A Note on The Weakly Nonlinear Ekman Layer: Thickness and Flux

by

Joseph Pedlosky

Woods Hole Oceanographic Institution

Woods Hole, MA 02543

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e-mail: jpedlosky@whoi.edu

Abstract

The first order effects of nonlinearity on the thickness and frictionally driven flux in the Ekman layer are described for the case of an Ekman layer in on a solid, flat plate driven by an overlying geostrophic flow as well as the Ekman layer on a free surface driven by a wind stress in the presence of a deep geostrophic current. In both examples a fluid is homogeneous. Particular attention is paid to the effect of nonlinearity in determining the thickness of the Ekman layer in both cases.

An analytical expression for the Ekman layer thickness as a function of Rossby number is given when the Rossby number is small. The result is obtained by insisting that the perturbation expansion of the Ekman problem in powers of the Rossby number remains uniformly valid. There are two competing physical effects. The relative vorticity of the geostrophic currents tend to reduce the width of the layer but the vertical velocity induced in the layer can fatten or thin the layer depending on the sign of the vertical velocity.

The regularized expansion is shown to give, to lowest order, expressions for the flux in agreement with earlier calculations.

1. Introduction

The theory of the Ekman layer is central to Geophysical Fluid Dynamics and its applications to both oceanic and atmospheric phenomena, Pedlosky (1987). The fundamental theory is linear and so it is not surprising that much effort has gone into extending the theory into the nonlinear domain. Some of the early work is reviewed by Greenspan (1968). See also Benton et. al. (1964). More recent analysis can be found in the work of Niiler (1974), Brink (1997), Hart (2000) and Thomas and Rhines (2002). Illuminating as these studies are, what is lacking is a clear and simple analytical formulation of the alteration in the thickness of the Ekman layer as a consequence of non zero Rossby numbers, i.e. of non negligible nonlinearity. In this note such a prediction is given for the two classical Ekman layer problems: the frictional layer satisfying the no-slip condition beneath a geostrophic inviscid interior and the stress driven boundary layer on a free surface.

Since in linear theory the boundary layer thickness is $\delta = \left(2\nu / f \right)^{1/2}$, where ν is the kinematic viscosity (or its turbulent equivalent) and f is the planetary vorticity, one might expect that the first effect of nonlinearity would be to replace f with the total vorticity, $f + \zeta$ where ζ is the relative vorticity. A positive relative vorticity would tend to make the boundary layer thinner. However, in the presence of a convergent frictional flux in the boundary layer due to that relative vorticity and the consequent vertical velocity, the boundary layer would be stretched and fattened by the vertical advection. So, the two

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effects of the relative vorticity are competing. What is the result? That is the object of the present study.

In section two we formulate the problem for the Ekman layer that satisfies the no-slip condition on a flat plane beneath a geostrophic current. Section 3 discusses the perturbation expansion and the condition that it remain uniformly valid in space and derives the correction to the layer thickness by removing secular terms in the perturbation expansion. After their removal the next order corrections to the Ekman velocities are calculated and the order Rossby number correction to the Ekman layer flux and vertical pumping velocity is calculated. Earlier studies, such as Hart (1995, 1996), in addition to the studies already mentioned earlier, have generally employed regular asymptotic expansions without concern about the uniformity of convergence. Although this allows an accurate calculation of the *integrated* effects of the nonlinearity on, for example, the pumping, they cannot provide a clear picture of the role of the nonlinearity on the structure of the layer, in particular its thickness.

Section 4 describes a similar procedure for the stress driven problem. In the first problem it is the vertical velocity in the Ekman layer that is the dominant factor in altering the linear Ekman layer thickness. In the stress driven problem there is a competition between the vertical velocity produced by the wind stress curl and the relative vorticity of the underlying geostrophic current.

2. The Ekman layer beneath a geostrophic flow.

We consider the boundary layer on a flat plane with a geostrophic flow in the x direction far from the plate. The plate is at $z = 0$. The geostrophic flow is taken to be a function of y the coordinate across the current but, for simplicity, all variables are

assumed independent of the x direction. We non-dimensionalize horizontal scales with L, the characteristic scale of variation of the current, and choose the linear Ekman layer thickness $\left(2\nu/f\right)^{1/2}$ as the vertical scale. The velocities are scaled with the characteristic value, U_0, of the geostrophic current. The vertical coordinate is ζ, i.e. the vertical coordinate scaled by the Ekman layer thickness. It is useful to also consider a "slow" spatial variable related to ζ to account for the effects of nonlinearity. We define that coordinate as $Z = \varepsilon\,\zeta$. All the variables in the boundary layer are then considered to be functions of both ζ and Z so that the vertical derivatives transform as,

$$\frac{\partial}{\partial\zeta} \rightarrow \frac{\partial}{\partial\zeta} + \varepsilon\frac{\partial}{\partial Z} \tag{2.1}$$

The steady equations of motion then become, to first order in Rossby number,

$$\varepsilon\left[vu_y + Wu_\zeta\right] - v = \frac{1}{2}\left[u_{\zeta\zeta} + 2\varepsilon u_{\zeta Z} + E\,u_{yy}\right]$$

$$\varepsilon\left[vv_y + Wv_\zeta\right] + u = -\frac{\partial p}{\partial y} + \frac{1}{2}\left[v_{\zeta\zeta} + 2\varepsilon v_{\zeta Z} + E\,v_{yy}\right]$$

$$\varepsilon E\left[vW_y + WW_\zeta\right] = -\frac{\partial p}{\partial\zeta} + \frac{E}{2}\left[W_{\zeta\zeta} + 2\varepsilon W_{\zeta Z} + EW_{yy}\right] \tag{2.2. a,b,c,d}$$

$$v_y + W_\zeta + \varepsilon W_Z = 0$$

where

$$\varepsilon = U_0\,/\,fL, \quad E = \left(\delta/L\right)^2.$$

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The vertical velocity has been scaled with $\dfrac{\delta}{L}U$, i.e. the relation between dimensional and non-dimensional velocities is,

$$[u^*, v^*, w^*] = U_0[u.v.\delta/_L W]$$ (2.3)

At $\zeta = 0$, all velocities vanish and for large vertical coordinate the horizontal velocity smoothly merges with $U(y)$. Far from the boundary layer, in the interior, v is $O(E)$ and so far from the boundary the vertical velocity must be independent of vertical coordinate.

The Rossby number ε is assumed small and a perturbation expansion is sought in the form, for all variables,

$$u = u_o + \varepsilon u_1 + ...$$ (2.4)

The lowest order problem is the linear problem whose solution is

$$u_o = U - A(y,Z)e^{-\zeta}\cos\zeta + B(y,Z)e^{-\zeta}\sin\zeta,$$

$$v_o = A(y,Z)e^{-\zeta}\sin\zeta + B(y,Z)e^{-\zeta}\cos\zeta$$ (2.5)

Note that the "constants" A and B in the solution are functions of both y and Z. On the lower boundary we have the conditions (suppressing the dependence on y),

$$A(0) = U, \quad B(0) = 0$$ (2.6)

The vertical velocity is obtained from (2.2 d) at lowest order in Rossby number, yielding,

$$W_o = C(Z) + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}[\sin\zeta + \cos\zeta] + \frac{1}{2}\frac{\partial B}{\partial y}e^{-\zeta}[\cos\zeta - \sin\zeta]$$ (2.7).

To satisfy the lower boundary condition on W we must have,

$$C(0) = -\frac{1}{2}\frac{\partial U}{\partial y}$$ (2.7a)

However, the condition that the interior vertical velocity must be independent of vertical coordinate implies that (2.7) holds for all Z. thus,

$$C = -\frac{1}{2}U_y \qquad\qquad (2.7b)$$

for all Z and

$$W_o = -\frac{1}{2}\frac{\partial U}{\partial y} + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}\left[\sin\zeta + \cos\zeta\right] + \frac{\partial B}{\partial y}e^{-\zeta}\left[\cos\zeta - \sin\zeta\right] \quad (2.8)$$

To determine the Z structure of the coefficients A and B we need to consider the $O(\varepsilon)$ problem. That can be written,

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = Ru + iRv,$$

$$\Lambda_1 = u_1 + iv_1,$$

$$\qquad\qquad\qquad\qquad (2.9\ a,b,c,d,e)$$

$$Ru \equiv 2\left[v_o u_{oy} + W_o u_{o\zeta}\right] - 2u_{o\zeta Z},$$

$$Rv \equiv 2\left[v_o v_{oy} + W_o v_{o\zeta}\right] - 2v_{o\zeta Z}$$

The right hand side of (2.9a) contains terms that are multiples of the solutions of the homogeneous operator of the left hand side, i.e. of the form $e^{-\zeta(1+i)}$. Those terms must be removed or the solutions for u_1, v_1 would have the form $\zeta e^{-\zeta(1+i)}$ and render the solution disordered when $\varepsilon\zeta$ is order unity, i.e. when Z is order one. To eliminate those terms we can use the derivatives with respect to Z coming from the final terms in Ru and Rv. This leads to a differential equation in Z,

$$\frac{\partial}{\partial Z}[A - iB] - [A - iB]\left(\overbrace{\ddot{C}}^{vertical\ advection} + \overbrace{\frac{(1+i)}{4}U_y}^{relative\ vorticity}\right) = 0 \qquad (2.10)$$

The term C comes from the vertical advection in the boundary layer and the remaining

term in the final parenthesis comes from the relative vorticity term in the advection.

Using (2.6) and (2.7b),

$$A - iB = Ue^{-U_y Z(1-i)/4} \qquad (2.11)$$

so that finally, we obtain for the lowest order solutions,

$$u_o = U\left[1 - e^{-\zeta\left(1 + \varepsilon U_y/4\right)}\cos\zeta\left(1 - \varepsilon U_y/4\right)\right]$$

$$v_o = Ue^{-\zeta(1 + \varepsilon U_y/4)}\sin\zeta(1 - \varepsilon U_y/4) \qquad (2.12\ a,b.c)$$

$$W_o = -\frac{U_y}{2}\left[1 - e^{-\zeta(1 + \varepsilon U_y/4)}\left\{\sin\zeta\left(1 - \varepsilon U_y/4\right) + \cos\zeta\left(1 - \varepsilon U_y/4\right)\right\}\right]$$

Thus, a positive value of the vorticity, i.e. $U_y < 0$, leads to a slower decay of the

boundary layer and thus a thicker boundary layer. The effect of the vertical advection

dominates the effect of the relative vorticity. In our non-dimensional units the boundary

layer thickness is

$$\delta_\varepsilon = \frac{\delta}{\left(1 + \varepsilon U_y/2\right)^{1/2}}$$

In dimensional units this is equivalent to a decay scale for the Ekman layer[1]

$$\delta^* = \left(\frac{2\nu}{f + U_y / 2} \right)^{1/2} \qquad\qquad (2.13)$$

The same result can be obtained heuristically by considering the asymptotic matching region where the boundary layer solution blends into the interior. If (2.2 a,b) are linearized around the interior flow, i.e. $u = U(y) + u', v = v', W = -\frac{1}{2}U_y + W'$ where the primed variables are considered small, the resulting linear problem yields the same decay scale for the boundary layer as (2.12 a, b).

In order to obtain the correction to the cross isobar flow at order Rossby number it is necessary to solve (2.9) *after secular terms have been removed*. This, after some algebra, and using the result (2.11) yields,

$$\Lambda_{1\zeta\zeta} - 2i\Lambda_1 = (1+i)e^{-2(\zeta + \gamma Z)}UU_y - iUU_y e^{-\zeta(1-i)}e^{-\gamma Z(1+i)}$$

$$\gamma \equiv \frac{U_y}{4} \qquad\qquad (2.14 \text{ a, b})$$

whose solution, satisfying the no-slip boundary conditions at the plate is,

$$\Lambda_1 = UU_y \left[\frac{1+3i}{10}\left\{ e^{-2(\zeta+\gamma Z)} - e^{-\zeta(1+i)} \right\} + \frac{1}{4}\left\{ e^{-\zeta(1-i)-\gamma Z(1+i)} - e^{-\zeta(1+i)} \right\} \right] \quad (2.15)$$

The total correction to the cross isobar flow comes from the imaginary part of Λ_1 and its integral in ζ is, to lowest order in ε,

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[1] The approximation $(1 + \varepsilon)^{1/2} \approx 1 + \varepsilon / 2$ has been used.

$$\varepsilon \int_0^\infty v_1 d\zeta = \varepsilon \frac{12}{40} UU_y \qquad (2.15)$$

Returning to (2.12b), the total cross isobar flow, *including the Rossby number dependence of v_0* is

$$\int_0^\infty (v_o + \varepsilon v_1) d\zeta = \frac{U}{2} \left[1 + \varepsilon \frac{7}{20} U_y \right] \qquad (2.16)$$

which agrees with the result of Benton *et. al.* (1964) and Hart (2000). The vertical velocity pumped into the interior from the Ekman layer follows immediately and, in agreement with the references cited above,

$$W(\infty) = -\frac{1}{2} U_y - \varepsilon \frac{7}{40} \left(U_y^2 + UU_{yy} \right) \qquad (2.17)$$

so that in the case of a constant shear, the Rossby number correction to the Ekman pumping velocity is *independent of the sign of the vorticity.*

3. The stress-driven Ekman layer.

The classical problem of the Ekman layer on a free surface, driven by an applied stress, has often been studied (see, for example, Stern, 1965 and more recently Thomas and Rhines, 2002). As in the example discussed in section 2 we are particularly interested in obtaining a simple analytic prediction for the nonlinear correction to the Ekman layer thickness. For simplicity, we again assume the flow is independent of the x direction and, is homogeneous, i.e. that the Burger number based on the horizontal scale of the stress and the depth of the fluid is negligibly small.

The governing equations again are (2.2 a, b, c, d). We imagine a wind stress in the x direction with a characteristic magnitude τ_O. The upper surface is at $z = 0$ and, scaling the depth with the linear Ekman layer thickness the dimensional depth variable lies in the range $-\infty \le \zeta \le 0$. We choose to scale the horizontal velocities with

$$U_{scale} = \frac{2\tau_o}{\rho f_o \delta}, \quad \delta = \left(\frac{2\nu}{f}\right)^{1/2} \tag{3.1}$$

We also assume that beneath the Ekman layer, at large negative ζ, there is a geostrophic current in the direction of the stress. In cases where the geostrophic flow is strong enough to directly affect the Ekman layer structure that alignment between the stress and the geostrophic velocity is most common. See for example the aforementioned example of Thomas and Rhines (2002). The boundary conditions are now,

$$u_\zeta = \tau(y), \quad v_\zeta = 0, \quad W = 0 \qquad \zeta = 0$$

$$\tag{3.2 a,b}$$

$$u \to u_g(y), \quad v \to 0, W \to W_\infty(y) \qquad \zeta \to -\infty$$

where W_∞ must be determined.

A similar expansion in powers of ε yields, for the lowest order problem,

$$\Lambda_{0\zeta\zeta} - 2i\Lambda_0 = 0$$
$$\Lambda_0 = u_o + iv_o$$
$$\Lambda_0 = A(Z)e^{\zeta(1+i)},$$
leading to
$$u_o = u_g(y) + e^\zeta \left(A_r \cos\zeta - A_i \sin\zeta\right),$$
$$v_o = e^\zeta \left(A_i \sin\zeta + A_r \cos\zeta\right)$$

$$\tag{3.3. a, b, c d, e)}$$

where A_r and A_i are the real and imaginary parts of $A(Z)$. The boundary conditions on $\zeta = Z = 0$ imply that,

$$A_r(0) = \tfrac{\tau}{2} \quad A_i(0) = -\tfrac{\tau}{2} \qquad\qquad (3.4 \text{ a,b})$$

The vertical velocity can be found from (2.2.d) and with the condition that the interior vertical velocity is independent of Z we obtain,

$$W_0 = -\frac{\tau_y}{2} - \frac{e^\zeta}{2}\frac{\partial}{\partial y}\left[(A_i - A_r)\cos\zeta + (A_i + A_r)\sin\zeta\right] \qquad (3.5)$$

The order ε problem again has the structure of (2.9). Carrying out the indicated calculations yields,

$$\frac{1}{2}\Lambda_{1\zeta\zeta} - i\Lambda = -e^{\zeta(1+i)}\frac{\partial}{\partial Z}\left[(A_r - A_i) + i(A_r + A_i)\right]$$

$$+ e^{\zeta(1+i)}\left[-\frac{\tau_y}{2}\{(A_r - A_i) + i(A_r + A_i)\} + \frac{u_{gy}}{2i}\{A_r + iA_i\}\right]$$

$$+ e^{\zeta(1-i)}\frac{u_{gy}}{2i}\left[-A_r + iA_i\right] + e^{2\zeta}\frac{(1+i)}{2}\left[A_i(A_{ry} + A_{iy}) + A_r(A_{ry} - A_{iy})\right]$$

$$(3.6)$$

The first and second terms on the right hand side both have the form in ζ of the homogeneous operator on the left hand side. To keep our expansion in ε uniformly valid those terms must vanish. This leads to a a simple differential equation for the Z structure of the coefficients A_r and A_i whose solution is,

$$A_r = \frac{\tau}{2^{1/2}} e^{-Z\left[\tau_y/2 + u_{gy}/4\right]} \cos(Zu_{gy}/4 + \pi/4)$$

$$(3.7)$$

$$A_i = -\frac{\tau}{2^{1/2}} e^{-Z\left[\tau_y/2 + u_{gy}/4\right]} \sin(Zu_{gy}/4 + \pi/4)$$

The order one velocity fields can then be written,

$$u_o = u_g + \frac{\tau}{2^{1/2}} e^{\zeta\left[1 - \varepsilon\left(\frac{\tau_y}{2} + \frac{u_{gy}}{4}\right)\right]} \cos\left\{\zeta(1 - \varepsilon u_{gy}/4) - \pi/4\right\},$$

$$(3.\,8\text{ a, b})$$

$$v_o = \frac{\tau}{2^{1/2}} e^{\zeta\left[1 - \varepsilon\left(\frac{\tau_y}{2} + \frac{u_{gy}}{4}\right)\right]} \sin\left\{\zeta(1 - \varepsilon u_{gy}/4) - \pi/4\right\}$$

Note that the geostrophic vorticity affects both the decay scale and the oscillatory variation in the Ekman layer. The wind stress curl affects *only* the thickness of the layer since it enters the dynamics solely in the vertical advection of the momentum. The vorticity of the geostrophic flow does not contribute to the vertical advection. This would require a strongly stratified flow whose geostrophic shear is large enough to produce a viscous stress of the same order as the applied stress.

The Ekman layer thickness, i.e. the characteristic decay scale is now,

$$\delta_\varepsilon = \frac{\delta}{\left(1 - \varepsilon\left\{\tau_y + u_{gy}/2\right\}\right)^{1/2}} \qquad (3.9)$$

or in dimensional units,

$$\delta_* = \left(\frac{2v}{f - \dfrac{2\tau_{*y_*}}{\rho f \delta} - \dfrac{u_{*gy_*}}{2}} \right)^{1/2} \qquad (3.10)$$

If τ_y is negative, i.e. if the wind stress curl is positive, the vertical velocity will be

positive and the vertical momentum advection will upward and this will make the

boundary layer thinner. If the geostrophic vorticity is positive, i.e. if $u_{gy} < 0$, the

boundary layer is again thinner although this time it is due to the enhanced total vorticity

and not the vertical advection effect. Note that the relative size of the two correction

terms in the denominator of (3.10) depends on the horizontal scales of the wind stress and

the geostrophic current and these need not be the same.

If we integrate equation (3.6) after potentially resonant terms have been removed

and use the stress condition on $\zeta = 0$,

$$\Lambda_{1_\zeta} + \Lambda_{0Z} = 0, \qquad \zeta = Z = 0 \qquad (3.11)$$

we obtain,

$$-i\int_0^\infty (u_1 + iv_1) d\zeta = (1+i)\frac{\tau \tau_y}{8} + \frac{iu_{gy}}{2^{3/2}(1-i)} e^{i\pi/4} + \frac{1}{2}\Lambda_{0Z} \quad (3.12)$$

from which we obtain,

$$\int_{-\infty}^0 \varepsilon v_1 d\zeta = -\frac{3}{8}\varepsilon\tau u_{gy} \qquad (3.13)$$

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When this is added to the vertical integral of v_0, using (3.8b) and keeping terms to order ε we obtain for the total Ekman flux perpendicular to the applied stress,

$$\int_{-\infty}^{0} (v_o + \varepsilon v_1)d\zeta = -\frac{\tau}{2} - \varepsilon\tau\left[\frac{\tau_y}{8} + \frac{u_{gy}}{2}\right] \qquad (3.14)$$

It is noteworthy that this important result for the flux can be obtained directly by integrating (2.2 a) with the nonlinear terms written in flux form, and using only the naïve linear solution (i.e. without including the variation in Z) to evaluate the nonlinear terms while using the boundary conditions to evaluate the stress term at the surface and the vertical advection at the base of the Ekman layer.

The vertical velocity at the base of the Ekman layer, to order ε is obtained from

$$W(-\infty) = \frac{\partial}{\partial y}\left(\int_{-\infty}^{0} v_o + \varepsilon v_1\right)d\zeta \qquad (3.15)$$

and is,

$$W(-\infty) = -\frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\tau}{1 - \varepsilon\{u_{gy} + \tau_y/4\}}\right] \qquad (3.16)$$

and these results for the Ekman flux and pumping velocity agree with Thomas and Rhines (2002).

4. Discussion

The effects of nonlinearity on the Ekman layer's thickness have been studied in the weakly nonlinear limit, i.e. small Rossby number. Although one might be simply tempted

to replace the planetary vorticity by the total, absolute vorticity to estimate the thickness, this neglects the relatively powerful effect of the vertical advection on the boundary layer thickness. For the boundary layer which allows a geostrophic flow to satisfy the no-slip condition on a solid boundary, the vertical advection dominates the vortex force effect in determining the thickness. A positive relative vorticity, although it augments the total vorticity, leads to a thickening of the boundary layer.

For a boundary layer driven by stress on a free surface the situation is even more extreme. In the absence of a strong geostrophic current beneath the layer, the *only* effect of nonlinearity is to either thicken or thin the layer depending on the sign of the vertical advection produced by the wind stress curl. A strong geostrophic flow, i.e. with velocity as large as boundary layer flow driven directly by the stress, does not affect the vertical advection and only contributes a vortex force such that positive vorticity makes the layer thinner as would be intuited by the naïve replacement of *f* by the total vorticity including the geostrophic flow.

As a check to the calculation, the fluxes in the Ekman layer and the vertical velocity pumped into or out of the layer have been calculated within the formalism described above and the results agree with the more naïve (and straight forward) calculations ignoring the role of nonlinearity on the layer thickness.

It is important to note the important simplification employed in this study. The geostrophic flow is rectilinear and an exact solution of the inviscid quasi-geostrophic equations. As such there is no advection of relative vorticity by the geostrophic flow and the effect of the geostrophic flow on the layer thickness and the pumping remains local. If the geostrophic flow is more complex, i.e. two dimensional and a function of both lateral

coordinates, this will no longer be the case. One expects then the effects of the flow to be

non-local and a shift between the local vorticity and the pumping to occur. Some results,

employing a regular iteration expansion are reported in Hart (1995). An approach similar

to the present paper for such cases would lead to a partial differential equation for the

alteration of layer thickness and the results of that study are not yet concluded.

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