Lecture 1: Fluid Equations

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1 Euler Equations of Fluid Dynamics

We begin with some notation; x is position, t is time, g is the acceleration of gravity vector, u(x,t) is velocity, ρ(x,t) is density, p(x,t) is pressure. The Euler equations of fluid dynamics are:

$$\rho_t + \nabla \cdot (\rho u) = 0 \qquad \text{Mass conservation} \tag{1}$$

$$\rho[u_t + (u \cdot \nabla u)] = -\nabla p - \rho g \qquad \text{Momentum equation} \tag{2}$$

$$\rho = \rho(p, S, T) \qquad \text{Equation of state.} \tag{3}$$

Here T= temperature and S= salinity in the ocean or humidity in the atmosphere. (1)-(3) represent five equations for five unknown functions (ρ, u, p). We assume T, S are given.

Note
(a) (1)-(3) hold in air with the ideal gas law
(b) (1)-(3) hold in water with the equation of state for water
(c) instead of (1)-(3), the equations of solid mechanics (elasticity, plasticity) hold in the interior of the Earth
(d) certain conditions must hold at the interfaces between air and water, water and solid, where the solutions are discontinuous

1.1 Boundary conditions

To study the motion of the water and its upper surface we assume that:
(a) the location of the bottom surface is known:

$$z = -h(x, y, t)$$

(b) the pressure in the air at the top surface is known:

$$p^{air}(x, y, t) = p^{air}[x, y, \eta(x, y, t), t].$$

At the top surface $z = \eta(x, y, t)$; η is to be determined.

• Kinematic condition

At each free surface, the normal velocity $u \cdot \nu$ of the fluid in the direction of the unit normal ν is equal to V, the normal velocity of the surface.

Bottom: $\nu = \frac{(h_x, h_y, 1)}{\sqrt{h_x^2 + h_y^2 + 1}}, \qquad V = \frac{-h_t}{\sqrt{h_x^2 + h_y^2 + 1}}$

$$uh_x + vh_y + w = -h_t \qquad \text{at} \quad z = -h(x,y,t) \tag{4}$$

$$-u\eta_x - v\eta_y + w = \eta_t \qquad \text{at} \quad z = \eta(x,y,t) \tag{5}$$

• Dynamic condition

At $z = \eta(x,y,t)$, the pressure in the water equals the pressure in the air (ignoring surface tension):

$$p[x,y,\eta(x,y,t),t] = p^{air}(x,y,t) \qquad \text{at} \quad z = \eta(x,y,t) \tag{6}$$

Note
(a) one condition at the bottom where $h(x,y,t)$ is known.
(b) two conditions at top where $\eta(x,y,t)$ is unknown.

1.2 Initial conditions

$$\rho(x,0) = R(x) \tag{7}$$
$$u(x,0) = U(x) \tag{8}$$
$$\eta(x,y,0) = N(x,y) \tag{9}$$

To solve the initial boundary value problem for motion of the water we must find $\rho(x,t)$, $u(x,t)$, $p(x,t)$, $\eta(x,y,t)$ satisfying (1)-(9) given g, an equation of state, $T(x,t)$, $S(x,t)$, h(x,y,t), $p^{air}(x,y,t)$, $R(x)$, $U(x)$, $N(x,y)$.

1.3 Hydrostatic equilibrium in a horizontally stratified ocean

Suppose

$$p^{air} = constant = p_0, \quad h = h(x,y), \quad T = T(z), \quad S = S(z). \tag{10}$$

Then a solution to the Euler equations is:

$$u = 0, \quad \eta = 0, \quad \rho = \rho(z), \quad p = p(z). \tag{11}$$

Equation (1) is satisfied as are the boundary conditions (4), (5) and the x, y components of (2). The z component of (2) becomes:

$$\frac{dp}{dz} = -g\rho[p(z), T(z), S(z)], \qquad z \leq 0 \tag{12}$$

with boundary condition

$$p(0) = p_0. \tag{13}$$

The solution of (12) and (13) is called the hydrostatic pressure.

To solve for $p(z)$ we need an equation of state. An approximate equation of state for seawater is:

$$\rho = \rho_0 - \alpha T + \beta S, \qquad \rho_0,\ \alpha,\ \beta\ = constants > 0. \tag{14}$$

The solution of (12) and (13) is:

$$p(z) = p_0 - g\left[\rho_0 z + \alpha \int_0^z T(z')dz' - \beta \int_0^z S(z')dz'\right], \qquad z \leq 0. \tag{15}$$

The solution (11) solves the initial boundary value problem when the initial values (7)-(9) are also also given by (11). How do we find solutions for different initial and boundary data?

1.4 Perturbation method for solving problems

Consider an equation depending on a parameter ϵ

$$F(u, \epsilon) = 0. \tag{16}$$

The unknown u may be a function of x, t or a collection of functions like ρ, u, p, η, and F may be a set of equations like (1)-(6) with initial conditions (7)-(9). The solution of (16), $u(\epsilon)$, will depend upon ϵ. If it is a smooth regular function of ϵ, we can expand it in a Taylor series:

$$u(\epsilon) = u(0) + \epsilon\dot{u}(0) + \mathcal{O}(\epsilon^2) \tag{17}$$

$$\dot{u} = \frac{du(\epsilon)}{d\epsilon}|_{\epsilon=0}.$$

Suppose that we can solve (16) when $\epsilon = 0$, i.e. we know $u(0)$. Then we can differentiate (16) with respect to ϵ and set $\epsilon = 0$:

$$F_u[u(0), 0]\dot{u}(0) + F_\epsilon[u(0), 0] = 0. \tag{18}$$

If we solve (18) for $\dot{u}(0)$ then (17) will give $u(\epsilon)$ with an error $\mathcal{O}(\epsilon^2)$. A better approximation can be obtained by keeping the term $\frac{\epsilon^2}{2}\ddot{u}(0)$ and differentiating (16) twice to get an equation for $\ddot{u}(0)$ etc.

This is the regular perturbation method, and ϵ is the perturbation parameter. Equation (18) is called the variational equation of (16). It is linear in the unknown $\dot{u}(0)$, so it is called the linearized equation (linearized about $u(0)$). It can also be derived by substituting (17) into (16) and expanding in powers of ϵ.

Note that ϵ can be any parameter, or set of parameters, which occurs in one or more of the differential equations or in the boundary or initial conditions. The linear operator $F_u[u(0), 0]$ in (18) is always the same. Only the inhomogeneous term $F_\epsilon[u(0), 0]$ depends on what the parameter is.

Here are some examples of how we can perturb hydrostatic equilibrium.

Let p_0, $h_0(x, y)$, ρ_0 be the equilibrium solution, and then set

$$p^{air} = p_0 + \epsilon p_1(x, y, t) \qquad \text{atmospheric disturbance} \tag{19}$$

$$h = h_0(x, y) + \epsilon h_1(x, y, t) \qquad \text{bottom motion (earthquake, landslide)} \tag{20}$$

$$\eta(x, t) = 0 + \epsilon \eta_1(x, y) \qquad \text{initial elevation or depression} \tag{21}$$

$$u(x, \eta, 0) = 0 + \epsilon u_1(x, y) \qquad \text{initial motion} \tag{22}$$

$$\rho(x, \eta, t) = \rho_0(z) + \epsilon \rho_1(x, y, z) \qquad \text{initial density anomaly.} \tag{23}$$

To get the linearized equations, we linearize about the equlibrium solution. We start with (1):

$$\rho_t + \nabla(\rho u) = 0.$$

We differentiate with respect to ϵ

$$\dot{\rho}_t + \nabla \cdot (\dot{\rho} u + \rho \dot{u}) = 0.$$

We set $\epsilon = 0$ and then $u = 0$, $\rho = \rho_0$. Hence

$$\dot{\rho}_t + \nabla \cdot (\rho_0(z) \dot{u}) = 0. \tag{$\dot{1}$}$$

Similarily, linearizing (2)-(9) we get

$$\rho_0 \dot{u}_t = -\nabla \dot{p} - \dot{\rho} g \tag{$\dot{2}$}$$

$$\dot{\rho} = \rho_p(\rho_0, T, S)\dot{p} + \rho_T \dot{T} + \rho_S \dot{S} \tag{$\dot{3}$}$$

$$\dot{u} h_{0x} + \dot{v} h_{0y} + \dot{w} = -\dot{h}_t \quad \text{at} \quad z = -h_0(x, y) \quad \text{BC at bottom} \tag{$\dot{4}$}$$

$$\dot{w} = \dot{\eta}_t \quad \text{at} \quad z = 0 \quad \text{BC at top} \tag{$\dot{5}$}$$

$$\dot{p} + p_z \dot{\eta} = p_1 \quad \text{at} \quad z = 0 \quad \text{BC at top} \tag{$\dot{6}$}$$

$$\dot{\rho}(x, 0) = \rho_1 \quad \text{at} \quad t = 0 \tag{$\dot{7}$}$$

$$\dot{u}(x, 0) = u_1 \quad \text{at} \quad t = 0 \tag{$\dot{8}$}$$

$$\dot{\eta}(x, 0) = \eta_1 \quad \text{at} \quad t = 0. \tag{$\dot{9}$}$$

Equations ($\dot{1}$) - ($\dot{9}$) are the linear equations for $\dot{\rho}$, $\dot{u}$, $\dot{p}$, $\dot{\eta}$ and constitute an initial boundary value problem.

2 Fluids of Constant Density

Now let's suppose that ρ is a constant. Then we can cancel ρ from the continuity equation to get

$$\nabla \cdot u = 0.$$

This condition states that u is divergence free. The momentum equation becomes

$$u_t + (u \cdot \nabla)u = -\frac{1}{\rho}p - g.$$

Recall we also had an equation of state, relating the density to temperature and salinity. Here this equation is simply $\rho = \rho_0$, a constant.
Since the density is constant, (15) becomes

$$p(z) = p_0 - g\rho_0 z.$$

Linearizing the equations around a rest state yields the linear system

$$\begin{aligned} \nabla \cdot \dot{u} &= 0 \\ \dot{u}_t &= -\frac{1}{\rho}\nabla \dot{p}. \end{aligned}$$

At the beginning of this section we assumed the fluid was of constant density, and showed that this implies a divergence free velocity field. If we assume instead that when we follow a fluid element the density does not change (rather than assume that the whole fluid has constant density), we get

$$\frac{D\rho}{Dt} = \rho_t + u \cdot \nabla\rho = 0.$$

The conservation of mass equation is then $\rho(\nabla \cdot u) = 0$. Thus if the fluid is incompressible, this also implies a divergence free velocity field.

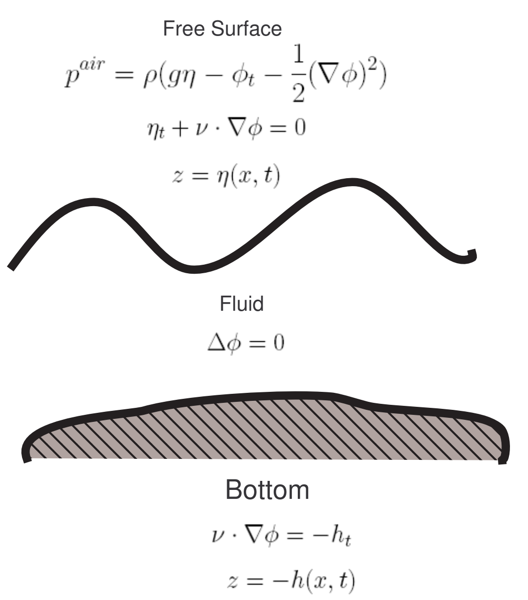

Figure 1: Irrotational flow of an incompressible ideal fluid

3 Irrotational Motion

Now let's introduce the quantity $\omega = \nabla \times u$, the vorticity of the fluid. If we assume that (or begin with) a fluid where this quantity is zero, then we say the motion is irrotational. When $u(x,t)$ is a curl-free vector field, then we can introduce a potential function

$$\phi(x,t) = \int_{x_0}^{x} u(s,t)ds.$$

Here this integral is evaluated along any curve C from x_0 to x. It is a calculus exercise to show that this potential is well defined only when we have an irrotational vector field (hint: use Stokes' Theorem). Notice now that $u = \nabla\phi$ so if an incompressible fluid is in irrotational motion, then conservation of mass gives

$$\nabla \cdot u = \nabla \cdot \nabla\phi = \Delta\phi = 0.$$

Thus ϕ solves Laplace's equation, or in other words, ϕ harmonic. Now consider the momentum equation

$$u_t + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nabla V.$$

Here V is a potential function for the force, f, acting on the fluid. Now we can plug in $u = \nabla\phi$, to get:

$$\nabla\phi_t + 1/2\nabla(\nabla\phi)^2 + \frac{1}{\rho}\nabla p - \nabla V = 0.$$

If ρ is a constant, we can integrate this equation, to get:

$$\phi_t + 1/2(\nabla\phi)^2 + \frac{1}{\rho}p - V = B(t).$$

This is called Bernoulli's Equation. Here $B(t)$ is an integration constant, which we can set to zero. If B was not zero then we could consider a new system where ϕ is redefined to be $\psi = \phi + \int_0^t B(s)ds$, and see that ψ solves Bernoulli's equation with zero right hand side. Since the gradient of ψ and ϕ agree, and it is this gradient we are interested in, we can safely set B to zero. Now we can solve for the velocity and pressure by solving for ϕ using Laplace's Equation, and then solving for p using Bernoulli's equation. The boundary value problem for ϕ and η is shown in Figure 1.

In Figure 1, we have a linear pde for ϕ with a linear boundary condition at the bottom boundary. On the surface, we have two nonlinear boundary conditions imposed at a location which depends upon the solution. In linearizing this system about a rest state, all the equations stay the same except those on the free surface. There we get

$$\begin{aligned} \dot\eta_t + \dot\nu \cdot \nabla\phi + \nu \cdot \nabla\dot\phi + (\nu \cdot \nabla\phi_z)\dot\eta &= 0 & \qquad z &= 0 \\ \dot{p}^{air} = \rho g\dot\eta - \dot\phi_t - \phi_{tz}\dot\eta &+ \frac{1}{2}\dot{(\nabla\phi)^2} & \qquad z &= 0. \end{aligned}$$

Since the rest state is $\phi = 0$ and $\eta = 0$, these linearized surface equations become

$$\begin{aligned} \dot\eta_t - \dot\phi_z &= 0 & \qquad z &= 0 \\ \dot\phi_t + g\dot\eta &= -\tfrac{1}{\rho}\dot{p}^{air} & \qquad z &= 0. \end{aligned}$$

Now we can differentiate the second equation and plug it into the first to get a single linear free surface condition

$$\dot{\phi}_{tt} + g\dot{\phi}_z = -\frac{1}{\rho}\dot{p}_t^{air}.$$

When p and ϕ are time harmonic with time dependence $\exp(-i\omega t)$ this becomes

$$-\frac{\omega^2}{g}\dot{\phi} - \frac{i\omega}{\rho g}\dot{p}^{air} - \dot{\phi}_z = 0 \qquad z = 0.$$

The equations for the fluid below the surface are

$$\begin{aligned} \dot{\phi}_n &= -\dot{h}_t && z = -h \\ \Delta\dot{\phi} &= 0 && -h \le z \le 0. \end{aligned}$$

These last three equations govern time harmonic small amplitude irrotational motion of an inviscid fluid of constant density.

Notes by Benjamin Akers and Tiffany Shaw.