# Lecture 4: Geometrical Theory of Diffraction (continued) and the Shallow Water Theory

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## 1 Introduction

In this lecture we will finish discussing the reflection from a boundary (Section 2). Next, in Section 3 we will switch to the geometric theory of diffraction without formal details (see [3]). Further, in Section 4, we will generalize the surface water wave theory to a case of non-uniform depth (see [2]). This theory predicts infinite amplitude at the shoreline, hence, in Section 5, the shallow water theory is introduced to fix that problem.

## 2 Reflection from a boundary

In the previous lectures we developed a method for solving the *Helmholtz equation*

$$\Delta U + k^2 n^2(X) U = 0. \tag{1}$$

It yields an asymptotic approximation like geometrical optics. It was applied to reflection by a parabolic cylinder. Now let us analyze reflection of waves, in water of constant depth $h = \text{const}$, by an arbitrary smooth boundary $B$, e.g. a vertical river bank. In this case the velocity potential is given by an incident contribution $\phi^i$ and a reflected part $\phi^r$. Since the normal velocity on the boundary vanishes, $\partial_\nu \phi|_B = 0$. The zero-order asymptotic solution is given by

$$\phi = \phi^i + \phi^r \sim z_0^i(x) e^{ikS^i} + z_0^r(x) e^{ikS^r}. \tag{2}$$

The condition of vanishing normal derivative yields

$$\left(ik \frac{\partial}{\partial n} S^i\right) e^{ikS^i} z_0^i(x) + e^{ikS^i} \frac{\partial}{\partial n} z_0^i(x) + \left(ik \frac{\partial}{\partial n} S^r\right) e^{ikS^r} z_0^r(x) + e^{ikS^r} \frac{\partial}{\partial n} z_0^r(x) \sim 0. \tag{3}$$

Upon equating the exponents in (3), we get

$$S^i = S^r \text{ on } B. \tag{4}$$

Then upon equating to zero the coefficient of $k$ in (3), we get

$$z_0^i \frac{\partial}{\partial n} S^i + z_0^r \frac{\partial}{\partial n} S^r = 0. \tag{5}$$

¿From (4), the tangential derivative $\frac{\partial s^i}{\partial \tau}$ equals $\frac{\partial s^r}{\partial \tau}$. Then from the eiconal equation, the normal derivatives are related by

$$\left(\frac{\partial S^i}{\partial n}\right)^2 = n^2 - \left(\frac{\partial S^i}{\partial \tau}\right)^2 = n^2 - \left(\frac{\partial S^r}{\partial \tau}\right)^2 = \left(\frac{\partial S^i}{\partial n}\right)^2 . \tag{6}$$

Thus $\partial S^r/\partial n = \pm \partial S^i/\partial n$. The "+" sign would yield $S^r(x) \equiv S^i(x)$, so we must choose the "−" sign

$$\frac{\partial S^r}{\partial n} = -\frac{\partial S^i}{\partial n}. \tag{7}$$

Then (5) yields

$$z_0^r = z_0^i \ \text{ on } \ B. \tag{8}$$

From $\partial S^r/\partial \tau = \partial S^i/\partial \tau$ and (7) we get *law of reflection*, familiar from geometrical optics.

Instead of a rigid boundary, we can also consider the *impedance boundary condition*

$$\frac{\partial}{\partial n}\phi + ikZ\phi = 0. \tag{9}$$

This condition is frequently used in electrodynamics, and in acoustics for compliant boundaries. In this case it follows that

$$s^r(X) = s^i(X), \ \ X \ \text{ on } \ B, \tag{10}$$

$$z_0^i \left(\tfrac{\partial s^i}{\partial \nu} + Z\right) + z_0^r \left(\tfrac{\partial s^r}{\partial \nu} + Z\right) = 0, \ \ X \ \text{ on } \ B, \tag{11}$$

$$z_m^i \left(\tfrac{\partial s^i}{\partial \nu} + Z\right) + z_m^r \left(\tfrac{\partial s^r}{\partial \nu} + Z\right) + \tfrac{\partial z_{m-1}^i}{\partial \nu} + \tfrac{\partial z_{m-1}^r}{\nu} = 0, \ \ m \geq 1, \ \ X \ \text{ on } \ B. \tag{12}$$

In terms of the incidence angle $\alpha$ and the impedance $Z$, (11) gives the reflection coefficient

$$\frac{z_0^r}{z_0^i} = \frac{n\cos\alpha - Z}{n\cos\alpha + Z}, \tag{13}$$

again for $X$ on $B$.

Fermat's principle of geometrical optics follows from the eiconal equation. It states that a ray travelling between two points takes the path with the shortest optical distance. For media with $n(x) = \text{const}$, we can find this path by imagining a string tightly spanned between these two points. Then the ray path will lie along this string.

To calculate the wave field numerically one often uses the finite element method. Due to the oscillatory nature of the wave field, this procedure requires very small elements. But we know that the leading order approximation has the form $z_0^i(x)e^{ikS^i} + z_0^r(x)e^{ikS^r}$. We can use this ansatz, calculating the phase by means of rays. After that the amplitudes $z_0^i(x)$ and $z_0^r(x)$ can be calculated by the finite element method. This allows us to use much larger elements.

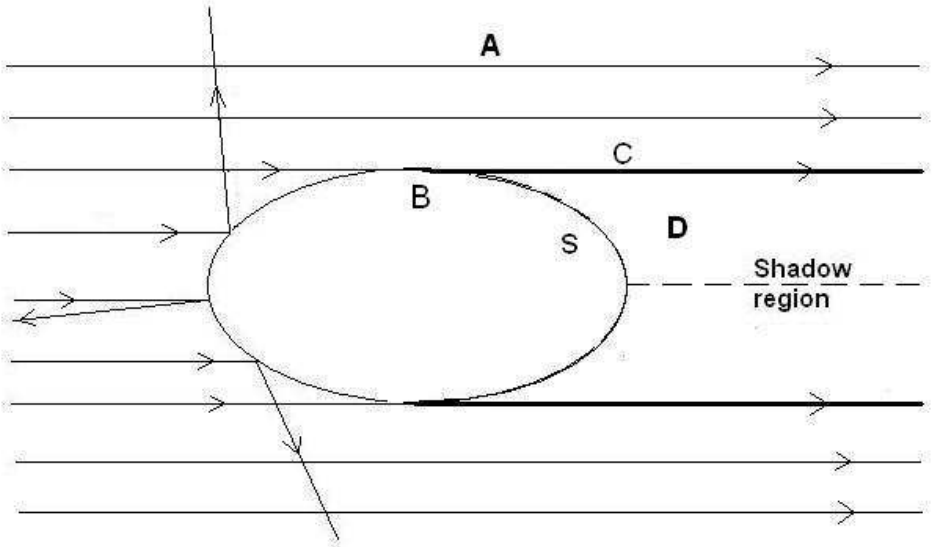


Figure 1: Scattering by a smooth object. Regions for asymptotic expansion: $A$ - region of ordinary geometrical optics, $B$ - point at which incident rays are tangent to the boundary, $S$ - the object boundary, $C$ - shadow boundary, $D$ - shadow region.

# 3 Geometrical theory of diffraction

The asymptotic expansion method presented in the previous lectures is incomplete because of phenomena which are usually not taken into account by ordinary geometrical optics. Let us for example consider water waves being scattered by an island which we assume to be an oval-shaped object with smooth boundary. From ordinary geometrical optics follows that there exists a *shadow region* $D$ (see Figure 1) in which the intensity of waves is zero. This region is separated from the region $A$, reached by incident and reflected rays, by a surface called the *shadow boundary*. Obviously, along this boundary the solution obtained by the ordinary geometrical optics method is discontinuous. However, this is in sharp contradiction with the fact that solutions of the Helmholtz equation (1) are smooth away from the boundary. Agreement of asymptotic solutions with actual solutions can be achieved by introducing boundary layer solution in the neighborhood of shadow boundaries. They can be found by using asymptotic expansions of certain exactly solvable problems, or constructed by boundary layer techniques. The construction of asymptotic solutions requires different expansions in different regions.

We first will consider asymptotic solutions in the shadow region $D$. The only rays which reach this region are rays diffracted by the boundary $S$. They are called *surface diffracted rays*. To construct them we introduce *surface rays* (or *creeping rays*) which propagate along the boundary $S$. The point $B$ (see Figure 2) on the boundary between the shadow region $D$ and the illuminated region $A$ acts as a source for these rays. Note that at this

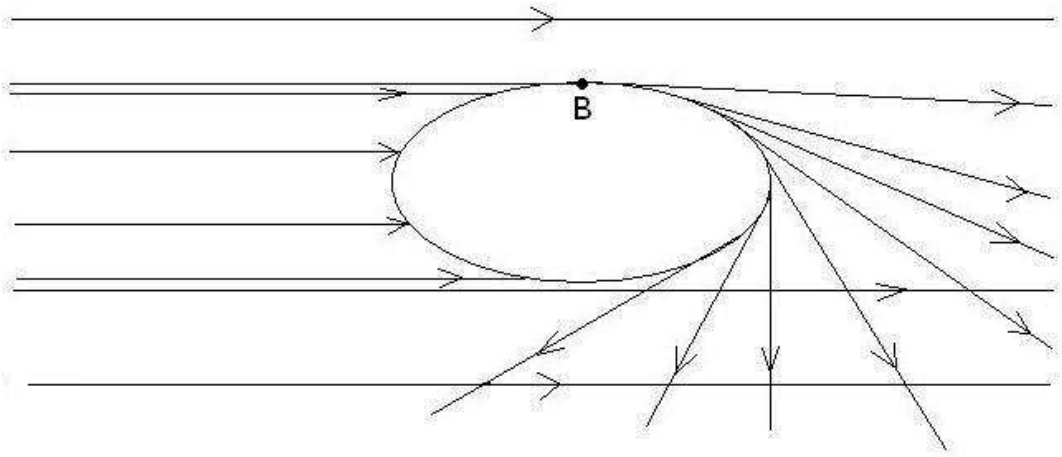


Figure 2: Surface diffracted rays omitted by the propagating surface ray starting at the point $B$.

point the incident ray is tangent to the boundary. Therefore, the incident ray splits into two branches. One branch goes along the shadow boundary $C$ while the other is the ray travelling along the boundary $S$. The latter radiates surface diffracted rays into the shadow region, and therefore the boundary $S$ acts as a secondary source. Because of this radiation, the intensity on the surface decays exponentially with distance along the ray. The surface ray travels infinitely many times around the boundary. Thus it sends an infinite number of surface rays to each point in the shadow, and also to each point in the illuminated region. Thus the complete wave field in the shadow region is an infinite sum of diffracted fields on surface diffracted rays.

The wave field on and near the shadow boundary can be obtained by using boundary layer theory. It yields Fresnel integrals which were used in the method of stationary phase. In the neighborhood of the separation point $B$ there is yet another kind of asymptotic solution, given by the Fock function.

The asymptotics of the wave field in the neighborhood of the boundary $S$ can be obtained from the exact solution of the Helmholtz equation for diffraction by a circular cylinder of radius $a$. In cylindrical coordinates a mode of the two dimensional wave field can be written as

$$u = e^{i\nu\theta} H_\nu^{(1)}(kr). \tag{14}$$

Here $H_\nu^{(1)}$, the Hankel function of first kind [1], is outgoing. Suppose the boundary condition is $u(r)|_{r=a} = 0$. This leads to the equation

$$H_\nu^{(1)}(ka) = 0. \tag{15}$$

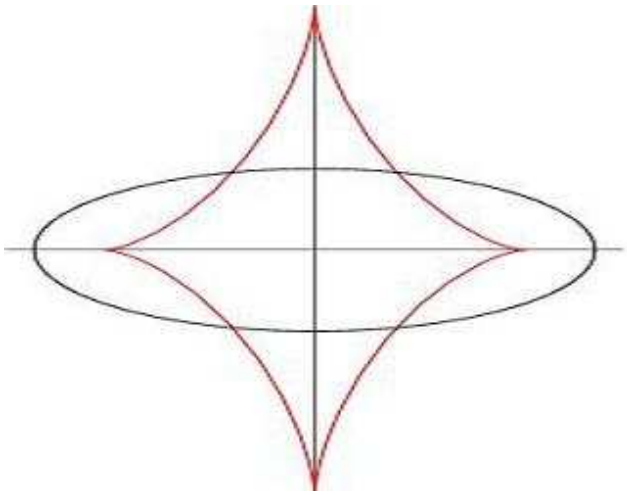

Figure 3: The Lamé curve.

This equation (15) has infinitely many complex roots $\nu_m$. The asymptotic behaviour of $\nu_m$ for $ka \gg 1$ is given by

$$\nu_m \sim ka + (ka)^{\frac{1}{3}} \tau_m e^{i\pi/3}. \tag{16}$$

When (16) is used for $\nu$ in (14), $u$ becomes

$$u_m = e^{i\nu_m \theta} H^{(1)}_{\nu_m}(kr) \sim e^{ika\theta + ie^{i\pi/3}\tau_m (ka)^{1/3}\theta} H^{(1)}_{\nu_m}(kr). \tag{17}$$

The result (17) shows that a single mode $u_m(r, \theta)$ decays exponentially with $\theta$ at a rate proportional to $(ka)^{1/3}$. For a noncircular boundary , the local decay rate can be obtained by replacing $a$ by the local radius of curvature $a(s)$ and setting $d\theta = a^{-1}(s)ds$. Then the exponent for the $n$-th mode becomes

$$iks + ie^{i\pi/3}\tau_m k^{1/3} \int_0^s \left(a(s')\right)^{-2/3} ds'. \tag{18}$$

The amplitude of each mode also involves diffraction coefficients at the point $B$, where the surface ray begins, and at the point $B'$ where it leaves the surface. Then the total field in the shadow is a sum of all the modes.

When a wave is diffracted by an axially symmetric object in three dimensions, the diffracted waves have a caustic along the axis. This yields a bright spot in the cross-section of the shadow. The wave field on and near this caustic can be expressed in terms of Bessel functions.

Now let us consider diffraction of a normally incident plane wave by a planar screen of arbitrary shape, with a smooth boundary $S$. Instead of a bright spot, there is a bright curved line in each normal cross-section of the shadow. The form of the bright line is given by the *evolute* of the curve $S$, and it is a caustic of the edge diffracted rays. For a planar curve, the evolute is the locus of centers of curvature of the normals to the curve. For example, for an ellipse the bright line is given by the Lamé curve, see Figure 3. To summarize: an asymptotic solution will usually consist of a sum of waves. Every wave

is constructed by means of rays. They are obtained by solving the ray equations. Then the phase $S$ and the amplitude $z_0$ are found along each ray by the formulae given above. The other $z_m$ can be found as solutions of the appropriate transport equations. This same approach is used in the next sections.

# 4 Surface waves on water of nonuniform depth

Previously we suggested that in water of nonuniform depth, we could determine the wave motion by using the reduced wave equation. That suggestion yields the correct phase, but not the correct amplitude. Therefore we shall now present an analysis which determines correctly both the phase and the amplitude[1].

As before, we assume that the water is inviscid, incompressible and in irrotational motion. It is bounded above by an unknown free surface $Z = \Re[e^{i\omega t}\eta(x,y)]$ ($\eta$ is the complex amplitude of the surface wave motion of angular frequency $\omega$) and bounded below by a rigid, non-uniform surface $Z = -H(x,y)$. The exact linear theory of surface waves yields for the free surface height

$$\eta(x,y) = \frac{i\omega}{g}\Phi(x,y,0),$$

where $\Phi(x,y,Z)$ is the *velocity potential* and $g$ is the gravitational acceleration[2].

The velocity potential satisfies (see Stoker [5])

$$\Delta\Phi = 0 \qquad \text{in} \qquad 0 \geqq Z \geqq -H(x,y), \tag{19}$$

$$\Phi_Z = \beta\Phi \qquad \text{on} \qquad Z = 0 \qquad (\beta = \omega^2/g), \tag{20}$$

$$\Phi_Z + H_x\Phi_x + H_y\Phi_y = 0 \qquad \text{on} \qquad Z = -H(x,y). \tag{21}$$

In the constant-depth case, the solution decays exponentially with depth, so short waves do not "feel" the bottom. To keep the influence of the depth variability, we rescale the vertical axis by introducing

$$z = \beta Z, \qquad h = \beta H, \qquad \phi(x,y,z) = \Phi(x,y,Z). \tag{22}$$

Then the problem (19)-(21) becomes that of finding solutions $\phi$ of the set of rescaled equations

$$\beta^2\phi_{zz} + \phi_{xx} + \phi_{yy} = 0 \qquad \text{in} \qquad 0 \geqq z \geqq -h(x,y), \tag{23}$$

$$\phi_z = \phi \qquad \text{on} \qquad z = 0, \tag{24}$$

$$\beta^2\phi_z + h_x\phi_x + h_y\phi_y = 0 \qquad \text{on} \qquad z = -h(x,y). \tag{25}$$

[1] It fails at the shoreline, where there is a boundary layer, see Section 5.

[2] The reason for using upper-case letters is to save the lower-case ones for the rescaled variables.

We seek solutions of (23)-(25) for large values of $\beta$.

Motivated by the constant-depth solution, we express $\phi$ in the form

$$\phi = A\cosh\left[k(z+h)\right]e^{i\beta S}. \tag{26}$$

Here $k = k(x,y)$, $S = S(x,y)$ and $A(x,y,z,\beta)$ are functions to be determined.

Plugging (26) into the system of equations (23)-(25) leads to

$$\beta^2\left((k^2-(\nabla S)^2)A\cosh\alpha + A_{zz}\cosh\alpha + 2kA_z\sinh\alpha\right) +$$

$$+i\beta\left((\nabla^2 S)A\cosh\alpha + 2\nabla S\cdot\nabla(A\cosh\alpha)\right) + \nabla^2(A\cosh\alpha) = 0, \tag{27}$$

$$A_z\cosh kh + kA\sinh kh = A\cosh kh \qquad z=0, \tag{28}$$

$$\beta^2 A_z + i\beta A\nabla h\cdot\nabla S + \nabla h\cdot\nabla A = 0 \qquad z=-h. \tag{29}$$

Here $\alpha = k(z+h)$ and $\nabla = (\partial/\partial x, \partial/\partial y)$.

Next we assume that $A$ admits the following asymptotic expansion for large $\beta$:

$$A(x,y,z,\beta) \sim A_0(x,y) + \sum_{n=1}^{\infty} A_n(x,y,z)/(i\beta)^n. \tag{30}$$

Again, motivated by the constant-depth case, we have assumed that the first term $A_0$ does not depend on the vertical coordinate. We also assume that termwise differentiation in (30) is allowed.

Inserting the asymptotic expansion (30) into (27)-(29), and equating coefficients of the corresponding powers of $\beta$, we obtain the following three systems of equations:

$$\begin{cases} (\nabla S)^2 = k^2 \\ (A_1)_{zz}\cosh\alpha + 2k(A_1)_z\sinh\alpha = 2\nabla S\cdot\nabla(A_0\cosh\alpha) + A_0\cosh\alpha\nabla^2 S \\ (A_n)_{zz}\cosh\alpha + 2k(A_n)_z\sinh\alpha = 2\nabla S\cdot\nabla(A_{n-1}\cosh\alpha) + A_{n-1}\cosh\alpha\nabla^2 S + \\ \qquad\qquad +\nabla^2(A_{n-2}\cosh\alpha) \qquad (n\geqq 2), \end{cases} \tag{31}$$

$$\begin{cases} k\tanh kh = 1 \\ (A_n)_z = 0 \quad \text{at} \quad z=0 \quad (n\geqq 1), \end{cases} \tag{32}$$

and, finally,

$$\begin{cases} (A_1)_z = A_0\nabla h\cdot\nabla S \quad \text{at} \quad z=-h \\ (A_n)_z = A_{n-1}\nabla h\cdot\nabla S + \nabla h\cdot\nabla A_{n-2} \quad \text{at} \quad z=-h \quad (n\geqq 2). \end{cases} \tag{33}$$

The first equation in (32) determines $k(x,y)$ as a function of the known depth $h(x,y)$. Then the eiconal equation $(\nabla S)^2 = k^2$ can be solved for $S(x,y)$ by the ray method discussed in the previous lectures.

In order to find the amplitude $A_0$, we use the identity

$$(A_n)_{zz}\cosh\alpha + 2k(A_n)_z\sinh\alpha = \frac{\left((A_n)_z\cosh^2\alpha\right)_z}{\cosh\alpha} \quad . \tag{34}$$

Inserting (34) into (31.2) we obtain

$$\left((A_1)_z\cosh^2\alpha\right)_z = (2\nabla S\cdot\nabla A_0 + A_0\nabla^2 S)\cosh^2\alpha + A_0\nabla S\cdot\nabla\cosh^2\alpha. \tag{35}$$

Now we integrate (35) from 0 to $z$, using the boundary conditions (32.2)

$$(A_1)_z\cosh^2\alpha = \frac{1}{2}\left(2\nabla S\cdot\nabla A_0 + A_0\nabla^2 S + A_0\nabla S\cdot\nabla\right)\left(k^{-1}[\sinh\alpha\cosh\alpha - \sinh kh\cosh kh] + y\right). \tag{36}$$

Solving (36) will give $A_1$ up to an additive function of $(x,y)$, if $A_0$ is known.

Next we set $z=-h$ in (36) and eliminate $(A_1)_z$, using the boundary condition (33.1). This leads to an equation for $A_0$

$$2A_0\nabla h\cdot\nabla S = -(2\nabla S\cdot\nabla A_0 + A_0\nabla^2 S)(\sinh^2 kh + h) + A_0\nabla S\cdot(\nabla h - \nabla\sinh^2 kh). \tag{37}$$

Equivalently, (37) can be written

$$\nabla S\cdot\nabla\left(A_0^2(\sinh^2 kh + h)\right) + \left(A_0^2(\sinh^2 kh + h)\right)\nabla^2 S = 0. \tag{38}$$

We note that $\nabla S\cdot\nabla = k(d/d\tau)$, where $\tau$ measures arc-length along a ray. Then the solution of (38) can be written in the form

$$A_0^2(\sinh^2 kh + h) = [A_0^2(\sinh^2 kh + h)]_{\tau_0}\exp\left(-\int_{\tau_0}^{\tau} k^{-1}\nabla^2 S d\tau\right). \tag{39}$$

In [4], Luneberg has shown that the exponential above is given by

$$\exp\left(-\int_{\tau_0}^{\tau} k^{-1}\nabla^2 S d\tau\right) = \frac{k(\tau_0)}{k(\tau)}\frac{da(\tau_0)}{da(\tau)}. \tag{40}$$

Here $da(\tau)$ is the width of an infinitesimally narrow strip of rays at $\tau$. Plugging (40) into (39), we finally get the following equation for $A_0$ along a ray:

$$A_0^2(\sinh^2 kh + h)k\,da = \text{const.} \tag{41}$$

Equation (41) simply expresses the fact that the energy flux is constant along a tube of rays. By using (41) for $A_0(x,y)$ in (26), we get the leading term in the asymptotic expansion of $\phi$ in water of variable depth.

The amplitude $A_0$ is infinite at the shoreline, where $h=0$. This means that there is a boundary layer at the shore. To analyze the solution in this layer, we use the shallow water theory, which is introduced in the next section.

# 5 Shallow water theory

The main reference for this section is Stoker's famous monograph [5]. For simplicity, we consider the 2-dimensional case with horizontal $x$-axis and vertical $z$-axis. The free surface is given by $z = \eta(x,t)$, while the bottom is $z = -h(x)$. The equation of continuity for the components $u(x,z,t)$ and $v(x,z,t)$ of the water velocity is

$$u_x + v_z = 0. \tag{42}$$

At the free surface we have both the kinematic condition

$$(\eta_t + u\eta_x - v)\,|_{z=\eta} = 0 \tag{43}$$

and the dynamic condition

$$p\,|_{z=\eta} = 0\,. \tag{44}$$

The bottom boundary condition is

$$(uh_x + v)\,|_{z=-h} = 0. \tag{45}$$

Integrating (42) gives

$$\int_{-h}^{\eta} u_x dz + v\,|_{-h}^{\eta} = 0. \tag{46}$$

Using the top and bottom boundary conditions in (46) leads to

$$\int_{-h}^{\eta} u_x dz + \eta_t + u\,|_{\eta} \cdot \eta_x + u\,|_{-h} \cdot h_x = 0\,. \tag{47}$$

We can rewrite (47) as

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u dz = -\eta_t\,. \tag{48}$$

Notice that up to this point no approximation has been introduced. The sole approximation of the shallow water theory is to ignore the vertical acceleration. This is assumed because the water is shallow. Hence the pressure is given as in hydrostatics, namely

$$p = g\rho(\eta - z). \tag{49}$$

Here $\rho$ is the water density and $g$ is the acceleration of gravity. Differentiating (49) with respect to $x$ gives

$$p_x = g\rho\eta_x. \tag{50}$$

Note that since $\eta_x$ is independent of $y$, so is $p_x$.

Next, we assume that $u$ is independent of $z$ at $t = 0$ (This is true if the water is initially at rest). This will imply that $u$ is independent of $z$ at all times, since its horizontal acceleration $\rho^{-1}p_x$ does not depend on $z$ either, as (50) shows. Then (48) becomes

$$[u(\eta + h)]_x = -\eta_t\,. \tag{51}$$

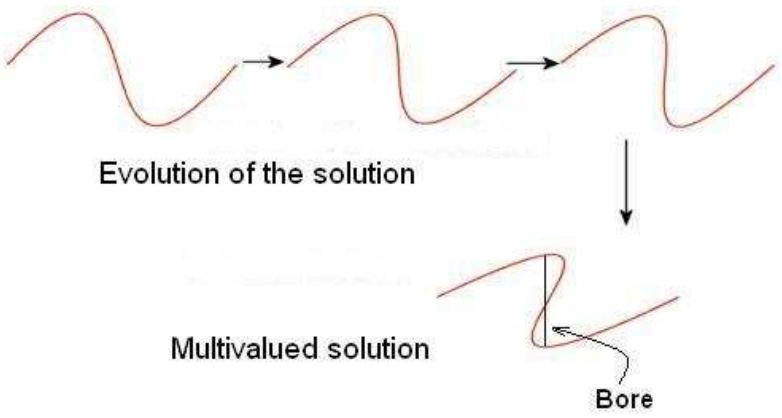


Figure 4: The "bore" formation.

The Eulerian equation of motion for $u(x,t)$ is

$$u_t + uu_x = -g\eta_x \,. \tag{52}$$

Eqs. (51) and (52) constitute the *(nonlinear) shallow water theory* for determining $u$ and $\eta$.

When $u$ and $\eta$ and their derivatives are small enough, we can linearize (51) and (52). This yields the *linear shallow water theory*, in which $u$ and $\eta$ satisfy

$$\begin{cases} u_t = -g\eta_x, \\ \\ (uh)_x = -\eta_t. \end{cases} \tag{53}$$

Eliminating $\eta$ from (53) yields

$$(hu)_{xx} - \frac{1}{gh}(hu)_{tt} = 0. \tag{54}$$

We have multiplied and divided by $h(x)$ to get the linear wave equation for the quantity $(hu)$. The propagation speed is $\sqrt{gh}$. If $h = \text{const}$, (54) is a linear wave equation just for $u$. The linear shallow water theory is used for the tides, where large wavelengths are involved.

The equations of the nonlinear shallow water theory admit an interesting analogy with the differential equations of gas dynamics. Let us define $\bar{\rho}$, the mass per unit area, by

$$\bar{\rho} = \rho(\eta + h). \tag{55}$$

¿From (55)

$$\bar{\rho}_t = \rho\eta_t. \tag{56}$$

Then the force per unit width $\bar{p} = \int_{-h}^{\eta} p\, dz$ is given by

$$\bar{p} = \frac{g\rho}{2}(\eta + h)^2 = \frac{g}{2\rho}\bar{\rho}^2. \tag{57}$$

Now we multiply (52) by $\rho(\eta + h)$ to get

$$\rho(\eta + h)(u_t + uu_x) = -g\rho(\eta + h)\eta_x\,. \tag{58}$$

Then using (55) and (57), we can write (58) as

$$\bar{\rho}(u_t + uu_x) = -\bar{p}_x + g\bar{\rho}h_x\,. \tag{59}$$

In terms of $\bar{\rho}$, we can write (51) as

$$(\bar{\rho}u)_x = -\bar{\rho}_t. \tag{60}$$

The equations (57), (59) and (60) are exactly the equations for the one dimensional flow of a compressible gas with adiabatic exponent $\gamma = 2$ and an external force $g\bar{\rho}h_x$. This force vanishes when the depth is uniform. The sound speed is

$$c = \sqrt{\frac{g\bar{\rho}}{\rho}} = \sqrt{g(\eta + h)}. \tag{61}$$

This is the speed of a small disturbance.

As in gas dynamics, the solutions of the nonlinear shallow water equation cease to be single valued at a finite time for certain initial conditions. They can be made single valued by introducing a discontinuity, called a "shock" in gas dynamics, and a "bore" in water waves. See Fig. 4. Such discontinuities can be observed in some rivers, and in kitchen sinks.

*Notes by Khachik Sargsyan and Walter Pauls.*

## References


[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover Publications, New York, 1965.

[2] J. B. Keller, *Surface waves on water of non-uniform depth*, J. Fluid Mech., 4 (1958).

[3] J. B. Keller and R. M. Lewis, *Asymptotic methods for PDEs*, Plenum Publishing, New York, 1995.

[4] R. K. Luneberg, *Mathematical theory of optics*, Brown University Lecture Notes, 1944.

[5] J. Stoker, *Water waves*, Wiley Interscience, 1957.