# A Rough Recipe for the Energy Balance of Quasi-Steady Internal Lee Waves

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Abstract. Recent fine- and microstructure observations indicate enhanced finescale shear and strain in conjunction with bottom intensified turbulent dissipation above regions of rough bathymetry. Such observations implicate the bottom boundary as an energy source for the finescale internal wavefield. They also pose a question. Can the vertical profile of turbulent dissipation be predicted from a model of wave generation which serves as input to a model describing how internal wave energy is transferred to dissipation scales as waves propagate away from their source region? This paper attempts to address that question in the context of quasi-steady lee wave generation and dissipation.

# 1 Introduction

The intensity, spatial distribution and causal physical mechanisms of diapycnal mixing in the deep ocean have been the subject of much speculation. Advective heat budgets in semi-enclosed basins (e.g. Hogg et al., 1982) typically return estimates of  $K \sim 1 - 10 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ , similar to estimates of  $K \sim 1 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$  obtained from vertical advection/diffusion models (Wyrtki 1962, Munk 1966). These estimates, however, do not appear to be appropriate for the stratified upper ocean, for which a purposeful tracer release experiment (Ledwell et al. 1993) and microstructure measurements (Gregg 1987) suggest  $K \sim 0.1 \times 10^{-4} m^2 s^{-1}$ . Validation studies (Polzin et al. 1995, Gregg 1989) of internal wave-wave interaction models indicate that the empirical Garrett and Munk (GM, Garrett and Munk 1975, as modified by Cairns and Williams 1976) spectral description of the background internal wave state supports only weak ( $K < 0.1 \times 10^{-4} m^2 s^{-1}$ ) mixing which is independent of the background stratification rate,  $N^2$ .

In order for internal wave driven mixing to explain the results of advective heat budgets in abyssal basins, the abyssal internal wave spectrum needs to depart substantially from the GM specification. As abyssal internal wave observations were used to help construct the GM model, significant spatial variability of the abyssal wavefield is implied. Neither departures of the abyssal internal wavefield from GM nor diffusivities in excess of  $1 \times 10^{-4}$  m<sup>2</sup> s<sup>-1</sup> are apparent above smoothly sloping abyssal plains and Continental Rise regions (Toole *et al.* 1994, Kunze and Sanford 1996, Polzin *et al.* 1997). Significant departures are, however, found above rough bathymetry (Polzin *et al.* 1997, Polzin and Firing 1997).

What, then, are the mechanisms which support enhanced mixing above rough topography? Polzin *et* 

al. (1997) proposed that the enhanced mixing above rough topography in the Brazil Basin was associated with a direct conversion of barotropic tidal energy into baroclinic tidal energy having horizontal scales which were characteristic of the bottom topographic roughness ( $\lambda_h < 6 - 10$  km). Linear internal wave kinematics  $(\lambda_v \sim \lambda_h N/\omega)$  implies the baroclinic response to be band-limited to vertical wavelengths ( $\lambda_v < 600 -$ 1000 m). If so band-limited, an internal wavefield can exhibit both relatively modest velocities and significant shear  $(S^2 > N^2)$ , where  $S^2$  is the shear variance and  $N^2$  is the buoyancy frequency squared) over large vertical length scales. Both are essential considerations. Energetic arguments suggest baroclinic tidal velocities should not exceed barotropic, and barotropic tidal velocities in the open ocean typically are only  $0.02 - 0.03 \ m \ s^{-1}$ . Turbulent production from internal wave breaking is quadratically dependent upon the length scale over which  $S^2 > N^2$  (Polzin, 1996). In the background wavefield,  $m^2 U^2 = S^2 = N^2$  for a vertical scale  $2\pi/\lambda_v = 1/m$  of 1 meter. If a single vertical scale characterizes both the shear and velocity fields of the baroclinic response, U/N = 20m for  $U = 0.02m \ s^{-1}$  and  $N = 0.001 \ s^{-1}$ . It is entirely possible to obtain turbulent dissipation rates that are two orders of magnitude larger than background values from the breaking of an internal tide having relatively modest velocities which are characteristic of the open ocean barotropic tidal velocity field. The trick is that the baroclinic response needs to have a small vertical scale. This small scale response is effectively and efficiently provided by flow over topographic roughness. Data obtained during an additional cruise to the Brazil Basin further supports this proposed mechanism (Ledwell et al. submitted). Depth-averaged dissipation data document a fortnightly modulation in the turbulence

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intensity that lags the amplitude of the barotropic tide by 1-2 days. The fortnightly modulation and small phase lag imply a spatially local balance between internal tide generation and dissipation, rather than generation of low modes at the shelf break of the Brazilian coast (Baines 1982) and subsequent propagation and scattering (Müller and Xu 1992) into smaller spatial scales above rough bathymetry.

Polzin and Firing (1997) infer depth averaged dissipation rates above rough topography on the Southeast Indian Ridge which have a similar enhancement as the data obtained above the Mid-Atlantic Ridge in the South Atlantic. They cite deep sub-inertial flows associated with the Antarctic Circumpolar Current as being responsible for the generation of small vertical scale internal lee waves. As with baroclinic tide generation, the issue is that wave energy resides at sufficiently small scales that waves dissipate near their source. How the magnitude and decay scale of turbulent production via wave breaking relates to the amplitude and spatial scales of the wavefield is defined by the competing and coupled effects of propagation in an inhomogeneous environment and nonlinear transfers. The amplitude and spatial scale of the wavefield can, in turn, be viewed as a product of wave generation at the bottom boundary.

The motivation for this study is to better define and understand those processes which contribute most significantly to the closure of the abyssal heat budget. The predisposition is to view generation of small vertical wavelength internal waves and their subsequent dissipation as the dominant process. The objective of this research is to develop quantitative, predictive models for the spatial evolution of the finescale wavefield and associated turbulent mixing. The problem of quasi-steady lee wave generation and dissipation is formulated in the following section and addressed below.

## 2 Problem statement

The turbulent fluxes associated with wave breaking can be viewed as the end result of a systematic, downscale transport of energy by adiabatic mechanisms at larger spatial scales. The downscale transport can result from a variety of mechanisms such as buoyancy scaling, wave-mean flow or wave-wave interactions. Wave-wave interactions are the most problematic of these. While the downscale transport of energy associated with either buoyancy or wave-mean flow interactions can be defined for a single internal wave, defining a transport model for fluxes associated with wave-wave interactions necessitates a statistical, spectral framework if the wavefield exhibits a broad-banded character in the vertical wavenumber domain.

A conceptual diagram of the problem appears in Figure 1. Internal waves are assumed to be generated at the bottom boundary (z = 0) by sub-inertial flow U(z) impinging upon topographic roughness. In view of the statistical framework, the bathymetry is described in terms of a 2-D spectrum H(k, l), where k and l are horizontal wavenumbers and the flow is assumed to be aligned in the 'k' direction. The sub-inertial flow is assumed here to be steady. A spectral model of wave generation is employed to specify the distribution of energy flux into the wavefield as a source function,  $S_{o}(k, l)$ , which can simply be viewed as a product of the bathymetric spectrum and a transfer function. A dispersion relation is then used to transfer the source function into the vertical wavenumber (m) and intrinsic frequency ( $\omega = \sigma - kU$ ) domain,  $S_o(m, \omega)$ . This source function then serves as a bottom boundary condition for an interior wavefield having both upward  $(E^+)$  and downward  $(E^-)$  propagating components, Section 8.

Evaluating the vertical evolution of the wavefield requires defining the relative effects of vertical propagation and transport in a vertical wavenumber (m), intrinsic frequency  $(\omega)$ , height above boundary (z) coordinate system. Here, transport is used to denote transfers of energy (or wave action) in the vertical wavenumber (F) and intrinsic frequency (G) domain associated with buoyancy scaling, wave-wave and wave-mean flow interactions. An equation governing the evolution of the wave spectrum is derived in Section 3, and transport models are discussed in Section 4.

Solutions for the internal wave spectrum are determined by solving the wavefield evolution equation subject to appropriate boundary conditions on  $E^+$  and  $E^-$ . In Section 6 a radiation condition is used. Solutions described in Section 7 assume an upward (downward) propagating wavefield reflects from the surface, z = H (and bottom, z = 0). Surface generation and scattering from non-uniform bathymetry or reflection from a sloping boundary are neglected. Complicating the analysis is the potential need to incorporate the interaction between upward  $[E^+(m, \omega, z)]$  and downward  $[E^{-}(m,\omega,z)]$  propagating waves. Direct transfers of energy between upward and downward propagating waves are ignored here. The two components of the wavefield are assumed to be simply coupled by the dependence of downscale transfers F upon the total wavefield,  $E^+ + E^-$ . Finally, estimates of turbulent fluxes can be obtained by evaluating the transport F using the solutions to the wavefield evolution equation.

#### **3** Wavefield Evolution Equation

The intent of this section is to derive an equation from which the relative effects of of propagation and transport on the amplitude of the wavefield can be assessed. The wavefield evolution equation represents a wave action conservation statement (e.g. Bretherton and Garrett 1968) formulated in the frequency, verti-



Figure 1. A conceptual process diagram. The shape of the solution in the vertical wavenumber domain is depicted at the center of the figure. Propagation of this spectrum into a region of increasing N(z) or decreasing U(z) implies a transport of energy to smaller scales depicted by the notation 'WKB'. Transport associated with wave-wave interactions simply results in a decrease of the amplitude of the spectrum with height. The evolution of the spectrum in this case is denoted by 'w-w'. See text for description of the other variables.

cal wavenumber domain. <sup>1</sup> It is the wave stress, rather than energy, which is conserved by waves propagating in a spatially varying background flow (Bretherton 1966, Jones 1967). The interaction of the wave stress with the background shear implies a transfer of energy between the two (Garrett 1968). The stress is then expressed as wave action. The hydrostatic approximation is assumed throughout as the transport models for buoyancy scaling and wave-mean flow interactions are based upon the WKB approximation.

Consider a point  $(m_2, \omega_2, z_2)$  in vertical wavenumber (m), frequency  $(\omega)$ , and vertical coordinate (z)space, Figure 2. The action density in the volume defined by the line segments  $\Delta m = m_3 - m_1$ ,  $\Delta \omega = \omega_3 - \omega_1$ ,  $\Delta z = z_3 - z_1$  is  $\Delta m \Delta \omega \Delta z \ E(m_2, \omega_2, z_2, t)/\omega_2$ , where  $E^{\pm}(m, \omega, z, t)$  is the vertical wavenumberfrequency energy density of either the upward (+)or downward (-) propagating wavefield, and  $\Delta m$ ,  $\Delta \omega$  and  $\Delta z$  are assumed to be small. The time rate of change of action density in the area,  $\Delta m \Delta \omega \Delta z \partial [E^{\pm}(m_2, \omega_2, z_2, t)/\omega_2] \partial t$ , is balanced by vertical fluxes of action through the surfaces defined by  $z = z_1$  and  $z = z_3$ ,  $[C_{gz}(m, \omega)E^{\pm}(m, \omega, z, t)/\omega]\Delta m \Delta \omega$ , where  $C_{gz} = \mp (\omega^2 - f^2)/\omega m$  is the vertical group velocity]; down-scale spectral transfers  $F^{\pm}(m, \omega, z, t)\Delta z \Delta \omega$ of action through the surfaces at  $m = m_1$  and  $m = m_3$ ; and spectral transfers in the frequency domain across  $\omega = \omega_1$  and  $\omega = \omega_3$ ,  $G^{\pm}(m, \omega, z, t)\Delta m \Delta z$ :

$$\Delta m \Delta \omega \Delta z \frac{\partial [E^{\pm}(m_2, \omega_2, z_2, t)/\omega_2]}{\partial t}$$
  
+  $\Delta m \Delta \omega C_{gz}(m_2, \omega_2, z_1) E^{\pm}(m_2, \omega_2, z_1, t)/\omega_2$   
-  $\Delta m \Delta \omega C_{gz}(m_2, \omega_2, z_3) E^{\pm}(m_2, \omega_2, z_3, t)/\omega_2$   
+  $\Delta \omega \Delta z F^{\pm}(m_1, \omega_2, z_2, t) - \Delta \omega \Delta z F^{\pm}(m_3, \omega_2, z_2, t)$   
+  $\Delta m \Delta z G^{\pm}(m_2, \omega_1, z_2, t) - \Delta m \Delta z G^{\pm}(m_2, \omega_3, z_2, t) = 0$   
(1)

Dividing by  $\Delta m \Delta \omega \Delta z$  and taking the limit as  $\Delta m$ ,  $\Delta \omega$  and  $\Delta z$  approach zero results in:

$$\frac{\partial [E^{\pm}/\omega]}{\partial t} \pm \frac{\partial [C_{gz}E^{\pm}/\omega]}{\partial z} + \frac{\partial F^{\pm}}{\partial m} + \frac{\partial G^{\pm}}{\partial \omega} = 0, \quad (2)$$

where  $C_{gz}$  has been assumed to be positive definite.

This equation defines the evolution of the vertical wavenumber-frequency action (and energy) density as a function of vertical coordinate and time. It has been

 $<sup>{}^{1}</sup>E^{\pm}(m,\omega,z)$  is the vertical wavenumber-frequency energy density with direction of energy propagation denoted by either + or –. The notation  $E^{\pm}(m,z)$  [or  $F^{\pm}(m,z)$ ] denotes integration over the frequency domain,  $E^{\pm}(m,z) = \int E^{\pm}(m,\omega,z)d\omega$ . For narrow frequency band solutions discussed below, this distinction is superfluous and the notation is simplified by dropping the explicit  $\omega$  dependence. Likewise, the absence of the  $(\pm)$  superscript denotes summation over both upward and downward propagating wavefields.



Figure 2. The energy balance for the vertical wavenumber-frequency energy density spectrum,  $E(m, \omega, z)$ , at vertical wavenumber  $m_2$ , frequency  $\omega_2$ and vertical coordinate  $z_2$ . The vertical flux of energy is  $C_{gz}E$ . Transports of energy to small scales are represented by F. Transports of energy in the frequency domain, G, are not depicted.

assumed that there is no local production or dissipation of action in the volume  $\Delta m \Delta z \Delta \omega$ . This effectively eliminates the direct exchange of energy between upward  $E^+$  and downward  $E^-$  propagating waves. The vertical wavenumber spectrum would seem to be illdefined because of vertical inhomogeneity implied by the z-dependence, but it is a well defined construct if one can invoke horizontal homogeneity and a dispersion relation. The need for a plane of statistical homogeneity currently appears to be one of the limitations of the theory. The theory, however, does not demand that the plane of homogeneity be vertical, and in that vein the description of wavefield evolution away from a planar slope is possible. But the description of the evolution of a wavefield from a point source, such as the offshore evolution of a baroclinic tide from the shelf break, is problematic.

# 4 Transport Models

The flux  $F(m, \omega)$  represents the transfer of wave action in vertical wavenumber space associated with a variety of physical mechanisms: wave-wave interactions, buoyancy scaling and wave-mean flow interactions. These mechanisms are dealt with successively in the following three subsections.

#### a. Wave-Wave Interactions

This sub-section is dedicated to defining an analytical representation for  $F^{\pm}(m,\omega)$  based upon the ray-tracing model of Henyey *et al.* 1986 (hereafter HWF). In the context of that model, the flux may be represented as:

$$F^{\pm}(m,\omega) = < \frac{E^{\pm}(m,\omega)}{\omega} \frac{d(m)}{dt} >$$
(3) with

$$\frac{dm}{dt} = -(kU_z + lV_z) \; .$$

The factors k and l represent horizontal wavenumbers of a test wave packet with vertical wavenumber m propagating in a time dependent background having larger vertical scales and with vertical shear  $(U_z, V_z)$ . The background in HWF was a stochastic representation of the GM model. HWF assume no correlation between the energy density and time rate of change of vertical wavenumber in (3), so that

$$F^{\pm}(m,\omega) = \frac{E^{\pm}(m,\omega)}{2\omega} < k_h > S(m)C(m), \quad (4)$$

where  $k_h = (k^2 + l^2)^{1/2}$  and S(m) is the rms shear;  $S(m)^2 = 2 \int_0^m m'^2 E_k(m') dm'$  with  $E_k(m)$  the vertical wavenumber kinetic energy density spectrum. The factor C(m) is expressed in HWF as (1-r)/(1+r), where r is the ratio of energy flux to higher and lower vertical wavenumber. HWF use Monte-Carlo simulations to estimate  $r(m_u)$ , where  $m_u$  is a high wavenumber limit, beyond which internal waves are considered to break. In extrapolating their numerical results to higher values of  $E(m_u)$ , HWF suggest that C(m) is a function only of S(m). Polzin *et al.* (1995) point out that if an inertial subrange exists, implying no convergence of energy flux in vertical wavenumber space,  $\int F(m,\omega)d\omega$ is independent of m and one can solve (4) for C(m)by integrating over  $\omega$  and specifying E(m) as the inertial subrange solution. Both HWF and McComas and Müller (1981b) ascribe inertial subrange behavior to the GM spectrum, so that with an  $m^{-2}$  dependence for  $E(m), C(m) \sim m^{1/2} \sim S(m)$ , and (4) may be rewritten as

$$F^{\pm}(m,\omega) = AmN^{-2}\phi(\omega)E^{\pm}(m,\omega)\int_0^m m'^2 E(m')dm'$$
(5)

with A = 0.10 and

$$\phi(\omega) = (\omega^2 + f^2)(\omega^2 - f^2)^{1/2}/\omega^3$$

In (5),  $\phi(\omega)$  accounts for the conversion from horizontal to vertical wavenumber and conversion from shear spectral density to energy density by invoking a linear dispersion relation. The specification of A = 0.10 renders (5) to be consistent with Polzin *et al.* (1995).

The expression (5) was utilized by Polzin *et al.* (1995) in a model/data validation study. When evaluated at  $m = m_c$ , where  $m_c$  is the wavenumber at which

$$S^{2}(m_{c}) = 2 \int_{0}^{m_{c}} {m'}^{2} E_{k}(m') dm' = 0.7N^{2} \qquad (6)$$

and with factors involving wave frequency estimated from the shear-strain ratio, (5) accurately predicts the rate of dissipation of turbulent kinetic energy  $\epsilon$  to within a factor of  $\pm 2$ , the approximate statistical uncertainty of the measurements.

The transport of wave action through the frequency domain associated with wave-wave interactions is assumed to be zero,

$$G^{\pm}(m,\omega) = 0. \tag{7}$$

#### b. Buoyancy Scaling

Variable stratification adds an additional complication. WKB scaling gives the change with N of vertical wavenumber for a single internal wave as (e.g. Leaman and Sanford, 1975):

$$m \cong \hat{m}N(z)/\hat{N},\tag{8}$$

where  $\hat{m}$  and  $\hat{N}$  are reference values of m and N. The change of vertical wavenumber implies a transport through the vertical wavenumber spectrum which needs to be accounted for in (2). The appropriate flux law is:

$$F^{\pm}(m,\omega) = \frac{E^{\pm}(m,\omega)}{\omega} \frac{\partial m}{\partial z} \frac{\partial z}{\partial t}$$
$$= \frac{\omega^2 - f^2}{\omega^2} E^{\pm}(m,\omega) \frac{\partial N}{\partial z} N^{-1}.$$
(9)

#### c. Wave-Mean Flow interactions

As with buoyancy scaling, internal wave propagation in a vertically inhomogeneous sub-inertial flow implies a transport. For internal waves propagating in a geostrophic background flow, invoking the WKB approximation returns the dispersion relation:

$$m \cong N(z)k_h/(\omega^2(z) - f^2)^{1/2},$$
 (10)

with

$$\omega(z) = \sigma - kU(z).$$

The derivation of (10) assumes the hydrostatic approximation,  $\zeta/f \ll U_z/N \ll k_h N/mf \sim O(1)$ , where  $\zeta$ is the relative vorticity (Polzin *et al.* 1996). The transport in the wavenumber domain is given by the product of the action and the time rate of change of vertical wavenumber for a single wave group:

$$F^{\pm}(m,\omega) = \frac{E^{\pm}(m,\omega)}{\omega} \frac{\partial m}{\partial \omega} \frac{\partial \omega}{\partial U} \frac{\partial U}{\partial z} \frac{\partial z}{\partial t}$$
$$= \frac{E^{\pm}(m,\omega)}{\omega} \frac{\partial \omega}{\partial U} \frac{\partial U}{\partial z}$$
(11)

The transport in the frequency domain is similarly:

$$G^{\pm}(m,\omega) = \frac{\partial\omega}{\partial U} \frac{\partial U}{\partial z} \frac{\partial z}{\partial t} \frac{E^{\pm}(m,\omega)}{\omega}.$$
 (12)

# **5** Approximations

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It is a rather simple affair to solve the system defined by (2), (5), (7), (9), (11) and (12) numerically, at least for a steady, unidirectional wavefield. If a numerical solution was one's intent, it is possible to define a more sophisticated and potentially realistic system of equations governing the spatial evolution of the internal wavefield.

For example, the spatial evolution of a wavefield may be influenced by the potential transfer of energy from waves propagating away from a source into those propagating towards the source. A representation of such transfers is given by the elastic scattering mechanism in the formalism of McComas and Müller (1981a):

$$\frac{\partial E^{\pm}(m,\omega_1)}{\partial t} = -\frac{\pi\omega_1 m^3}{N^2}$$
$$\epsilon [E^{\pm}(m,\omega_1) - E^{\mp}(m,\omega_1)] [E^{\pm}(2m,\omega_2) + E^{\mp}(2m,\omega_2)],$$

in which the hydrostatic, non-rotating limit has been invoked and  $\omega_1 > \omega_2$ . Representation of such phenomena within the eikonal framework is unclear: The transport associated with individual waves is a random variable and (5) is merely an expression for the mean transport in an isotropic wavefield. There is no gaurantee that (5) is an adequate representation for evaluating the spatial evolution of an anisotropic wavefield.

The transport (5) does not appear to be a complete representation of transports even within the eikonal model. Sun and Kunze (1999, accepted) suggest the need to incorporate a term proportional to the vertical divergence in the evolution of the test wave vertical wavenumber (3). They also indicate a need for relatively sophisticated filtering in the wavenumber domain in order to ensure validity of the methodology. If the intent was to determine numerical solutions to (2), such additional details could be incorporated into the transport terms.

The frequency domain transports associated with wave-wave interactions has been neglected (7). This is certainly an idealization. The transports associated with the GM spectrum are to lower frequency in the weak interaction approximation and to higher frequency in the eikonal model. Not much can be done about this issue unless the equilibrium frequency spectrum is defined and a rule is developed for how quickly perturbed spectra will relax. Implementing such a scheme begs the question of which, if either, of the two models is an accurate representation of wave-wave interactions.

Rather than add such complexity, the intent here is to simplify the transport terms such that (2) can be solved analytically. At this point in time, this implies:

$$\frac{\partial E}{\partial t} = 0, \tag{13}$$

$$\int_0^m m'^2 E(m') \, dm' \sim m^3 E(m) \text{ and} \qquad (14)$$

 $\omega = \hat{\omega} U(z) / \hat{U}, \tag{15}$ 

where  $\hat{\omega}$  and  $\hat{U}$  are reference values of  $\omega$  and U.

The approximation (14) states that the shear variance in the wavenumber band 0 < m' < m is dominated by contributions at  $m' \sim m$ . The relation (14) is exact if  $m^2 E(m)$  is independent of m, which is approximately true at high wavenumbers for the Garret and Munk spectrum. While this represents an approximation of the transport in (5), the reader should note that the author does not attach inordinate significance to the specific functional dependence of F upon m in (5), despite apparent agreement between (5) and dissipation data in the validation study of Polzin *et al.* (1995). Roughly the same degree of agreement was found between (5) and an equivalent representation of the transport under the weak interaction approximation, with the exception that the triad based transport estimate tended to overestimate the observed dissipation rates by a factor of 2-3. That estimate was based upon a scale argument. No numerical results exist to provide a more definitive result. It is an open question which, if either, of these two theories represents a better description of the transport.

With this question in mind, it is worth noting that the two theories share important similarities. The transport F depends upon buoyancy frequency as  $N^2$ and is quadratically nonlinear in amplitude,  $F \sim E^2$ . Both theories predict faster interaction rates for higher frequency waves. Both theories suggest the GM spectrum is in equilibrium with respect to the nonlinear interactions at high wavenumber, with the implication that spectra which are perturbed from the GM specification of  $E(m) \sim m^{-2}$  are relaxed back to that power law. The approximation of the shear variance in (14) retains these features common to both models and the estimate of transport at high wavenumber is not quantitatively altered if  $m_c \gg m_o$ .

The specification of the depth dependence of the intrinsic frequency in (15) is clearly an idealization. In the context of steady flow over topography, it describes the linear propagation of the largest scales in the problem. This approximation can not be construed to imply linear propagation of a steady wavefield at all scales, however. Physically, (15) states that the intrinsic frequency is independent of vertical scale, which would be true if the energy appearing at small scales obtained its characteristic frequency by transport from larger scales at that depth and the transport in the frequency domain associated with wave-wave interactions was negligible. Mathematically, the approximation (15) eliminates  $\omega$  as an independent variable. The parametric specification of  $\omega$  in terms of U(z) is accounted for by taking 'z' derivatives in the spatial flux divergence term.

Finally, (15) represents the wavefield as a narrow frequency band process. The existence of a broadfrequency band wavefield is fundamental to the triad description of wave-wave interactions and the importance of such a frequency spectrum has not been addressed within the eikonal framework. In terms of the dynamics of wave-wave interactions, it could matter greatly if the wavefield was narrow- or broad-banded. However, the problem of wave generation at the bottom boundary. be it either tidal or lee wave, characteristically involves a response at more than a single frequency (e.g. Bell 1975a,b). The idealization of narrow frequency band solutions is an *ad hoc* characterization of the generation process rather than an extrapolation of theoretical results for the flux representation which were based upon a broad-frequency band wavefield. The derivation above does not exclude the possibility of a broad-band frequency wavefield.

# 6 A Uni-directional Solution

With the approximations given in Section (5), the wavefield evolution equation for a uni-directional (upward propagating) wavefield becomes:

$$\frac{\partial}{\partial z} [[\frac{\omega^2 - f^2}{\omega^2 m}] E^+(m, z)] + \frac{\partial}{\partial m} [[[\frac{\omega^2 - f^2}{\omega^2} \frac{N_z}{N} - \frac{U_z}{U}] E^+(m, z)] + \frac{a(\omega^2 + f^2)(\omega^2 - f^2)^{1/2}}{N^2 \omega^3} m^4 E^{+2}(m, z)] = 0.$$
(16)

A solution to (16) is:

$$E^{+}(m,z) = b^{+}A^{+}(z)\frac{\omega^{2}}{\omega^{2} - f^{2}}\alpha(m,z)\left[1 - \frac{\alpha(m,z)}{2}\right]$$
(17)

with

$$\begin{split} \alpha(m,z) &= \frac{m_o^2 N^2 (\omega_o^2 - f^2)}{m^2 N_o^2 (\omega^2 - f^2)}, \\ A^+(z) &= \frac{1}{1 + b^+ \int_0^z \beta(z') dz'}, \text{ and} \\ \beta(z) &= 2a N^2 \frac{\omega(\omega^2 + f^2)}{(\omega^2 - f^2)^{3/2}} \frac{(\omega_o^2 - f^2)^2}{(\omega^2 - f^2)^2} \frac{m_o^4}{N_o^4}. \end{split}$$

The solution is characterized by an energy containing wavenumber of  $m = m_o N(\omega_o^2 - f^2)^{1/2}/N_o(\omega^2 - f^2)^{1/2}$ , which varies with height in accordance with WKB scaling. The constants  $N_o$  and  $U_o$  represent the N(0) and U(0), respectively. The action flux associated with this solution,  $\int_0^\infty C_{gz} E^+(m, z) \omega^{-1} dm$ , is independent of height in the absence of wave-wave interactions [i.e. a = 0]. The basic character of this solution in the absence of wave-mean flow interactions is described in Polzin (1999).

# 7 A Bi-directional Solution

The addition of an upper boundary results in separate equations for the upward and downward propagating wavefields. These are coupled through the wave-

#### wave interaction transports:

$$\begin{aligned} &\frac{\partial}{\partial z} [[\frac{\omega^2 - f^2}{\omega^2 m}] E^+(m, z)] + \frac{\partial}{\partial m} [[[\frac{\omega^2 - f^2}{\omega^2} \frac{N_z}{N} - \frac{U_z}{U}] E^+(m, z)] + \\ &\frac{a(\omega^2 + f^2)(\omega^2 - f^2)^{1/2}}{N^2 \omega^3} m^4 E^+(m, z) [E^+(m, z) + E^-(m, z)]] = 0 \\ &\text{and} \end{aligned}$$

$$\frac{\partial}{\partial z} [[\frac{\omega^2 - f^2}{\omega^2 m}] E^-(m, z)] + \frac{\partial}{\partial m} [[[\frac{\omega^2 - f^2}{\omega^2} \frac{N_z}{N} - \frac{U_z}{U}] E^-(m, z)] + \frac{a(\omega^2 + f^2)(\omega^2 - f^2)^{1/2}}{N^2 \omega^3} m^4 E^-(m, z) [E^+(m, z) + E^-(m, z)]] = 0$$
(18)

A solution can be found by specifying

$$\begin{split} E^+(m,z) &= b^+ A^+(z) \frac{\omega^2}{\omega^2 - f^2} \alpha(m,z) [1 - \frac{\alpha(m,z)}{2}] \text{ and} \\ E^-(m,z) &= b^- A^-(z) \frac{\omega^2}{\omega^2 - f^2} \alpha(m,z) [1 - \frac{\alpha(m,z)}{2}], \text{ which} \\ \text{reduces (18) to a system of ordinary differential equations,} \end{split}$$

$$b^{+}A_{z}^{+} + \beta(z)A^{+}b^{+}(A^{+}b^{+} + A^{-}b^{-}) = 0 \text{ and}$$
  
$$b^{-}A_{z}^{-} - \beta(z)A^{-}b^{-}(A^{+}b^{+} + A^{-}b^{-}) = 0.$$
(19)

The boundary conditions are applied to the difference between the upward and downward propagating spectra. Application of the boundary conditions is simplified by expressing (19) in terms of the sum  $(S = A^+b^+ + A^-b^-)$  and difference  $(D = A^+b^+ - A^-b^-)$  amplitude functions. The corresponding coupled equations are:

$$S_z + \beta(z)SD = 0 \text{ and}$$
$$D_z + \beta(z)S^2 = 0.$$
 (20)

The non-constant coefficient  $\beta(z)$  can be eliminated by defining a stretched coordinate  $q(z) = \int_0^z \beta(z') dz'$ . The resulting sum and difference equations can be combined to obtain a second order nonlinear equation for D:

$$D_{qq} + 2DD_q = 0. (21)$$

This equation can be solved by noting that (i) it is "exact", and integration returns a Ricatti equation, (ii) the Ricatti equation can be transformed into a Bernoulli equation with the substitution  $D(q) = D_2(q) + \gamma_1$ , where  $\gamma_1$  is a constant, and (iii) transforming the Bernoulli equation for  $D_2(q)$  into a first order, linear equation (e.g. Bender and Orzag 1978). After application of the upper boundary condition that  $D(q = q_H = \int_0^H \beta(z')dz') = 0$  [i.e. perfect reflection at the upper boundary],

$$D(q) = -\gamma tan[\gamma(q - q_H)], \text{ and}$$
  
 $S(q) = \gamma/cos[\gamma(q - q_H)].$  (22)

The coefficient  $\gamma$  is then determined by equating the difference spectrum with a source spectrum at the bottom boundary.

#### 8 Bottom Boundary Condition

The bottom boundary condition is viewed here to be set by the process of quasi-steady lee-wave generation. The scales of interest are typically small. A vertically propagating response is obtained only for intrinsic frequencies f < kU < N, implying horizontal wavelengths  $2\pi U/N < \lambda_h < 2\pi U/f$ , or 600 - 6000m for  $U = 0.1ms^{-1}$ ,  $f = 1 \times 10^{-4} s^{-1}$ , and  $N = 1 \times 10^{-3} s^{-1}$ . Faulting and vulcanism at mid-ocean ridge crests are the dominant sources of seafloor roughness having these scales. This roughness has been characterized in terms of a 2-D, anisotropic spectral representation, H(k,l)(Goff and Jordon 1988). Extant models of quasi-steady lee-wave generation (e.g. Bretherton 1969) can then be used with such 2-D representations of seafloor topography to define the vertical wavenumber, intrinsic frequency domain wave response, which can then be mapped onto the interior solution. This mapping process is not straightforward, as the solutions to the wave generation problem may not be expressible in terms of the solution to the nonlinear wave evolution equation. This relationship will be described in detail at a later time.

For now, however, it suffices to note the following: First, the internal wave dispersion relation is  $m = k_h (N^2 - \omega^2)^{1/2} / (\omega^2 - f^2)^{1/2}$ . In the case of steady flow normally incident upon bathymetry,  $\omega = -kU$ ,  $k_h = k$ , and the dispersion relation in the hydrostatic, non-rotating limit reduces to m = N/U. The internal wave response will exhibit a well defined peak at m = N/U under these conditions. An increased response at slightly higher vertical wavenumber is anticipated with the addition of rotation and non-normal incidence. Second, for flow normally incident upon 1-D topography, the vertical flux of energy associated with the internal wavefield is

$$E_{flux} = \frac{2U(0)}{\pi} \int_{f/U(0)}^{N(0)/U(0)} [N(0)^2 - k^2 U(0)^2]^{1/2} [k^2 U(0)^2 - f^2]^{1/2} H(k) dk$$
(23)

(e.g. Bell 1975a). The amplitude of the solution to the wavefield evolution equation can then be defined by equating  $E_{flux}$  with the vertical energy flux of the wavefield at the bottom boundary,

$$\int_{m_o/\sqrt{2}}^{\infty} C_{gz} [E^+(m,0) - E^-(m,z)] dm = D(0) \frac{\omega_o}{2} = E_{flux}$$
(24)

# 9 Comparison

Numerous full water depth lowered ADCP (LADCP) profiles of relative velocity have been collected coincident with CTD profiles along WOCE hydrographic lines. The velocity profiles, in principle, resolve oceanic currents having vertical wavelengths from full water



Figure 3. Bathymetry and depth-averaged current vectors along the I8S section line. The analysis focuses upon a group of 12 profiles centered about  $55^{\circ}$  S. These stations are denoted with bold current vectors. The bathymetric contour interval is 1000 m, with thick contours every 2000 m.

depth down to about 50 m (Kunze *et al.* in preparation), and the barotropic component can be estimated by a method similar to utilized with shipboard Dopplers (Fischer and Visbeck, 1993). These data afford the opportunity to investigate the spatial characteristics of the finescale internal wavefield and groundtruth the theoretical development presented in the previous sections. Application of finescale parameterizations permit corresponding estimates of the turbulent dissipation rate ( $\epsilon$ ) and diapycnal eddy diffusivity ( $K_{\rho}$ ).

Finestructure estimates were made with data from the I8S hydrographic section, Figure 3. The section extends from  $30^{\circ}$  S,  $90^{\circ}$  E on the Broken Plateau, across the SE Indian Ridge to  $64^{\circ}$  S,  $82^{\circ}$  E, south of the Kerguelen Plateau. This section cuts across the Antarctic Circumpolar Current (ACC) in region of particularly strong eddy energy and topographic influence. A group of 12 stations centered about 55° S is examined. Further discussion of the data appears in Polzin and Firing (1997).

Shear spectra from  $55^{\circ}$  S differ markedly from the background GM internal wavefield, Figure 4. Middepth shear spectra reveal (i) significantly larger high wavenumber spectral levels, (ii) excess clockwise (CW) versus counter-clockwise (CCW) phase rotation with depth and (iii) a low wavenumber roll-off at smaller vertical wavelengths than the GM model. Mid-depth



Figure 4. Buoyancy frequency normalized shear spectra. The thin solid line represents the observed LADCP spectrum from depths of 1000-3560m. The thin, short-dashed lines are the rotary (CW and CCW) components. The thick solid line depicts the LADCP shear spectrum after correction for the effects of spatial averaging. The GM spectrum is represented by the thin, long-dashed line. The thick, long-dashed lined is the analytical solution.

(1000–3500 m) spectral levels are 7–8 times larger than GM for vertical wavelengths 1000  $< \lambda_v < 100$  m. The GM model is vertically isotropic, implying no net vertical energy flux. The observed enhancement of CW phase rotation with depth implies an excess of upward energy propagation in the southern hemisphere for nearinertial internal waves (Leaman and Sanford, 1975). The GM shear spectrum is characterized by an increasing spectral level for vertical wavelengths larger than mode 3 and is independent of vertical wavelength at smaller scales. Buoyancy scaling and the observed N<sup>2</sup> profile imply mode 3 is equivalent to a vertical wavelength of 4000 m over the depth range in question. In contrast, the observed spectrum is bandwidth limited to vertical wavelengths smaller than 1000 m.

Estimates of turbulent dissipation and diffusivity were made with the finestructure data using (5) following the method outlined in Polzin *et al.*, 1995. In this observational approach, turbulent production is equated with the spectral energy transport at high vertical wavenumber,

$$\epsilon = (1 - R_f)F(m_c) \text{ and } K_{\rho} = R_f F(m_c)/N^2,$$
 (25)

where  $R_f \sim 0.2$  is the flux Richardson number which expresses the partitioning of turbulent production into potential energy fluxes and dissipation. Below 1000m depth, the average diapycnal eddy diffusivity is  $2 \times 10^{-4}$  m<sup>2</sup> s<sup>-1</sup>, 20 times larger than that for a GM



Figure 5. Vertical profiles of finestructure derived (thick) and theoretical (thin) diffusivity. The dashed line represents a region where no data are available and the diffusivity estimate has been interpolated. The thin horizontal line at the bottom of the finestructure profile represents the average bottom depth. No data are available in the bottom most 130m of each profile. Finestructure based diffusivity estimates were calculated in a height above bottom coordinate system in the deepest 1400m.

wavefield, Figure 5. Shear spectral levels and diffusivity estimates decay towards GM levels above 1000 m. The heightened diffusivity corresponds to shear spectral levels a factor of 5–8 above GM. The enhanced spectral levels, excess of CW variance and heightened levels of turbulent mixing inferred from (25) are interpreted as the product of quasi-steady internal lee wave generation associated with the flow of large depth-averaged velocities over rough bathymetry and the consequent breaking of the waves as they propagate away from the bottom boundary.

Vertical variability in turbulent production associated with wave breaking is defined by the competing and coupled effects of propagation in an inhomogeneous environment and nonlinear transports. Solutions to the wavefield evolution equation detailed above suggest that profiles of turbulent quantities can be estimated once a bouyancy profile N(z), a velocity profile U(z) and bottom boundary conditions on the energy containing scale  $m_o$ , intrinsic frequency  $\omega_o$ , and vertical flux of energy associated with the generation process,  $E_{flux}$ , are specified. The velocity and buoy-



Figure 6. Vertical profile of  $N^2$  defined as an average over the group of twelve stations. The thin line represents a polynomial fit and is used in the theoretical analysis.

ancy profiles are obtained by averaging the station data and fitting with low order polynomials, Figure 6 and 7. The latter three constants represent the output of a generation model. These constants depend quantitatively upon issues such as whether the bathymetry is characterized as either 1- or 2-D and the direction of the background current relative to the orientation of the bathymetry. Defining an adequate generation model to deal with such complexities and producing an entirely theoretical estimate for the diffusivity profile will be addressed at a later date. Here it is simply noted that a qualitatively robust feature of wave generation by steady flow over bathymetry represented by a red spectrum is a dominant response at a vertical scale slightly smaller than U/N. Given the smoothed velocity and buoyancy profiles, the model parameter  $m_o$  is identified as  $m_o = \frac{N}{U}(\omega^2 - f^2)^{1/2} = 0.0011 cpm$ , with  $\omega_o = 1.5f$ , from a fit to the observed mid-depth shear spectrum, Figure 4. The amplitude is set here by specifying the net vertical flux of energy at the bottom boundary as being proportional to the depth integrated dissipation rate inferred from the observed finestructure,  $E_{flux} = (3mW/m^2)/\rho$ . This value for the energy flux is somewhat smaller than would be anticipated on the basis of flow normally incident upon 1-D topography (23), where the the bathymetric spectrum is prescribed in terms of a 1-D representation of the 2-D model spectrum proposed by Goff and Jordan



Figure 7. Vertical profile of horizontal velocity U averaged over the group of twelve stations. The thin line represents a polynomial fit and is used in the theoretical analysis. The velocity data were analyzed in a depth coordinate system. The bottom (z = 0) is defined as the greatest depth for which data are available for eleven of the twelve stations.

(1988):  $H(k) \propto 1/(k_0^2 + k^2)^{-\nu}$ . Appropriate values for the South East Indian Ridge crest at 100° E are  $k_0 = 0.002 \text{ m}^{-1}$  and rms height of 60 m (Goff *et al.* 1997) with  $\nu = 1.45$  (John Goff, personal communication 1999). With a value of  $\nu = 1.5$ , (23) returns an estimate of  $E_{flux} = (11mW/m^2)/\rho$ . While there maybe some spatial variability in the spectral parameters and uncertainty in the adequacy of the generation model, it is reassuring that the theoretical estimate of  $E_{flux}$ is much larger than that identified from the finescale observations.

After applying the bottom boundary condition to determine the constant  $\gamma$ , the theoretical estimate of the diffusivity profile is simply,

$$K(z) = \lim_{m \to \infty} \frac{R_f \omega F(m, z)}{N^2}$$
(26)

The degree of agreement with the diffusivity profile inferred from the finestructure estimates is amazing, Figure 5.

# 10 Summary

The ultimate goal of this research is the specification of turbulent flux profiles in terms of limited inputs: a background velocity profile, a buoyancy profile and a spectrum of bottom topography. This problem has been reduced here to the specification of three constants: an energy containing scale, a frequency, and an energy flux. Future work will focus on an entirely theoretical prescription for these constants.

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