Lecture 3: Asymptotic Methods for the Reduced Wave Equation

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1 The Reduced Wave Equation

Let $v(t, \mathbf{x})$ satisfy the wave equation

$$\Delta v - \frac{1}{c^2(\mathbf{X})} \frac{\partial^2 v}{\partial t^2} = 0, \tag{1}$$

where $c(\mathbf{X})$ is the *propagation speed* at the point $\mathbf{X}$. Separate variables, letting $v(t, \mathbf{X}) = g(t) u(\mathbf{X})$. Then

$$c^2(\mathbf{X}) \frac{\Delta u(\mathbf{X})}{u(\mathbf{X})} = \frac{g''(t)}{g(t)} = -\omega^2. \tag{2}$$

So

$$g''(t) + \omega^2 g(t) = 0 \tag{3}$$

and

$$\Delta u + \frac{\omega^2}{c^2(\mathbf{X})} u = 0. \tag{4}$$

Here the constant ω is the *angular frequency*. Equation 4 is known as the *reduced wave equation* or the *Helmholtz equation*. Introduce a constant reference speed c_0, and define the *index of refraction* $n(\mathbf{X}) = c_0/c(\mathbf{X})$ and the *propagation constant* (or *wave number*) $k = \omega/c_0$. Then the reduced wave equation (4) becomes

$$\Delta u + k^2 n^2(\mathbf{X}) u = 0. \tag{5}$$

2 Leading order asymptotics

When $n(\mathbf{X})$ is constant, the reduced wave equation has the plane wave solution

$$u(\mathbf{X}, \mathbf{K}) = z(\mathbf{K}) e^{in\mathbf{K} \cdot \mathbf{X}}. \tag{6}$$

Here the *propagation vector* $\mathbf{K}$ is any constant vector such that $|\mathbf{K}| = k$, and the amplitude $z(\mathbf{K})$ is a constant. In the case of $n(\mathbf{X})$ not constant, the plane wave solution motivates looking for solutions to (5) of the form

$$u(\mathbf{X}) = z(\mathbf{X}, k) e^{iks(\mathbf{X})}. \tag{7}$$

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Here $z(\mathbf{X}, k)$ is the amplitude and $s(\mathbf{X})$ the phase. Substituting this form into the reduced wave equation (5) yields

$$-k^2 \left[(\nabla s)^2 - n^2 \right] z + 2ik\nabla s \cdot \nabla z + ikz\Delta s + \Delta z = 0. \tag{8}$$

We are interested in looking at the asymptotic behavior of solutions to the reduced wave equation (5) as $k \to \infty$. To explore this, we suppose $z(\mathbf{X}, k)$ has an *asymptotic expansion* of the form

$$z(\mathbf{X}, k) \sim \sum_{m=0}^{\infty} z_m(\mathbf{X})(ik)^{-m} = \sum_{m=-\infty}^{\infty} z_m(\mathbf{X})(ik)^{-m}, \ z_m = 0 \ \text{for} \ m = -1, -2, ... \tag{9}$$

Here $\sim$ denotes *asymptotic equality*. The asymptotic expansion above means that for each $n \geq 0$

$$z(\mathbf{X}, k) = \sum_{m=0}^{n} z_m(\mathbf{X})(ik)^{-m} + o\left(k^{-n}\right), \tag{10}$$

where the notation $o\left(k^{-n}\right)$ denotes a term for which $\lim_{k \to \infty} k^n |o\left(k^{-n}\right)| = 0$. Note that an asymptotic expansion may not converge! However, by truncation of the series we get an approximation with an error which tends to zero as $k \to 0$. Substituting the asymptotic expansion (9) for $z(\mathbf{X}, k)$ into (8) yields

$$\sum_m (ik)^{1-m} \left\{ \left[(\nabla s)^2 - n^2 \right] z_{m+1} + [2\nabla s \cdot \nabla z_m + z_m \Delta s] + \Delta z_{m-1} \right\} \sim 0. \tag{11}$$

The coefficient of each power of k must be zero. For $m = -1$ this gives

$$\left[(\nabla s)^2 - n^2 \right] z_0 = 0, \tag{12}$$

since $z_m = 0$ for $m = -1, -2, ...$. Assuming $z_0 \neq 0$, this implies the *eikonal equation* for the phase s,

$$(\nabla s)^2 = n^2(\mathbf{X}). \tag{13}$$

$m = 0$ yields the *transport equation* for the leading order amplitude z_0,

$$2\nabla s \cdot \nabla z_0 + z_0 \Delta s = 0. \tag{14}$$

$m = 1, 2, ...$ yield further transport equations for determining the other z_m. We shall concentrate on the leading order amplitude z_0 in what follows. The leading order solution $z_0(\mathbf{X}) e^{iks(\mathbf{X})}$ is known as the *geometrical optics field*.

3 Phase, Wavefronts, and Rays

Surfaces of constant phase, defined by $s(\mathbf{X}) = $ constant, are called *wavefronts*. Curves orthogonal to the wavefronts are called *rays* (or more generally, *characteristics*), and are used to solve for $s(\mathbf{X})$. We write the equation of a ray in terms of a parameter σ in the form

$$\mathbf{X} = (x_1, x_2, x_3) = \mathbf{X}(\sigma) \tag{15}$$

Orthogonality of the ray and the wavefronts implies

$$\frac{dx_j}{d\sigma} = \lambda \frac{\partial s}{\partial x_j}, \tag{16}$$

where $\lambda(\mathbf{X})$ is an arbitrary proportionality factor, and $j = 1, 2, 3$. Now, dividing the above expression by λ, differentiating with respect to σ, and using the summation convention we have that

$$\frac{d}{d\sigma} \left(\frac{1}{\lambda} \frac{dx_j}{d\sigma} \right) = \frac{d}{d\sigma} \left(\frac{\partial s}{\partial x_j} \right) = \frac{dx_i}{d\sigma} \frac{\partial^2 s}{\partial x_i \partial x_j} = \lambda \frac{\partial s}{\partial x_i} \frac{\partial^2 s}{\partial x_i \partial x_j} = \frac{\lambda}{2} \frac{\partial}{\partial x_j} \left(\frac{\partial s}{\partial x_i} \frac{\partial s}{\partial x_i} \right). \tag{17}$$

Then, using the eikonal equation (13) on the right hand side we have that

$$\frac{1}{\lambda} \frac{d}{d\sigma} \left(\frac{1}{\lambda} \frac{dx_j}{d\sigma} \right) = \frac{\partial}{\partial x_j} \left(\frac{n^2}{2} \right). \tag{18}$$

Furthermore, substituting the orthogonality equation (16) into the eikonal equation (13) yields

$$\frac{dx_j}{d\sigma} \frac{dx_j}{d\sigma} = \lambda^2 n^2. \tag{19}$$

The four equations given by (18) and (19) are known as the *ray equations*. The three equations given by (18) are second order ordinary differential equations for the rays $\mathbf{X}(\sigma)$, and (19) gives the variation of σ along the ray. The rays are determined solely by $n(\mathbf{X})$ once the initial values for (18) are specified and the arbitrary proportionality factor $\lambda(\mathbf{X})$ chosen.

Since λ is arbitrary, we may choose it as we please. When $\lambda = n^{-1}$ the ray equations become

$$n \frac{d}{d\sigma} \left(n \frac{dx_j}{d\sigma} \right) = \frac{\partial}{\partial x_j} \left(\frac{n^2}{2} \right), \tag{20}$$

$$\frac{dx_j}{d\sigma} \frac{dx_j}{d\sigma} = 1. \tag{21}$$

(21) implies that σ is simply the arc length along the ray. When $\lambda = 1$ with σ replaced by τ, the ray equations become

$$\frac{d^2 x_j}{d\tau^2} = \frac{\partial}{\partial x_j} \left(\frac{n^2}{2} \right), \tag{22}$$

$$\frac{dx_j}{d\tau} \frac{dx_j}{d\tau} = n^2. \tag{23}$$

(22) has a natural interpretation in terms of classical mechanics, with the left hand side being an acceleration and the right hand side being the gradient of a potential. Also, from (21) and (23) we can see σ is related to τ by

$$d\sigma = \sqrt{dx_j dx_j} = nd\tau. \tag{24}$$

$c_0 \tau$ is known as the *optical length* along a ray.

4 Ray solution

The eikonal equation (13) can be solved for the phase s. Using the orthogonality (16) we have for the derivative of s along a ray

$$\frac{d}{d\sigma}s[\mathbf{X}(\sigma)] = \frac{\partial s}{\partial x_j}\frac{dx_j}{d\sigma} = \lambda\frac{\partial s}{\partial x_j}\frac{\partial s}{\partial x_j} = \lambda n^2. \tag{25}$$

This can be integrated to give the solution for s

$$s[\mathbf{X}(\sigma)] = s[\mathbf{X}(\sigma_0)] + \int_{\sigma_0}^{\sigma} \lambda\left[\mathbf{X}(\sigma')\right] n^2\left[\mathbf{X}(\sigma')\right] d\sigma'. \tag{26}$$

The transport equation (14) can be solved for the leading order amplitude z_0. Again using orthogonality, we find that

$$\nabla s \cdot \nabla z_0 = \frac{\partial s}{\partial x_j}\frac{dz_0}{dx_j} = \frac{1}{\lambda}\frac{dx_j}{d\sigma}\frac{dz_0}{dx_j} = \frac{1}{\lambda}\frac{d}{d\sigma}z_0\left[\mathbf{X}(\sigma)\right]. \tag{27}$$

Thus the transport equation (14) becomes a first order ordinary differential equation along the ray

$$\frac{2}{\lambda}\frac{dz_0}{d\sigma} + z_0\Delta s = 0. \tag{28}$$

Given initial conditions (28) can also be integrated to solve for z_0. However, there is a more direct way to solve for z_0. Note that (14) implies

$$\nabla \cdot \left(z_0^2 \nabla s\right) = z_0\left(2\nabla z_0 \cdot \nabla s + z_0\Delta s\right) = 0. \tag{29}$$

Introduce a region R bounded by a tube of rays containing the given ray, and by two wavefronts $W(\sigma_0)$ and $W(\sigma)$ at the points σ_0 and σ of the given ray (Figure 1). Then the gradient of the phase, ∇s, is parallel to the sides of the tube and normal to its ends. Integrating (29) over R and using the divergence theorem yields

$$0 = \int_R \nabla \cdot \left(z_0^2 \nabla s\right) dV = \int_{W(\sigma)} z_0^2 \nabla s \cdot \mathbf{N} da - \int_{W(\sigma_0)} z_0^2 \nabla s \cdot \mathbf{N} da. \tag{30}$$
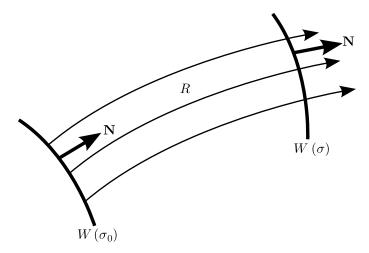
Here $\mathbf{N}$ is a unit vector orthogonal to the wavefront and da is an element of area on the wavefront. From the eikonal equation (13) we have that $\nabla s \cdot \mathbf{N} = n$. Then, by shrinking the tube of rays to the given ray we obtain the solution for z_0 from (30)

$$z_0^2(\sigma) n(\sigma) da(\sigma) = z_0^2(\sigma_0) n(\sigma_0) da(\sigma_0). \tag{31}$$

This can be written more conveniently in terms of the *expansion ratio* $\xi(\sigma)$ with respect to a reference point σ_1 on the ray, defined by

$$\xi(\sigma) = \frac{da(\sigma)}{da(\sigma_1)}. \tag{32}$$

Figure 1: Ray tube and wavefronts defining region R.

The expansion ratio measures the expansion of the cross-section of a tube of rays, and is simply the Jacobian of the mapping by rays of $W(\sigma_1)$ on $W(\sigma)$. (31) then becomes

$$z_0(\sigma) = z_0(\sigma_0) \left[\frac{\xi(\sigma_0) n(\sigma_0)}{\xi(\sigma) n(\sigma)} \right]^{1/2}. \tag{33}$$

Importantly we note that the amplitude $z_0(\sigma)$ varies inversely as the square root of $n\xi$ along a ray. Thus for n constant, as rays converge the amplitude z_0 increases, and as rays diverge z_0 decreases.

5 Case of Homogeneous Media

A homogeneous medium is defined as one where the propagation speed $c(\mathbf{X})$, and thus $n(\mathbf{X}) = c_0/c(\mathbf{X})$, are constants. If $\lambda = n^{-1}$, the ray equations (22) become

$$\frac{d^2 x_j}{d\tau^2} = \frac{\partial}{\partial x_j} \left(\frac{n^2}{2} \right) = 0. \tag{34}$$

(34) gives that the rays are straight lines. The equation (26) for the phase s becomes

$$s(\sigma) = s(\sigma_0) + n(\sigma - \sigma_0). \tag{35}$$

To determine the amplitude $z_0(\sigma)$ using (33), we need to determine the expansion ratio $\xi(\sigma)$. To calculate the expansion ratio, look at two intersecting rays which form an infinitesimal angle $d\theta_1$, as in Figure 2. Now take any two wavefronts $W(0)$ and $W(\sigma)$ intersecting these two rays. Denote the distance between them as σ, and the distance to the intersection point to be ρ_1, the radius of curvature of $W(0)$. Then we can calculate the infinitesimal area ratio to be

$$\xi(\sigma) = \frac{da(\sigma)}{da(0)} = \frac{(\rho_1 + \sigma)d\theta_1}{\rho_1 d\theta_1} = \frac{\rho_1 + \sigma}{\rho_1}, \tag{36}$$
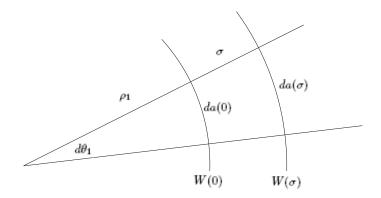
Figure 2: Calculating the expansion ratio.

$\rho_1 + \sigma$ is the radius of curvature of $W(\sigma)$. (36) states that the expansion ratio is just the ratio of the radii of curvature of the two wavefronts. (33) then becomes

$$z_0\left(\sigma\right) = z_0\left(\sigma_0\right)\left[\frac{\rho_1 + \sigma_0}{\rho_1 + \sigma}\right]^{1/2}. \tag{37}$$

To use this analysis works in three dimensions, we must take twp cross sections which slice the wavefront into its curves of maximal and minimal curvature. In 3-D this analysis gives the expansion ratio

$$\xi(\sigma) = \frac{(\rho_1 + \sigma)(\rho_2 + \sigma)}{\rho_1 \rho_2}. \tag{38}$$

The 3-D analogue of (37) is

$$z_0(\sigma) = z_0(\sigma_0)\left[\frac{(\rho_1 + \sigma_0)(\rho_2 + \sigma_0)}{(\rho_1 + \sigma)(\rho_2 + \sigma)}\right]^{1/2}. \tag{39}$$

6 An Initial Value Problem for the Eikonal Equation

Here we consider the solution of the eikonal equation with initial data $s(x)$ given at x on a manifold M, ie a point, line or surface. The eikonal equation is

$$(\nabla s)^2 = n^2. \tag{40}$$

To make $s(x)$ unique, we impose the condition that the solution is outgoing. Mathematically, this condition can be expressed as

$$\nabla s \cdot \mathbf{N} > 0, \qquad \text{with } \mathbf{N} = \text{The unit outward normal from } M. \tag{41}$$

Here M is the initial surface from which the solution is outgoing.

We will solve this problem using the method of characteristics. When the initial data is given at a point p, we can define the solution, using the previous theory, on each ray emanating from p.

When the initial data is given on a curve C we must determine the angle at which rays emanate. This can be done by parameterizing the curve by arc length, η. Now the initial condition $s|_C = s_0(\eta)$ with a parameterization $\mathbf{X}_0(\eta)$ of the curve C, yields the equation $s_0(\eta) = s(\mathbf{X}_0(\eta))$. Differentiation yields

$$\nabla s \cdot \frac{d\mathbf{X}_0}{d\eta} = \frac{ds_0}{d\eta}. \tag{42}$$

Now we use the vector identity $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \beta$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors and β is the angle between them. Then (42) gives that the angle $\beta(\eta)$ between the tangent vector $\frac{d\mathbf{X}_0}{d\eta}$ to the curve C and the direction ∇s of the ray is given by

$$\cos(\beta(\eta)) = \frac{1}{|\nabla s|} \frac{ds_0}{d\eta} = \frac{1}{n[\mathbf{X}_0(\eta)]} \frac{ds_0}{d\eta}. \tag{43}$$

We have now shown how to solve the initial value problem with initial data at a point p or on a curve C in an infinite domain. Similar analysis works when initial data is given on a surface. Next we will consider what happens when the domain has boundaries.

7 Reflection From a Boundary

In order to consider reflection from a boundary B, we must first prescribe a boundary condition. We will take the general impedance boundary condition, with impedance Z:

$$\frac{\partial u}{\partial \nu} + ikZ(\mathbf{X})u = 0, \qquad \mathbf{X} \text{ on } B, \ \nu = \nabla B. \tag{44}$$

Notice that the limits $Z \to 0$ and $Z \to \infty$ yield the simpler Neumann and Dirichlet boundary conditions. To satisfy the boundary condition (44), we must introduce a reflected wave, u^r in addition to the incident wave, u^i. Here we will try the same type of expansion for u^r that we have been using for u^i

$$u^r \sim e^{iks^r} \sum_{m=0}^{\infty} z_m^r (ik)^{-m}. \tag{45}$$

Now plugging $u = u^r + u^i$ into the boundary condition, we see immediately that the phases must match on the boundary for these waves to add to zero

$$s^r(\mathbf{X}) = s^i(\mathbf{X}), \qquad \mathbf{X} \text{ on } B. \tag{46}$$

Now collecting powers of (ik) we get the following equation for the leading order amplitudes

$$z_0^i \left(\frac{\partial s^i}{\partial \nu} + Z \right) + z_0^r \left(\frac{\partial s^r}{\partial \nu} + Z \right) = 0. \tag{47}$$

Thus

$$z_0^r = - \left(\frac{\frac{\partial s^i}{\partial \nu} + Z}{\frac{\partial s^r}{\partial \nu} + Z} \right) z_0^i. \tag{48}$$

We have found the phase and amplitude of the reflected wave on the boundary. Then we can use the previous method to construct the reflected wave.

8   Reflection From a Parabolic Cylinder

To illustrate how this method works we will consider the example of waves reflected by a parabolic cylinder. Physically we could envision this to be the example of waves hitting a vertical cliff with a parabolic profile when viewed from above. Here we will consider the
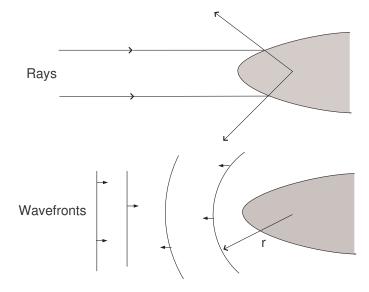

Figure 3: Reflection from a parabolic cylinder.

simple problem of an incoming plane wave e^{ikx}, with the x axis the axis of symmetry of the parabola. We will also take a uniform bottom, so that $n \equiv 1$, and the cliff face to be rigid, $\frac{\partial u}{\partial \nu} = 0$ for $\mathbf{x} \in B$. Now using the fact that parabolas focus all rays to a point, we see that the rays reflected from the boundary of the cliff will all emanate from the focus of the parabola. Thus the wavefronts will be circles centered at the focus of the parabola (Figure 3). If we define our coordinate system with origin at the focus of the parabola, we get the phase of the reflected wave to be

$$s(r) = s_0 + r. \tag{49}$$

Here r is the distance from the origin, as in polar coordinates. We can also determine the reflected amplitude. Here matching the incident and reflected amplitude on the boundary and using equation (37) give

$$z_0(r) = \sqrt{\frac{r_0(\theta)}{r}}. \tag{50}$$

Here $r = r_0(\theta)$ is the equation of the parabola. Thus we get the leading order solution to be

$$u = u^i + u^r \sim e^{ikx} + \sqrt{\frac{r_0(\theta)}{r}} e^{ik(s_0+r)}. \tag{51}$$

Similar analysis enables us to determine the higher order terms in u^r.

Notes by Ben Akers and John Rudge.

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