Problem Set 1

Grading strategy: we give 40 points for the first problem, and 15 points each for the remaining 4 problems.

1. **The age of the elements:**

^{235}U decays faster than ^{238}U due to its shorter half-life. The ratio of the two is subsequently decreasing with time. We have the two radioactive decay equations, given by

$$^{235}\mathrm{U} = {}^{235}\mathrm{U}_0 e^{-\lambda_{235}t}$$

and

$$^{238}\mathrm{U} = {}^{238}\mathrm{U}_0 e^{-\lambda_{238}t}$$

Dividing the first equation by the second, we now have

$$\left(\frac{^{235}\mathrm{U}}{^{238}\mathrm{U}}\right) = \left(\frac{^{235}\mathrm{U}_0}{^{238}\mathrm{U}_0}\right)\frac{e^{-\lambda_{235}t}}{e^{-\lambda_{238}t}} = \left(\frac{^{235}\mathrm{U}_0}{^{238}\mathrm{U}_0}\right)e^{-(\lambda_{235}-\lambda_{238})t}$$

or

$$R(t) = R_0 e^{-(\lambda_{235}-\lambda_{238})t}$$

Where R(t) is the ratio of ^{235}U to ^{238}U at some time t, and R_0 is the initial ratio.. This just looks like a regular radioactive decay equation (but is not identical). Thus we can take the natural log of both sides to get

$$\ln\left(\frac{R}{R_0}\right) = -(\lambda_{235}-\lambda_{238})t$$

or

$$t = \frac{\ln\left(\frac{R}{R_0}\right)}{(\lambda_{238}-\lambda_{235})}$$

Using the minimum, most likely, and maximum values for R_0 of 1.05, 1.35, and 1.65, a present day ratio of 7.257x10^{-2}, and decay constants of 9.849 x 10^{-10} and 1.551x10^{-10} y^{-1} for ^{235}U and ^{238}U, we obtain ages of 6.00, 6.30, and 6.54 Ga.

[20% for this]

We can do the same thing for the ^{232}Th and ^{238}U pair, using a decay constant for ^{232}Th of 4.948x10^{-11}, for starting ^{232}Th/^{238}U ratios of 1.45, 1.65, and 1.85 we get 9.61, 8.44, and 7.30 Ga for the maximum, most likely, and minimum ages. Note the larger spread in ages despite the smaller percentage uncertainty in the initial ratio. This is due to the longer half-lives.

[20% for this]

The main weaknesses of this second estimate are (a) that we are dealing with an elemental rather than isotopic ratio, which is easier to change by normal geological or cosmochemical processes (also note the considerable uncertainty and difficulty in ascertaining what the terrestrial Th/U ratio really is), and (b) the half-lives are comparable to or in the case of ^{232}Th longer than the time-scales measured, so the answer is more sensitive to the uncertainties in the ratios (both starting and finishing).

[20% for this]

We now consider the possibility that nucleosynthesis was likely an on-going process (witness the evidence of extinct short-lived radioactivities in meteoritic CAIs). We can then derive an equation for the end-member scenario of continuous nucleosynthesis. We don't know the absolute production rates, but the relative rates can be obtained by taking the steady-state situation where production is exactly balanced by decay. This can be stated in a differential equation that accounts for production and (radioactive) loss: $\frac{dN}{dt} = P - \lambda N$, whose solution for $t>t_0$ for the two isotopes of interest is

$$N_{235} = \frac{P_{235}}{\lambda_{235}}\left(1 - e^{-\lambda_{235}(t-t_0)}\right)$$

$$N_{238} = \frac{P_{238}}{\lambda_{238}}\left(1 - e^{-\lambda_{238}(t-t_0)}\right)$$

(also noting that $N = 0 \text{ for } t \le t_0$) and dividing one by the other:

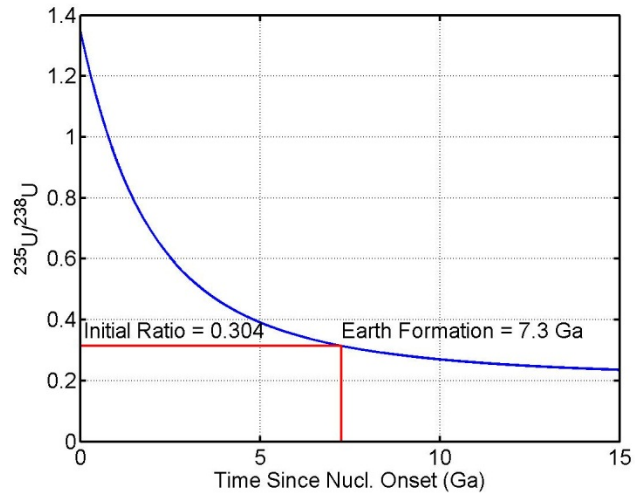
$$\frac{N_{235}}{N_{238}} = \frac{P_{235}\lambda_{238}\left(1 - e^{-\lambda_{235}\Delta t}\right)}{P_{238}\lambda_{235}\left(1 - e^{-\lambda_{238}\Delta t}\right)}$$

for $t>t_0$ of course. Here we've used the shorthand $t - t_0 = \Delta t$. Since we know the nucleosynthetic production ratio and the decay constants for these isotopes, we have a steady state ratio of

$$\frac{N_{235}}{N_{238}} = 0.213\frac{\left(1 - e^{-\lambda_{235}\Delta t}\right)}{\left(1 - e^{-\lambda_{238}\Delta t}\right)}$$

Given enough time, the ratio of ^{235}U/^{238}U will decrease from the original 1.35 to 0.213 because the exponential terms eventually drop to zero. If you compute the earth's starting ^{235}U/^{238}U ratio by taking the present day and "reverse decaying" by the age of the earth (t = 4.55 Ga) you get

$$\left(\frac{N_{235}}{N_{238}}\right)_0 = \left(\frac{N_{235}}{N_{238}}\right)_{Today} \frac{e^{\lambda_{235}t}}{e^{\lambda_{238}t}} = 0.304$$

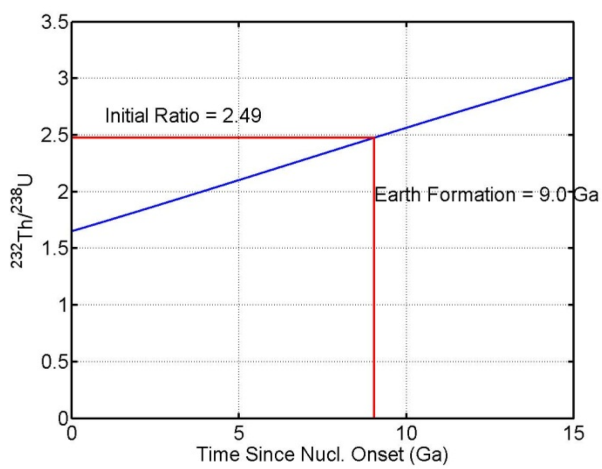


You can then graphically find the point where the evolutionary curve crosses the initial value like the graph on the right here, namely about 7.3 Ga. This would imply that nucleosynthesis, at least for our local galaxy began just a few billion years after the big bang.

[20% for this]

We can generate a similar plot for the evolution of $^{232}Th:^{238}U$. With the onset of nucleosynthesis, the starting ratio would 1.65, and given the equation

$$\frac{N_{232}}{N_{238}} = \frac{P_{232}\lambda_{238}\left(1-e^{-\lambda_{232}\Delta t}\right)}{P_{238}\lambda_{232}\left(1-e^{-\lambda_{238}\Delta t}\right)} = 5.172\frac{\left(1-e^{-\lambda_{232}\Delta t}\right)}{\left(1-e^{-\lambda_{238}\Delta t}\right)}$$



which would look like the figure to the right. Note that the plot looks close to linear: this is due to the very long half-life of Th, which is comparable to the age of the universe. Reading off the graph (or more precisely from the numbers themselves), we get the time lapse to be a little longer than the U isotope story. Given all the uncertainties and assumptions, they are in basic agreement.

I tend to believe these results more since nucleosynthesis has likely been going on a long time in our galaxy (there's other evidence for this). I favor the U-isotope results, because of their greater sensitivity (the short half-life of ^{235}U), their smaller mass difference, and the fact that we really don't know the earth's Th:U ratio very well. After all, they are different elements with different chemistries.

[20% for this]

2. **Build your own H-Bomb:**

The equations are:

$$
\begin{aligned}
&{}^{6}_{3}\mathrm{Li}+\mathrm{n} \rightarrow {}^{4}_{2}\mathrm{He}+{}^{3}_{1}\mathrm{H} \\
&{}^{7}_{3}\mathrm{Li}+\mathrm{n} \rightarrow {}^{4}_{2}\mathrm{He}+{}^{3}_{1}\mathrm{H}+\mathrm{n} \\
&{}^{3}_{1}\mathrm{H}+{}^{2}_{1}\mathrm{H} \rightarrow {}^{5}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He}+\mathrm{n}
\end{aligned}
$$

The isotopic masses are ^{6}Li = 6.015122, ^{7}Li = 7.016014, ^{2}H = 2.014102, ^{3}H = 3.016049, ^{4}He = 4.002603, n = 1.008665 g mol^{-1}. We can then calculate the mass deficit for each reaction (that is, the difference between the sum of the masses on the LHS and the RHS of each eqution), and hence the energy yields on a molar basis. Note that since the number of protons and neutrons on each sides of the equations remain the same, we don't have to include the masses of any electrons in the budgets.

$$
\begin{aligned}
&{}^{6}_{3}\mathrm{Li}+\mathrm{n} \rightarrow {}^{4}_{2}\mathrm{He}+{}^{3}_{1}\mathrm{H} \quad 6.015122 + 1.008665 - 4.002603 - 3.016049 = 0.005135\ \mathrm{g\ mol}^{-1} \\
&{}^{7}_{3}\mathrm{Li}+\mathrm{n} \rightarrow {}^{4}_{2}\mathrm{He}+{}^{3}_{1}\mathrm{H}+\mathrm{n} \quad 7.016014 + 1.008665 - 4.002603 - 3.016049 - 1.008665 = -0.002638\ \mathrm{g\ mol}^{-1} \\
&{}^{3}_{1}\mathrm{H}+{}^{2}_{1}\mathrm{H} \rightarrow {}^{5}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He}+\mathrm{n} \quad 3.016049 + 2.014102 - 4.002603 - 1.008665 = 0.018883\ \mathrm{g\ mol}^{-1}
\end{aligned}
$$

Notice that the second reaction is slightly endothermic (although it will proceed "in the heat of the moment"), but combined with the third reaction, will be net exothermic. So we have two net reactions occurring: the ^{6}Li^{1}H pair (1 & 3) at 0.024018 g mol^{-1} and the ^{7}Li^{2}H pair (2 & 3) at 0.016245 g mol^{-1}. The natural ^{6}Li/^{7}Li ratio is 0.0821 so assuming that the lithium is completely consumed, for every mole of deuterium consumed (reaction 3), there are 0.076 moles ^{6}Li and 0.924 moles of ^{7}Li consumed. So we have for every mole of LiD consumed an appropriately weighted net mass deficit of 0.076 x 0.024018 + 0.924 x 0.016245 = 0.016836 g mol^{-1}. Using E=mc^{-2} and being careful to convert the mass to Kg, we have the fusion energy yield for 1 mol of LiD

E = 1.684 x 10^{-5} kg x (3 x 10^{8} m s^{-1})^{2} = 1.516 x 10^{12} kg m^{2}s^{-2} = 1.516 x 10^{12} Joules.

Since we need 4.18 x 10^{15} joules, we have

Mass = (4.18 x 10^{15} joules)/(1.516 x 10^{12} joules/mol) = 2,757 moles

or (6.941 + 2.014)g x 2.757 x 10^{3} = 24.7 kg LiD.

Now the net mass conversion rate was 0.0168936/(6.941+2.014) = 0.19% which seems remarkably efficient to me. Put another way, a 1 MT warhead converts ~47 g to energy.

3. **The SN1987A calculation:**

We know that SN1987A was 168,000 light-years away, which corresponds to a distance of 1.59×10^{23} cm. The surface area of a sphere centered on SN1987A with this radius is $3.18 \times 10^{47} cm^2$, so the total efflux of neutrinos must have been 1.6×10^{58}. Since only 10% involved conversion of protons to neutrons, the number of protons converted to neutrons was 1.6×10^{57} (given the crudeness of the calculation, we ignore the fact that only 99.5% of the protons converted). Since the core was largely Ni-Fe, the average n:p ratio was roughly 1.1 to 1.15, so this corresponds to a core of 3.4×10^{57} nucleons, which is 5.7×10^{30} kg. One solar mass is 1.99×10^{30} kg, so this corresponds to 2.86 solar masses, which is above the Chandrasekhar limit and below that required for a black hole (just barely!), so a neutron star must have formed.

4. The meteorite problem:

1st part: You calculate atomic abundances by summing up all isotope ratios (e.g. Nuevo Laredo: 50.28+34.86+67.97+1 = 154.11), and then renormalizing the individual isotope ratios to this sum (i.e. 32.6% ^{206}Pb, 22.6% ^{207}Pb, 44.1% ^{208}Pb and 0.65% ^{204}Pb). The respective relative abundances for the Canyon Diablo meteorite are (18.8%, 20.6%, 58.6% and 2.0%).

2nd part: The atomic weight is simply the sum of the absolute atomic abundances multiplied by the atomic weights of each Pb isotope (note: these should be the precise atomic weights, not the nominal ones (e.g. 205.97, not 206, for ^{206}Pb). The atomic weight of Pb in Nuevo Laredo is 207.07 g per mole. The atomic weight of Pb in Canyon Diablo is 207.32 g per mole. Such a difference is measureable and led to the finding that the atomic weights of different lead samples on Earth are variable, even before the concept of isotopes was developed.

3rd part: The ^{206}Pb/^{207}Pb age of these five meteorites is determined iteratively from the slope of a type II regression (isochron). The slope is

$$m = {}^{235}U/{}^{238}U\ [(e^{\lambda 238t}-1) / (e^{\lambda 235t}-1)]$$

The first term is 1/137.88, the ratio of the atomic abundances of ^{235}U to ^{238}U. You should get an age slightly younger (~4.51 Gyr, depending on the type of regression you have been using) than the age of the solar system (4.57 Gyr). The data in the data table is the original data from the classic paper by Clair Patterson (1956) that for the first time determined the age of the Earth precisely, after he realized that a reasonable sample of terrestrial lead plots on the meteorite isochron. His “reasonable” average terrestrial lead came from a deep-sea sediment (he figured they’d represent well-mixed input from the continents). Within the error envelope this age, though less precise than modern re-assessments, is still correct.

5.The $^{234}U/^{238}U$ problem:

1st part: The first part was meant to be very simple. It takes at least 5 half-lives for an excess to decay. A more conservative number (depending on the precision of your measurement) were 10 half-lives. As the half-life of the daughter (^{234}U) matters (245,250 years), it will take between ~1.2 and ~2.4 million years for the present excess ^{234}U in seawater to decay away.

2nd part: You can solve this in different ways. First, consider a steady-state situation where there is no decay. Then the uranium isotope ratio of river input must equal the ratio in seawater (1.14). With decay, the unsupported ^{234}U (i.e. surplus over secular equilibrium, S.E., activity ratio of 1.00) that is decaying every year in seawater is equivalent to the flux that rivers need to supply extra ^{234}U to seawater. ^{238}U decay can be neglected because the decay constant is so small compared to the decay constant of ^{234}U. This extra flux of ^{234}U can be thought of as a radiogenic production term similar to that we got to know in lecture 3 – production of radiogenic isotopes by simple decay.

$D^* = N\,(e^{\lambda t}-1)$

The question is: how many unsupported ^{234}U atoms in seawater decay per year? This annual production term (t = 1 yr) can be translated into excess activity by taking the ratio of U input into the ocean by rivers to the U inventory of the ocean into account. This is equivalent to the residence time of U in seawater (τ_{usw}) which is given by the product of the concentration ratio of U in the ocean to that in rivers (12) and the residence time of water in the ocean relative to the input of riverwater into the ocean, i.e. how long it would take rivers to refill the ocean (30,000 yr). The excess ^{234}U activity (A) rivers need to bring to the ocean is then

$A_{4x} = (AR_{sw} - AR_{s.e.})\,(e^{\lambda t}-1)\,\tau_{usw}$

Then, the activity ratio of rivers needed to keep the ocean at steady state with respect to $A_4/A_{8r} = (A_{4sw} + A_{4x}) / A_{8sw}$

You can also set up a full mass balance for uranium atoms (N) in seawater, taking all inputs (fluxes [J] and $^{234}U/^{238}U$ ratios [R], outputs, production from ^{238}U ($\lambda_8 N$)and decay of ^{234}U into account.

$dNR_{sw}/dt = J_r\,R_r - J_o R_{sw} + \lambda_8 N - \lambda_4 N R_{sw}$

The second term, the flux of U out of the ocean (J_o, into marine sediments) can be neglected because we are considering the ocean at steady state (dN/dt = 0) and assume no fractionation of U isotopes during this process. Remember that activity is $A = \lambda N$. Multiplying by $\lambda_{234}/\lambda_{238}$ yields

$N\,dA_{sw}/dt = J_r\,(A_r - A_{sw}) + \lambda_4 N + \lambda_4 N A_{sw}$ or $dR_{sw}/dt = R_r/\tau_{usw} + \lambda_4 - R_{sw}\,(1/\tau_{usw} + \lambda_4)$

Where $R_r = (R_{sw}\,(1/\tau_{usw} + \lambda_4) - \lambda_4)\,\tau_{usw}$

The activity ratio of river input into seawater that is required to keep the seawater activity ratio at 1.14 is 1.28_3. Interestingly, this is higher than the globally averaged activity ratio of rivers of ~1.2. Where could that extra ^{234}U in seawater be coming from if it is not coming from dissolved uranium transported by rivers to the ocean?