GFD 2013 Lecture 2: Scaling Laws

Paul Linden; notes by Daniel Lecoanet and Karin van der Wiel

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1 Introduction

These notes focus on scaling laws describing the evolution of a gravity current. For the moment, we will restrict our attention to a rectangular, finite volume release in a channel (see Figure 1). Given the reduced gravity of the released material, initially given by

$$g_0' = g\frac{\rho_L - \rho_U}{\frac{1}{2}(\rho_L + \rho_U)}, \quad (1)$$

the initial length L_0, and height D of the released material, we would like to describe the evolution of the reduced gravity g', length L, and height h as a function of time. We assume throughout the height of the ambient medium H is very large, and that we are in the Boussinesq approximation.

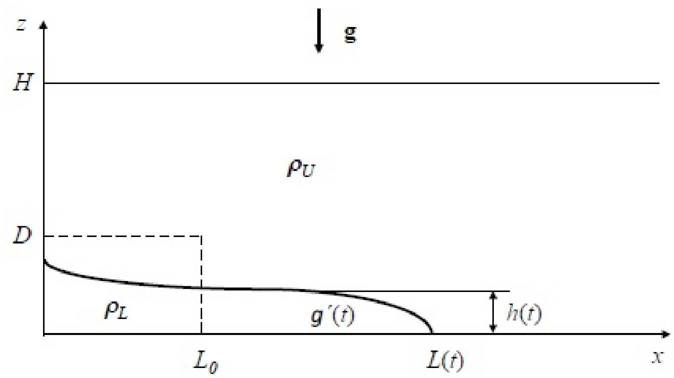

Figure 1: A schematic of the release of a finite volume of dense (ρ_L) fluid into a less dense (ρ_U) stationary environment of depth H. The dense fluid is initially held behind a lock gate at $x = L_0$, and the initial depth is D.

We will find that the gravity current evolution can be described by three different regimes: a constant velocity (or "slumping") regime, a "self-similar" regime, and a viscous regime. In both the constant velocity and self-similar regime, the buoyancy force is balanced by inertia. The initial wave propagation speed is $\sqrt{g_0'D}$, so it takes a time $\sim L_0/\sqrt{g_0'D}$ for the gravity current to realize it has finite extent. Before this time, the gravity current spreads out with a constant velocity, but afterwards (during the self-similar regime) it spread more slowly due to only having finite buoyancy. At late times, the velocity can become small enough that viscosity becomes important. In this viscous regime, the spreading becomes even slower than in the self-similar regime.

In these notes, we describe these phenomena using three techniques. First, we will use dimensional analysis, along with insights from experiments, to describe these three regimes. Next, we note that experiments had measured constant Froude numbers in gravity currents prior to the viscous regime, and use this to derive the constant velocity and self-similar regimes. Lastly, we will describe the force balances in the three regimes.

Although we focus on finite volume release in a channel, this analysis can be easily extended to axisymmetric flows, and constant flux releases. These effects will alter the scaling laws, and sometimes even change the order or presence of different regimes of gravity current evolution. Finally, we will briefly mention some of the experimental results which support or refute these simple scaling laws.

2 Dimensional Arguments

We will begin by describing the constant velocity regime. Before waves can propagate the length L_0 of the gravity current, the flow does not know it has finite length. Thus, the only dimensional quantities in the problem are the initial reduced gravity, g_0', the initial layer depth D, and time t. Then we must have that the velocity of the gravity current $U = \dot{L}$ is

$$U = \left(g_0' D\right)^{1/2} F f\left(t/T_a\right), \tag{2}$$

where

$$T_a = \sqrt{\frac{D}{g_0'}}, \tag{3}$$

is a free-fall time, F is a dimensionless number, and f is some function. The experimental observation that U initially stays about constant suggests that $f(x) \to 1$ for $x \gg 1$. Then, we can identify F as the Froude number. Thus, for $t \gg T_a$, we have

$$U = \left(g_0' D\right)^{1/2} F, \tag{4}$$

$$L = L_0 + \left(g_0' D\right)^{1/2} F t. \tag{5}$$

We expect this to be valid on the intermediate timescale

$$T_a \ll t \ll T_V, \tag{6}$$

where T_V is defined below.

On the timescale

$$T_V = \frac{L_0}{\sqrt{g_0' D}} \tag{7}$$

several things change. First, this is the timescale on which waves will propagate along the layer, allowing communication along the entire length of the gravity current, adding a new dimensional parameter L_0. Also, the length of the gravity current about doubles on this timescale. Whereas for $t \ll T_V$ we could approximate the gravity current as having height about D, for $t \gtrsim T_V$, the depth of the layer must change to conserve volume. By dimensional analysis, we now have that

$$U = \left(g_0' D\right)^{1/2} F f\left(t/T_a, t/T_V\right). \tag{8}$$

On timescales much longer than T_V (but much shorter than the viscous timescale T_ν defined below), we can posit that the only important dimensional quantities are the time and the total initial negative buoyancy,

$$B_0 = g_0' L_0 D. \tag{9}$$

Then, by dimensional analysis we have

$$U \sim \left(\frac{B_0}{t}\right)^{1/3} \tag{10}$$

$$L \sim B_0^{1/3} t^{2/3}. \tag{11}$$

On this intermediate timescale (between T_V and viscous timescale T_ν), the flow forgets its initial condition, i.e., this similarity solution is an attractor.

Finally, at very late times, viscosity becomes important. The viscous time T_ν is given by when the wave propagation time $L/\sqrt{g'h}$ equals the viscous time h^2/ν. Viscous evolution occurs for $t \gg T_\nu$. We assume that in the viscous regime g' does not change, i.e., stays equal to g_ν', so that the volume $V_\nu = hL$ stays constant. Thus, the viscous time is

$$T_\nu = \left(\frac{V_\nu^4}{g_\nu'^2 \nu^3}\right)^{1/7}. \tag{12}$$

It is also convenient to write this as

$$T_\nu = \frac{\nu L_\nu^2}{g_\nu' h_\nu^3}, \tag{13}$$

where $L_\nu = L(T_\nu)$ and $h_\nu = h(T_\nu)$, because we will later find that

$$t \sim \frac{\nu L^2}{g_\nu' h^3}, \tag{14}$$

although this cannot be derived via dimensional analysis.

3 Constant Froude Number

We will now exploit the experimental evidence that the Froude number stays about constant in the constant velocity and self-similar regimes. Thus, we assume that

$$F_h = \frac{U(t)}{\sqrt{g'(t)h(t)}} \tag{15}$$

stays constant. Furthermore, we assume that the buoyancy flux is constant,

$$g'(t)h(t)L(t) = c_B B_0, \tag{16}$$

where c_B is a geometrical factor which is equal to one for a rectangle. Implicit in this second expression is the assumption that g' is constant in space. Putting these together, we find

$$\frac{L(t)}{L_0} = \left[1 + \frac{3}{2}F_h\sqrt{c_B B_0}\frac{t}{T_V}\right]^{2/3}. \tag{17}$$

Now consider the limits of small or large t. If $t \ll T_V$, then we have

$$\frac{L}{L_0} \approx 1 + F_h c_B^{1/2}\frac{t}{T_V} \qquad\qquad t \ll T_V. \tag{18}$$

However, for large t, we have

$$\frac{L}{L_0} \approx \left[\frac{3}{2}F_h c_B^{1/2}\frac{t}{T_V}\right]^{2/3} \qquad\qquad t \gg T_V. \tag{19}$$

It is easy to check this is consistent with the results from the section of dimensional analysis.

4 Force Balance

If one considers the three forces that act on the fluid in the channel, expressions for the shifts between the three regimes will follow. Assume a channel where there is a constant source of fluid at $x = 0$, proportional to t^α. For $\alpha = 0$ this is the case of the finite volume release, $\alpha = 1$ describes the case of a constant volume flux into the channel (See Figure 2). Conservation of volume will give $hL \sim q_0 t^\alpha$ and the length of the current $L \sim Ut$. The total buoyancy is the reduced gravity times the input flux, $B = g'q$, with dimensions $[B] = L^3T^{-3}$.

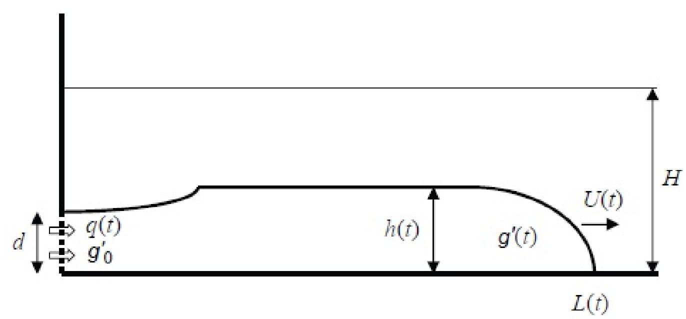

Figure 2: A schematic of a flux release in a channel of depth H. Fluid of reduced gravity g' is introduced at a rate of $q(t)$ at the end of the channel.

As mentioned, in both the constant velocity regime and the self-similar regime there is a balance between buoyancy and inertial forces. The force balance is thus between

$$F_b = \int \frac{\partial p}{\partial x}dV \sim \rho\frac{g'h}{L}q_0 t^\alpha \tag{20}$$

and

$$F_i = \rho \int u u_x dV \sim \rho \frac{L}{t^2} q_0 t^\alpha. \tag{21}$$

First consider the initial constant velocity regime, where $h \approx D$. In this case, equating F_b and F_i gives

$$L \sim \sqrt{g' D} t. \tag{22}$$

For the self-similar regime, we can take $hL \sim q_0 t^\alpha$, so the force balance implies

$$L \sim \left(g' q_0\right)^{1/3} t^{(\alpha+2)/3}. \tag{23}$$

For the finite volume release ($\alpha = 0$) $L \sim t^{2/3}$ which defines the self-similar regime. For the case of a constant volume flux into the channel ($\alpha = 1$) this describes the constant velocity regime as $L \sim t$.

Another possibility is a balance between buoyancy and viscous forces. With

$$F_\nu = \rho\nu \int \nabla^2 u dV \sim \rho\nu \frac{L}{th^2} q_0 t^\alpha \sim \rho\nu \frac{L^3}{q_0} L t^{-\alpha-1}, \tag{24}$$

this gives

$$L \sim \left(\frac{g' q_0^3}{\nu}\right)^{1/5} t^{(3\alpha+1)/5} \tag{25}$$

This describes the viscous regime for the finite volume release ($\alpha = 0$, $L \sim t^{1/5}$) and a decelerating flow for the constant volume flux case as $L \sim t^{4/5}$.

Next we find the time at which the system changes from the inertia–buoyancy balance to the viscosity–buoyancy balance. We divide the inertial force by the viscous force. Taking into account the different length scales for the two regimes (Equations 23 and 25), this gives:

$$\frac{F_i}{F_\nu} \sim \left[\left(\frac{q_0^4}{g'^2 \nu^3}\right)^3 t^{(4\alpha-7)/3}\right]^{2/3}. \tag{26}$$

So the time when all three forces are equal is

$$t_T = \left(\frac{q_0^4}{g'^2 \nu^3}\right)^{1/(4\alpha-7)} = J^{1/(4\alpha-7)}. \tag{27}$$

In Figure 3 the relative magnitude of inertial and viscous forces is plotted against time for four different cases. In the case where $J > 1$ the current starts in an inertial–buoyancy balance. If $\alpha > \alpha_c = 7/4$ (the upper curve), the increase in source flux maintains this balance for all times. When $\alpha < \alpha_c$ (the lower curve), the relative magnitude of the viscous force increases and the current enters a viscous–buoyancy balance for $t > t_1$. When $J < 1$ the current is initially in a viscous–buoyancy balance and remains there when $\alpha < \alpha_c$ (the lower curve), but becomes inertial if $\alpha > \alpha_c$ (the upper curve) for $t > t_2$.

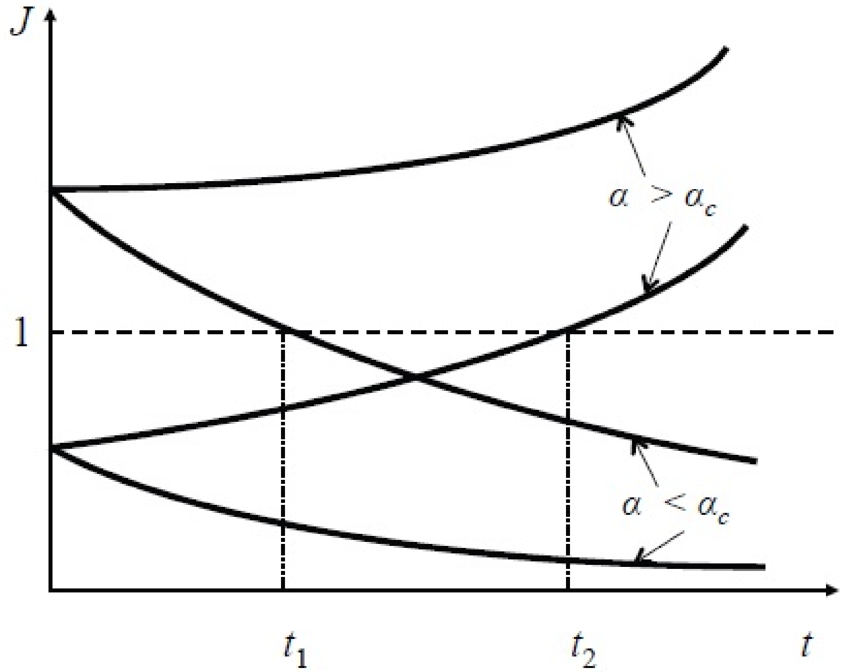

Figure 3: The relative magnitude of the inertia force F_i to the viscous force F_ν plotted against time.

5 Axisymmetric flow

For finite volume not in a channel one can consider the case of axisymmetric flow (See Figure 4). The initial buoyancy is $B_0 = g_0' D R_0^2 \pi$ with dimensions $L^4 T^{-2}$. It follows that

$$R(t) = B_0^{1/4} t^{1/2}, \tag{28}$$

which is the self-similar regime. Experiments have not given evidence for there being a constant velocity regime in the case of axisymmetric flow.

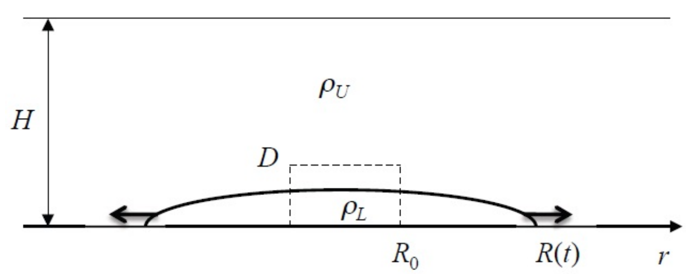

Figure 4: A schematic of the axisymmetric release of a finite volume of dense fluid into a less dense stationary environment of depth H. The dense fluid is initially held in a cylinder of radius R_0 and depth D.

6 Experimental results

Figure 5 shows the evolution in time of front positions from three lock release experiments (plotted dimensionlessly). One can see that at the start of all three experiments the front position scales linearly with time (i.e. $L \sim t$); this is the constant velocity regime. Two of the three experiments then go into a regime where $L \sim t^{2/3}$, the self-similar regime. Finally, all experiments end in the viscous regime as $L \sim t^{1/5}$.

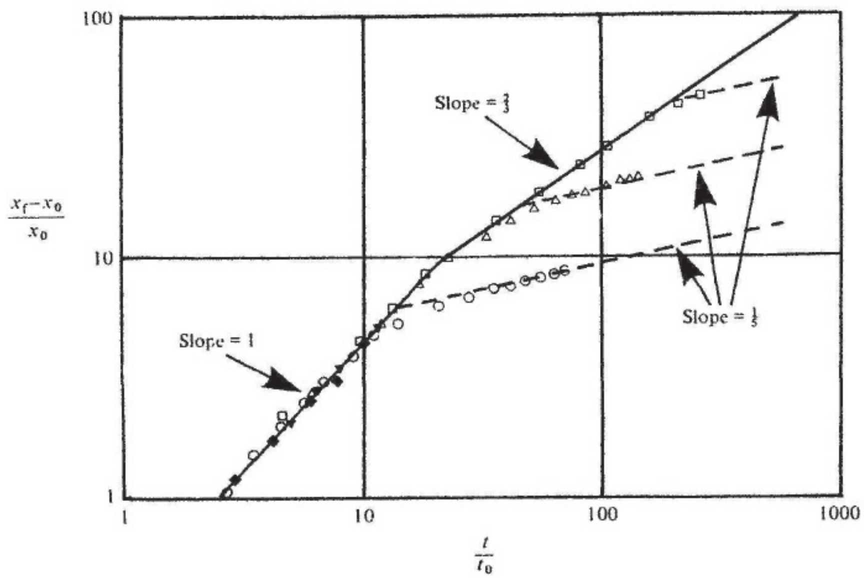

Figure 5: A log-log plot of dimensionless front positions against dimensionless time for 3 full-depth lock releases. Taken from [1].

References

[1] J. W. Rottman and J. E. Simpson, *Gravity currents produced by instantaneous release of a heavy fluid in a rectangular channel*, J. Fluid Mech., 135 (1983), pp. 95–110.