psi_a$, $\tilde{\Phi}_2 = \Psi_b$, $\tilde{\Phi}_3 = \Psi_c$, and $\tilde{\Phi}_4 = \Psi_d$. In this simple case it means that all $\tilde{\Phi}_k$ have been determined, but in more complicated cases there can be $\tilde{\Phi}_k$ in the interior of the domain which are not directly specified by the boundary condition. Even in these more complicated cases, the minimisation is now just an easy to solve matrix problem.

Similarly we can solve the dual problem (13) where the current $\mathbf{j}$ is specified on the boundary rather than the potential Φ. The dual problem yields a corresponding asymptotic upper bound for the inverse map Λ^{-1}, and thus an asymptotic lower bound for Λ. As in the single saddle case it turns out that the asymptotic lower bound for Λ is the same as the asymptotic upper bound, and thus we get an asymptotic equality.

Notes by Khachik Sargsyan and John Rudge.

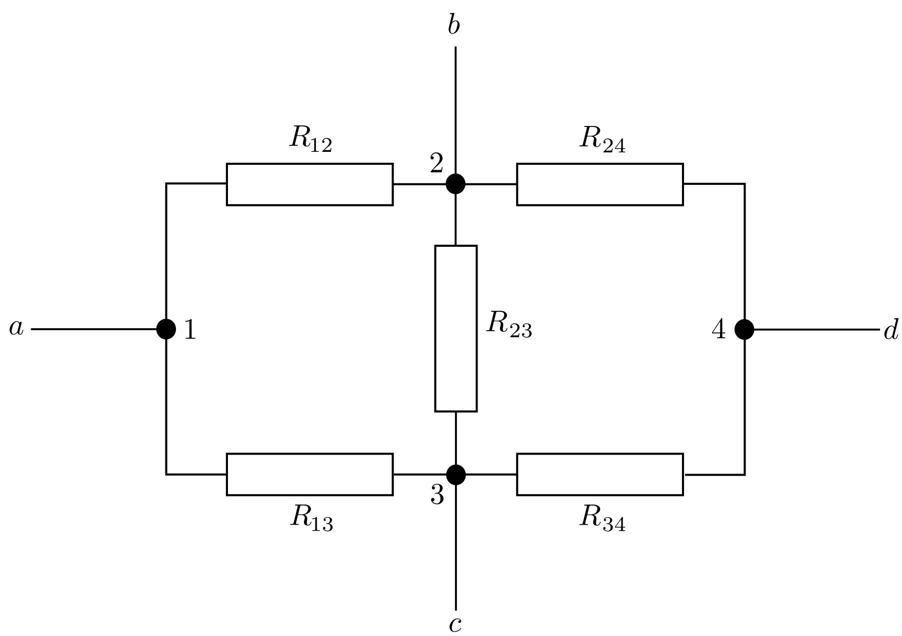

Figure 3: Resistor network corresponding to the domain in Figure 2. Current flows into or out of the network over the boundaries a, b, c, d.