Extratropical Rossby Waves in the Presence of Buoyancy Mixing

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Abstract

The propagation of Rossby waves on a mid-latitude β-plane is investigated in the presence of density diffusion with the aid of linear hydrostatic theory. The search for wave solutions in a vertically bounded medium subject to horizontal (vertical) diffusion leads to an eigenvalue problem of second (fourth) order. Exact solutions of the problem are obtained for uniform background stratification (N), and approximate solutions are constructed for variable N using the Wentzel-Kramers-Brillouin method. Roots of the eigenvalue relations for free waves are found and discussed.

It is found that the barotropic wave of adiabatic theory is also a solution of the eigenvalue problem as this is augmented with density diffusion in the horizontal or vertical direction. The barotropic wave is undamped as fluid parcels in the wave move only horizontally and are therefore insensitive to the vortex stretching induced by mixing. On the other hand, density diffusion modifies the properties of baroclinic waves of adiabatic theory. In the presence of horizontal diffusion the baroclinic modes are damped but their vertical structure remains unaltered. The ability of horizontal diffusion to damp baroclinic waves stems from its tendency to counteract the deformation of isopycnal surfaces which is caused by the passage of these waves. The damping rate increases (i) linearly with horizontal diffusivity and (ii) nonlinearly with horizontal wave number and mode number. In the presence of vertical diffusion the baroclinic waves suffer both damping and a change in vertical structure. In the long-wave limit the damping is critical (wave decay rate numerically equal to wave frequency) and increases as the square roots of vertical diffusivity and zonal wave number. Density diffusion in the horizontal or vertical direction reduces the amplitude of the phase speed of long, westward propagating waves. Observational estimates of eddy diffusitivies suggest that horizontal and vertical mixing strongly attenuates baroclinic waves in the ocean, but that vertical mixing is too weak to notably modify the vertical structure of the gravest modes.

1 Introduction

Planetary or Rossby waves are long-period oscillations in the oceans and atmosphere, whose restoring mechanism is provided by the variation of the Coriolis parameter with latitude. In the oceans the anisotropy of energy transmission of these waves is responsible for a major feature of the general circulation, i.e., the concentration of small-scale energy in the western portion of oceanic gyres (e.g., *Pedlosky* (1987)). Rossby waves also constitute the prevalent mechanism by which the ocean adjusts on annual to decadal scales to perturbations in the atmosphere. They have been suggested to play a role in important oceanic phenomena such as the response to suddenly applied wind stress (e.g., *Anderson and Gill* (1975)), the variability of equatorial motions (e.g., *Cane and Sarachik* (1976)), the establishment of abyssal flows (e.g., *Kawase* (1987)), and the changes in meridional overturning circulation (e.g., *Johnson and Marshall* (2002)).

Interest in Rossby waves in oceanography was stimulated over the last decade by their inference from observations of sea surface height (SSH) by satellite altimetry (e.g., *Chelton and Schlax* (1996)). The SSH anomalies were shown to propagate westward at speeds decreasing strongly with latitude, which is consistent with the latitude dependence of the phase speed of long baroclinic modes predicted by standard linear theory. However, the westward propagation of SSH anomalies outside the band 10°S–10°N appeared systematically faster than predicted (*Chelton and Schlax*, 1996). A wealth of studies followed, which attempted to interpret the reported disagreement in terms of effects absent from the standard linear theory of Rossby waves (e.g., *Killworth et al.* (1997), *Qiu et al.* (1997), *Dewar* (1998), *de Szoeke and Chelton* (1999), *Killworth and Blundell* (1999), *Tailleux and McWilliams* (2001), *LaCasce and Pedlosky* (2004), *Killworth and Blundell* (2005)). *Zhang and Wunsch* (1999) revisited the analysis of North Pacific data by *Chelton and Schlax* (1996) by using a different processing method and contended that a significant fraction of the data is actually consistent with linear theory. On the other hand, a more recent analysis of altimetric data with higher resolution concluded that much of the mesoscale SSH variability outside the tropics consists of nonlinear eddies (*Chelton et al.*, 2007).

In this paper, the propagation of Rossby waves in a mid-latitude ocean is investigated in the presence of small-scale, turbulent transport of buoyancy ('mixing'). Our study is specifically motivated, not by the anomalous propagation postulated by *Chelton and Schlax* (1996), but by the following two elements. First, theories of the steady circulation emphasize the role of mixing in restricting the westward extension, in the form of arrested Rossby waves, of pressure anomalies established on the eastern boundary of oceanic basins (e.g., *Kawase* (1987); *Edwards and Pedlosky* (1995)). Despite this recognized role, the influence of mixing on progressive waves has not received much attention in theoretical work (for a study with a reduced gravity model see however *Deshayes and Frankignoul* (2005)), although such influence was considered for the low latitudes, e.g., for the equatorial undercurrent (e.g., *McCreary* (1981)). Second, various studies showed that mixing can influence the steady state response of ocean circulation to surface buoyancy forcing. For example, scaling arguments suggest that the nature of vertical mixing could determine the very sign of the response in a single-hemisphere ocean (e.g., *Nilsson and Walin* (2001)). The effects of mixing on the time-dependent response, however, remains largely unexplored from the theoretical viewpoint. Indeed, it is friction, not buoyancy mixing, which is traditionally viewed as a major damping mechanism for Rossby waves (e.g., *Pedlosky* (1987); *Qiu et al.* (1997)). Here the diabatic term in the vorticity balance is retained in order to examine how this term fundamentally modifies the properties of the progressive waves of adiabatic theory.

It is probably worth being explicit about some of the limitations of this paper. A first limitation is the reliance on simplified dynamics in the form of the linearized hydrostatic equations (*LeBlond and Mysak*, 1978). The hydrostatic system, however, allows us to filter waves of no direct interest, such as inertial-gravity waves, and hence to isolate the effects of mixing on Rossby waves which do remain solutions of the dynamical equations. Another limitation is the assumption that buoyancy mixing operates as Fickian diffusion. Results from tracer release experiments seem to support this assumption for vertical mixing but question it for horizontal mixing at mesoscales (e.g., *Ledwell et al.* (1998)). Whereas the representation of mixing as Fickian diffusion is almost universal in dissipative theories, there still appears to be no fundamental justification for it. Other limitations include the

omissions of a background flow (e.g., *Killworth et al.* (1997); *Dewar* (1998); *de Szoeke and Chelton* (1999)) and bottom topography (e.g., *Killworth and Blundell* (1999)). It is felt that the influences of mixing on Rossby waves should first be considered in isolation of these yet important aspects of the general circulation.

The paper is organized as follows. In section 2, the problem posed by the determination of Rossby waves in a mid-latitude ocean subject to density diffusion is introduced. The vorticity equation of the problem is derived, and the vertical structure equation and its boundary conditions for the study of free waves in a vertically bounded medium are obtained. In section 3, the effects of horizontal density diffusion on Rossby waves are determined. The analysis is repeated for vertical density diffusion in section 4. Exact solutions of the problem are first derived for the case where the background density stratification is uniform. Approximate solutions are then constructed for variable stratification using the WKB method. The oceanographic significance our results are discussed in section 5. Conclusions follow in section 6.

2 Eigenvalue Problem

In this section, the problem posed by the determination of Rossby waves in a mid-latitude ocean subject to density diffusion is presented. The analysis assumes a continuously stratified ocean on a β-plane, which is bounded at the bottom and the surface (for reference, the case of an unbounded ocean is briefly considered). On the other hand, the wave disturbance is assumed to occur far from lateral boundaries, so it is not affected by reflection at these boundaries.

2.1 Vorticity Equation

The dynamical framework is provided by the linearized hydrostatic β-plane equations, which are traditionally used to study small-amplitude, long-wave motions in the ocean (*LeBlond and Mysak*, 1978). Here these equations are extended to include horizontal and

vertical diffusion of density. They are expressed in dimensionless form in order to clarify the assumptions underlying the construction of solutions using the WKB approach. It is postulated that the wave motions to be determined can be characterized by a time scale T, a horizontal (vertical) scale L (D), and a horizontal (vertical) velocity scale U ($(D/L)U$). Thus, the variables of the dynamical equations are normalized based on the following relations:

$$t = Tt', \tag{1a}$$

$$(x, y) = L(x', y'), \tag{1b}$$

$$z = Dz', \tag{1c}$$

$$(u, v) = U(u', v'), \tag{1d}$$

$$w = \frac{D}{L}Uw'. \tag{1e}$$

Here the primed variables are dimensionless, t is time, and (u, v, w) are the (x, y, z) components of velocity, where x is the zonal coordinate (counted positively eastward), y the meridional coordinate (northward), and z the vertical coordinate (upward). The scaling for pressure p is based on the geostrophic balance and the scaling for density ρ relies on the thermal wind:

$$p = \rho_o U f_o L p', \tag{2a}$$

$$\rho = \frac{\rho_o U f_o L}{gD}\rho', \tag{2b}$$

where ρ_o is a constant density, f_o the Coriolis parameter at the mid-latitude of the basin, and g the acceleration of gravity. The Coriolis parameter is normalized based on

$$f = f_o f', \tag{3}$$

where $f' = 1 + \beta' y'$, $\beta' = \beta L/f_o$, and β is the gradient of planetary vorticity.

The equations of motion in dimensionless form are then (omitting the primes for neatness):

$$u_x + v_y + w_z = 0, \tag{4a}$$

$$Ru_t - fv = -p_x, \tag{4b}$$

$$Rv_t + fu = -p_y, \tag{4c}$$

$$\rho = -p_z, \tag{4d}$$

$$R\rho_t - B(z)w = E_h\nabla_h^2\rho + E_v\rho_{zz}, \tag{4e}$$

where a subscript denotes partial differentiation and $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian. The dimensionless numbers in (4b–c) and (4e) are

$$R = \frac{1}{f_o T}, \tag{5a}$$

$$B(z) = \frac{N(z)^2 D^2}{f_o^2 L^2}, \tag{5b}$$

$$E_h = \frac{\kappa_h}{f_o L^2}, \tag{5c}$$

$$E_v = \frac{\kappa_v}{f_o D^2}, \tag{5d}$$

where $N(z)$ is the buoyancy frequency of the background density field and κ_h (κ_v) is the horizontal (vertical) diffusivity. The Rossby number R is proportional to the ratio of the rotation period to the wave period, the Burger number B sets the background stratification, and E_h (E_v) represents the effect of horizontal (vertical) density diffusion.

A vorticity equation in terms of pressure can be derived from the dynamical equations (4a–e) using the same manipulations as for adiabatic theory (*LeBlond and Mysak*, 1978). The operator $\mathcal{L} = R^2\partial^2/\partial t^2 + f^2$ is applied twice to the continuity equation (4a),

$$\mathcal{L}[\mathcal{L}u_x + \mathcal{L}v_y + \mathcal{L}w_z] = 0. \tag{6}$$

The two combinations $R\partial/\partial t(4b) + f(4c)$ and $R\partial/\partial t(4c) - f(4b)$ are then calculated. Differentiating the first combination with respect to x and the second with respect to y gives

$$\mathcal{L}u_x = -\left(Rp_{xxt} + fp_{xy}\right), \tag{7a}$$

$$\mathcal{L}v_y = -Rp_{yyt} + fp_{xy} + \beta p_x - 2\beta f v. \tag{7b}$$

Finally, the hydrostatic and density equations (4d–e) lead to $w = (-Rp_{zt} + E_h\nabla_h^2 p_z + E_v p_{zzz})/B(z)$, so

$$\mathcal{L}w_z = \mathcal{L}\left(\frac{-Rp_{zt} + E_h\nabla_h^2 p_z + E_v p_{zzz}}{B(z)}\right)_z. \tag{8}$$

The vorticity equation is obtained by adding (7a), (7b) and (8),

$$\mathcal{L}\left[R\left(\nabla_h^2 p\right)_t + \mathcal{L}\left(\frac{Rp_{zt} - E_h\nabla_h^2 p_z - E_v p_{zzz}}{B(z)}\right)_z\right] + \beta\left[f^2 p_x - 2fRp_{yt} - R^2 p_{xtt}\right] = 0. \quad (9)$$

This form of the vorticity balance includes earlier forms as special cases. For example, if $E_h = E_v = 0$, it becomes the (dimensionless) vorticity equation of adiabatic theory (*LeBlond and Mysak* (1978), p. 124, eq. (15.21')). Assuming steady state, $B_z = 0$, and $E_h = 0$, it becomes $E_v p_{zzzz} - (B\beta/f^2)p_x = 0$, which is the vorticity balance of the abyssal circulation model of *Edwards and Pedlosky* (1995). Note the straightforward interpretation of equation (9) in the steady state limit. In this limit, the equation simplifies to $-f^4[(E_h\nabla_h^2 p_z + E_v p_{zzz})/B(z)]_z + \beta f^2 p_x = 0$, showing that horizontal and vertical mixing provides the vortex stretching for meridional motion in the field of variable planetary vorticity.

2.2 Dispersion Relation for Unbounded Medium

A wave solution of the vorticity equation (9) is first sought for an ocean that is vertically unbounded. The analysis essentially assumes that the horizontal boundaries (the bottom and the surface) are distant from the region of the wave disturbance. Although this is a restrictive assumption, it constitutes a useful first step in our discussion. For simplicity, the background stratification is taken as uniform (more specifically, B is taken as constant over the vertical scale of the wave).

A plane wave solution of (9) is considered,

$$p = \Re\,\hat{p}e^{i(kx+ly+mz-\omega t)}, \quad (10)$$

where $\Re$ implies that the real part of the following expression is taken, $\hat{p}$ is the wave amplitude, (k, l, m) are the (x, y, z) components of the wave vector, and $\omega = \omega_r + i\omega_i$ is the complex wave frequency. Inserting (10) into (9) leads to a dispersion relation,

$$R(\omega_r + i\omega_i) = \frac{1}{BK^2 + f^2m^2}\left[-B\beta k - i\left(E_h K^2 + E_v m^2\right) f^2 m^2\right], \quad (11a)$$

where $K = \sqrt{k^2 + l^2}$ is the horizontal wave number. In dimensional form,

$$\omega_{r*} + i\omega_{i*} = \frac{1}{N^2 K_*^2 + f_*^2 m_*^2}\left[-N^2\beta_* k_* - i\left(\kappa_h K_*^2 + \kappa_v m_*^2\right) f_*^2 m_*^2\right], \tag{11b}$$

where the subscript '*' designates a dimensional variable and $\omega_* = \omega/T$. The real part of (11a–11b) is the dispersion relation for Rossby waves in adiabatic theory. The imaginary part is negative definite and proportional to (E_h, E_v) or (κ_h, κ_v), which indicates that the wave is damped by density diffusion. The damping increases with the horizontal and vertical wave numbers as well as with the meridional coordinate. Note than when $m = 0$, the wave is barotropic (p independent of depth) and undamped. Indeed, for $m = 0$, the displacement of fluid parcels in the wave is strictly horizontal, so the parcels remain unaffected by the vortex stretching due to mixing (Eq. 8).

2.3 Vertical Structure Equation and Boundary Conditions

The plane wave (10) is relevant only in situations where the vertical scale of the wave is much less than the ocean depth. Below this restriction is removed by considering a wave solution of the form

$$p = \Re\, \hat{p}(z) e^{i(kx+ly-\omega t)}, \tag{12}$$

where $\hat{p}(z)$ is a vertical structure function. Inserting (12) into the vorticity balance (9) gives a vertical structure equation:

$$iE_v\left[\frac{\hat{p}_{zzz}}{B(z)}\right]_z - \left(R\omega + iE_h K^2\right)\left[\frac{\hat{p}_z}{B(z)}\right]_z + \frac{R\omega K^2 + \beta k}{f^2}\hat{p} = 0. \tag{13}$$

This differential equation is subject to boundary conditions at the bottom ($z = -1$) and at the surface ($z = 0$). It is assumed that the vertical velocity is zero at both boundaries ($w = 0$ at $z = -1, 0$). When recast in terms of $\hat{p}$, the conditions of no normal flow are

$$iE_v\hat{p}_{zzz} - \left(R\omega + iE_h K^2\right)\hat{p}_z = 0 \qquad \text{at } z = -1, 0. \tag{14a}$$

Two more conditions are necessary to determine wave motions in the presence of vertical mixing. Here the vertical density flux is set to zero at both boundaries ($E_v \partial\rho/\partial z = 0$ at $z = -1, 0$),

$$\hat{p}_{zz} = 0 \qquad \text{at } z = -1, 0. \tag{14b}$$

Note that the vertical structure equation (13) subject to conditions (14a–14b) constitutes a eigenvalue problem of second order if $E_v = 0$ and of fourth order if $E_v \neq 0$.

3 Horizontal Density Diffusion

In this section, the influence of horizontal density diffusion on Rossby waves in an ocean of finite vertical extent is investigated. The simple case with uniform background stratification is first considered. The more complicated situation with variable stratification is then addressed.

3.1 Constant Stratification

For $E_v = B_z = 0$, the vertical structure equation (13) becomes

$$-\frac{R\omega + iE_hK^2}{B}\hat{p}_{zz} + \frac{R\omega K^2 + \beta k}{f^2}\hat{p} = 0. \tag{15}$$

This equation is more conveniently written as

$$\hat{p}_{zz} + \lambda\hat{p} = 0, \tag{16}$$

where

$$\lambda = -\frac{B}{f^2}\frac{R\omega K^2 + \beta k}{R\omega + iE_hK^2}. \tag{17}$$

The boundary conditions for this equation are the conditions of no normal flow at the bottom and the surface (14a), i.e.,

$$\hat{p}_z = 0 \qquad \text{at } z = -1, 0. \tag{18}$$

The general solution of the eigenvalue problem (16, 18) is

$$\hat{p}(z) = C_1 \cos\left(\sqrt{\lambda}z\right) + C_2 \sin\left(\sqrt{\lambda}z\right). \tag{19}$$

The eigenvalues λ satisfy the relation

$$\lambda \sin\sqrt{\lambda} = 0. \tag{20}$$

The solution $\lambda = 0$ corresponds to a barotropic mode ($\hat{p}$ independent on depth) with dispersion relation

$$R\left(\omega_r + i\omega_i\right) = -\frac{\beta k}{K^2}. \tag{21}$$

On the other hand, the solutions $\lambda_n = n^2\pi^2$, with $n = 1, 2, \ldots$, define baroclinic modes ($\hat{p}$ dependent on depth and with zero depth-integral) with dispersion relation

$$R\left(\omega_r^{(n)} + i\omega_i^{(n)}\right) = -\frac{\beta k}{K^2 + \left(n\pi f/\sqrt{B}\right)^2} - iE_h\left(\frac{n\pi f}{\sqrt{B}}\right)^2 \frac{K^2}{K^2 + \left(n\pi f/\sqrt{B}\right)^2}. \tag{22a}$$

In dimensional form,

$$\omega_{r*}^{(n)} + i\omega_{i*}^{(n)} = -\frac{\beta_* k_*}{K_*^2 + (n\pi/L_D)^2} - i\kappa_h\left(\frac{n\pi}{L_D}\right)^2 \frac{K_*^2}{K_*^2 + (n\pi/L_D)^2}, \tag{22b}$$

where $L_D = (N/f_*)D$ is the internal radius of deformation. The boundedness of the medium is manifested by the well-known result that, compared to (11–12), the vertical wave number is now quantized as an integral multiple of π/L_D. As for a vertically unbounded ocean, the barotropic mode is undamped, whereas the baroclinic modes suffer decay by horizontal density diffusion. The decay rate is large for small wavelengths compared to the deformation radius and increases with mode number (Figure 1).

3.2 Variable Stratification

For $E_v = 0$ and $B_z \neq 0$, the vertical structure equation (13) can be written as

$$-R\varpi\hat{p}_{zz} + R\varpi\frac{B_z}{B}\hat{p}_z + \left(R\omega K^2 + \beta k\right)\frac{B}{f^2}\hat{p} = 0, \tag{23}$$

where $R\varpi = R\omega + iE_hK^2$ is a modified frequency and the dependence of B upon depth is implicit. The boundary conditions for this equation are again the conditions of no normal flow (18). An approximate solution of (23) is constructed from the WKB method (e.g., *Bender and Orszag* (1978)). The vertical structure function $\hat{p}(z)$ is given the form

$$\hat{p}(z) \sim \exp\left[\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n(z)\right], \tag{24}$$

where the symbol $\sim$ means that the series is not necessarily convergent and ϵ is a small parameter. Inserting this expression for $\hat{p}(z)$ into the vertical structure equation (23)

gives

$$-R\varpi\left[\frac{(S_0')^2}{\epsilon^2}+2\frac{S_0'S_1'}{\epsilon}+\frac{S_0''}{\epsilon}+\ldots\right]+R\varpi\frac{B_z}{B}\left[\frac{S_0'}{\epsilon}+\ldots\right]+\left(R\omega K^2+\beta k\right)\frac{B}{f^2}=0, \quad (25)$$

where the primes indicate differentiation with respect to depth. Observations show that the change in stratification at middle latitudes is of first order over the depth of the ocean, i.e., $B_z/B = \mathrm{O}(1)$. The first order or 'eikonal' equation of the WKB approximation is therefore taken as

$$-R\varpi\frac{(S_0')^2}{\epsilon^2}+\left(R\omega K^2+\beta k\right)\frac{B}{f^2}=0. \quad (26)$$

With the choice $\epsilon=\sqrt{R}$, the solution of this equation is

$$S_0(z)=\pm i\sqrt{-\frac{R\omega K^2+\beta k}{\varpi f^2}}\int_{-1}^{z}\sqrt{B(z')}dz', \quad (27)$$

to within an additive constant. The two sign possibilities yield the first contribution to $\hat{p}(z)$ in the form of a linear combination of

$$\cos\left[\sqrt{-\frac{R\omega K^2+\beta k}{\varpi f^2}}\int_{-1}^{z}\sqrt{B(z')}dz'\right] \quad \text{and} \quad \sin\left[\sqrt{-\frac{R\omega K^2+\beta k}{\varpi f^2}}\int_{-1}^{z}\sqrt{B(z')}dz'\right]. \quad (28)$$

The second order or 'transport' equation of the WKB approximation is

$$-R\varpi\left(2\frac{S_0'S_1'}{\epsilon}+\frac{S_0''}{\epsilon}\right)+R\varpi\frac{B_z}{B}\frac{S_0'}{\epsilon}=0. \quad (29)$$

Its solution is

$$S_1(z)=\ln\left[\frac{R\omega K^2+\beta k}{B(z)\varpi f^2}\right]^{-1/4}, \quad (30)$$

apart from another additive constant. This expression provides the second contribution to $\hat{p}(z)$. An approximate solution $\hat{p}(z)$ is obtained from the first two contributions in (24),

$$\hat{p}_{\mathrm{WKB}}(z)=B(z)^{1/4}\left[C_1\cos\left(\int_{-1}^{z}\sqrt{\ell B(z')}dz'\right)+C_2\sin\left(\int_{-1}^{z}\sqrt{\ell B(z')}dz'\right)\right], \quad (31)$$

where

$$\ell=-\frac{R\omega K^2+\beta k}{R\varpi f^2}. \quad (32)$$

The application of the boundary conditions (18) to the general solution (31) yields a system of two linear algebraic equations with two unkowns (C_1, C_2). A non-trivial solution exists if the determinant of the system vanishes, which leads to the eigenvalue relation

$$r_1 - r_2 = 0, \quad (33)$$

where

$$r_1 = \frac{1}{4}\left(\frac{B_z}{B}\right)_{-1}\left[\frac{1}{4}\left(\frac{B_z}{B}\right)_0 \sin\left(\int_{-1}^{0}\sqrt{\ell B(z')}dz'\right) + \sqrt{\ell B(0)}\cos\left(\int_{-1}^{0}\sqrt{\ell B(z')}dz'\right)\right] \quad (34a)$$

$$r_2 = \sqrt{\ell B(-1)}\left[\frac{1}{4}\left(\frac{B_z}{B}\right)_0 \cos\left(\int_{-1}^{0}\sqrt{\ell B(z')}dz'\right) - \sqrt{\ell B(0)}\sin\left(\int_{-1}^{0}\sqrt{\ell B(z')}dz'\right)\right] \quad (34b)$$

Here $(B_z/B)_{-1}$ and $(B_z/B)_0$ designate the value of B_z/B near the bottom and the surface, respectively. The specific values ℓ satisfying (33) for arbitrary $B(z)$ determine, in the WKB approximation, the wave solutions of the form (12) which are supported by the diabatic system (4a–e) for $E_h \neq 0$, $E_v = 0$, and $B_z \neq 0$. Such values are easily found for the particular but oceanographically relevant case where stratification is uniform near the bottom and the surface ($B_z/B = 0$ at $z = -1, 0$). In this case the eigenvalue relation (33) becomes simply

$$\ell \sin\left(\int_{-1}^{0}\sqrt{\ell B(z')}dz'\right) = 0. \quad (35)$$

The root $\ell = 0$ gives the wave mode $R(\omega_r + i\omega_i) = -\beta k/K^2$. In contrast to the case with uniform stratification, this mode is not strictly barotropic at the level of the WKB approximation (31) owing to the presence of the factor $B(z)^{1/4}$. The roots

$$\int_{-1}^{0}\sqrt{\ell B(z')}dz' = n\pi, \quad (36)$$

where $n = 1, 2, \ldots$, define baroclinic modes with dispersion relation

$$R\left(\omega_r^{(n)} + i\omega_i^{(n)}\right) = -\frac{\beta k}{K^2 + (n\pi f/\overline{B^{1/2}})^2} - iE_h\left(\frac{n\pi f}{\overline{B^{1/2}}}\right)^2 \frac{K^2}{K^2 + (n\pi f/\overline{B^{1/2}})^2} \quad (37a)$$

or

$$\omega_{r*}^{(n)} + i\omega_{i*}^{(n)} = -\frac{\beta_* k_*}{K_*^2 + (n\pi f_*/\overline{N})^2} - i\kappa_h\left(\frac{n\pi f_*}{\overline{N}}\right)^2 \frac{K_*^2}{K_*^2 + (n\pi f_*/\overline{N})^2}. \quad (37b)$$

Here $\overline{(\cdot)}$ designates an integral from the bottom to the surface. The dispersion relation (37a–37b) includes as special cases (i) the relation for variable N in the adiabatic situation (*Chelton et al.*, 1998), and (ii) the relation for uniform N and $E_h \neq 0$ or $\kappa_h \neq 0$ (Eqs. 22a–22b). The variations of wave damping with wave number and mode number are identical to those for uniform stratification (Figure 1), provided that the internal deformation radius is identified with $(\overline{N}/|f_*|)D$. Note that for both uniform and variable N, the presence of horizontal density diffusion does not modify the vertical structure of the wave.

4 Vertical Density Diffusion

In this section, the influence of vertical density diffusion on Rossby waves in a vertically bounded ocean is explored. Again, the case with uniform background stratification is first considered, followed by the more complicated situation with variable $B = B(z)$.

4.1 Constant Stratification

4.1.1 General Solution

For $E_h = B_z = 0$, the vertical structure equation (13) becomes

$$iE_v\hat{p}_{zzzz} - R\omega\hat{p}_{zz} + \frac{B}{f^2}\left(R\omega K^2 + \beta k\right)\hat{p} = 0. \tag{38}$$

This equation is re-written as

$$i\epsilon\hat{p}_{zzzz} - \hat{p}_{zz} - \lambda\hat{p} = 0, \tag{39}$$

where

$$\epsilon = \frac{E_v}{R\omega}, \tag{40a}$$

$$\lambda = -\frac{B}{f^2}\frac{R\omega K^2 + \beta k}{R\omega}. \tag{40b}$$

Note that, in general, $\epsilon = \epsilon_r + i\epsilon_i$ and $\lambda = \lambda_r + i\lambda_i$ are complex with real and imaginary parts given by

$$\epsilon_r = +\frac{E_v}{R}\frac{\omega_r}{\omega_r^2 + \omega_i^2}, \tag{41a}$$

$$\epsilon_i = -\frac{E_v}{R}\frac{\omega_i}{\omega_r^2 + \omega_i^2}, \tag{41b}$$

and

$$\lambda_r = -\frac{B\beta k}{Rf^2}\frac{\omega_r}{\omega_r^2 + \omega_i^2} - \frac{BK^2}{f^2}, \tag{42a}$$

$$\lambda_i = +\frac{B\beta k}{Rf^2}\frac{\omega_i}{\omega_r^2 + \omega_i^2}. \tag{42b}$$

A solution of (39) of the form

$$\hat{p}(z) = e^{\sqrt{r}z} \tag{43}$$

is sought, where $\sqrt{r}$ is complex. Inserting this form into (39) leads to the characteristic equation

$$i\epsilon r^2 - r - \lambda = 0. \tag{44}$$

The roots of this equation are

$$r_{\pm} = -\frac{i}{2\epsilon}\left(1 \pm \sqrt{1 + 4i\epsilon\lambda}\right), \tag{45}$$

so the general solution of (39) is

$$\hat{p}(z) = C_1 e^{\sqrt{r_+}z} + C_2 e^{-\sqrt{r_+}z} + C_3 e^{\sqrt{r_-}z} + C_4 e^{-\sqrt{r_-}z}. \tag{46}$$

4.1.2 Free Wave Solutions

The impermeability and insulation conditions (14a–14b) are imposed at the bottom and at the surface. Applying these conditions to the general solution (46) yields a system of four linear algebraic equations with four unknowns (C_1, C_2, C_3, C_4). The determinantal equation of the system is

$$\begin{vmatrix} (i\epsilon r_+ - 1)\sqrt{r_+}e^{-\sqrt{r_+}} & -(i\epsilon r_+ - 1)\sqrt{r_+}e^{\sqrt{r_+}} & (i\epsilon r_- - 1)\sqrt{r_-}e^{-\sqrt{r_-}} & -(i\epsilon r_- - 1)\sqrt{r_-}e^{\sqrt{r_-}} \\ (i\epsilon r_+ - 1)\sqrt{r_+} & -(i\epsilon r_+ - 1)\sqrt{r_+} & (i\epsilon r_- - 1)\sqrt{r_-} & -(i\epsilon r_- - 1)\sqrt{r_-} \\ r_+ e^{-\sqrt{r_+}} & r_+ e^{\sqrt{r_+}} & r_- e^{-\sqrt{r_-}} & r_- e^{\sqrt{r_-}} \\ r_+ & r_+ & r_- & r_- \end{vmatrix} = 0.$$

This equation provides by Laplace expansion an eigenvalue relation,

$$r_+ r_- \left(r_1 + r_2\right) = 0, \tag{47}$$

where

$$r_1 = 2\sqrt{r_+}\sqrt{r_-}\left(i\epsilon r_+ - 1\right)\left(i\epsilon r_- - 1\right)\left(1 - \cosh\sqrt{r_+}\cosh\sqrt{r_-}\right), \tag{48a}$$

$$r_2 = \left[r_+\left(i\epsilon r_- - 1\right)^2 + r_-\left(i\epsilon r_+ - 1\right)^2\right]\sinh\sqrt{r_+}\sinh\sqrt{r_-}. \tag{48b}$$

The roots of (47) define the wave motions which are supported by the diabatic system (4a–e) for $E_h = 0$, $E_v \neq 0$, and $B_z = 0$. Two roots are easily found by inspection. The first is $r_+ = 0$ or $r_- = 0$, which leads to $\lambda = 0$. The wave corresponding to this root has the familiar dispersion relation $R(\omega_r + i\omega_i) = -\beta k/K^2$. The second root is defined by $r_+ = r_- = r$ $(\neq 0)$, since the eigenvalue relation (47) then becomes

$$2r^3 (i\epsilon r - 1)^2 \left[1 - \cosh^2\sqrt{r} + \sinh^2\sqrt{r}\right] = 0. \tag{49}$$

The equality $r_+ = r_-$ implies that $\sqrt{1 + 4i\epsilon\lambda} = 0$ and leads therefore to the system

$$\epsilon_r \lambda_i + \epsilon_i \lambda_r = \frac{1}{4}, \tag{50a}$$

$$\epsilon_r \lambda_r - \epsilon_i \lambda_i = 0. \tag{50b}$$

In contrast to the first root, the wave solution satisfying this system must have a frequency with a non-vanishing imaginary part ($\lambda_i \neq 0$ and $\epsilon_i \neq 0$). The system is conveniently expressed in terms of ϵ_r, λ_r, and the ratio $\sigma = \omega_i/\omega_r$ which, for negative ω_i, is the ratio of wave decay rate to wave frequency:

$$2\sigma\epsilon_r\lambda_r + \sigma\epsilon_r \frac{BK^2}{f^2} = -\frac{1}{4}, \tag{51a}$$

$$\left(1 - \sigma^2\right)\epsilon_r\lambda_r - \sigma^2\epsilon_r \frac{BK^2}{f^2} = 0. \tag{51b}$$

Equation (51b) can be divided by ϵ_r and solved for λ_r. Then the real part of ω can be deduced from the defining relation (42a) and the imaginary part of ω can be found by inserting the expression for λ_r in (51a), which gives

$$R(\omega_r + i\omega_i) = -\frac{\beta k}{K^2}\frac{1 - \sigma^2}{1 + \sigma^2} - 4iE_v \frac{BK^2}{f^2}\frac{\sigma^2}{1 - \sigma^2}, \tag{52a}$$

or

$$\omega_{r*} + i\omega_{i*} = -\frac{\beta_* k_*}{K_*^2}\frac{1 - \sigma^2}{1 + \sigma^2} - 4i\kappa_v \frac{N^2 K_*^2}{f_*^2}\frac{\sigma^2}{1 - \sigma^2}. \tag{52b}$$

For fixed (k, l) or (k_*, l_*) the frequency of the wave decreases in amplitude with increasing values of the ratio $\sigma = \omega_i/\omega_r = \omega_{i*}/\omega_{r*}$ (Figure 2a). The imaginary part of ω is always negative for $\sigma^2 < 1$, which indicates wave damping (Figure 2b). For fixed σ, the decay rate increases linearly with E_v or κ_v and quadratically with the horizontal wave number.

It is revealing to re-write the real part of the dispersion relation (52a) as

$$\left[k - \frac{\beta}{-2R\omega_r}\frac{1-\sigma^2}{1+\sigma^2}\right]^2 + l^2 = \left[\frac{\beta}{2R\omega_r}\frac{1-\sigma^2}{1+\sigma^2}\right]^2. \tag{53}$$

Consider a wave with $k > 0$ and $\omega_r < 0$ (Figure 3). In order for the wave to preserve its orientation $\mathbf{K}/K$ the horizontal scale of the wave must increase as the ratio of wave decay rate to wave frequency increases. Conversely, in order for the wave to maintain its horizontal scale, the horizontal wave vector must become more zonal as σ is enhanced.

The dispersion relation (52a–52b) is only implicit in the sense it does not express wave frequency as a function of wave vector. A complete description of the wave, however, can be obtained for the long-wave limit ($K \to 0$). In this limit, equations (51a–51b) are simultaneously satisfied if

$$\sigma = \pm 1, \tag{54a}$$

$$\epsilon_r \lambda_r = \mp \frac{1}{8}, \tag{54b}$$

where σ and $\epsilon_r \lambda_r$ must have opposite signs. Condition (54a) implies that the imaginary part of ω is numerically equal to its real part. A positive ω_i is rejected, as the pressure p would then be unbounded in time according to (12). With $\sigma = \pm 1$ and $\omega_i < 0$, the long wave as defined by the second root of the eigenvalue relation (47) is critically damped, in the sense that the wave is attenuated at a rate which is equal to its frequency. Condition (54b) together with the definitions (41a, 42a) allows us to determine the decay rate of this wave

$$R\omega_i = -\frac{1}{|f|}\sqrt{2E_v B\beta|k|}, \tag{55a}$$

or

$$\omega_{i*} = -\frac{N}{|f_*|}\sqrt{2\kappa_v \beta_* |k_*|}. \tag{55b}$$

The decay rate increases as $\sqrt{E_v}$ or $\sqrt{\kappa_v}$ and as $\sqrt{|k|}$ or $\sqrt{|k_*|}$, which is to be contrasted with the results for an unbounded medium (Eqs. 11a–11b).

It is instructive to examine the modulus of the eigenfunction $r_+ r_-(r_1 + r_2)$ (Eq. 47) for different combinations of values for the damping factor σ and the diabatic factor ϵ

(Figures 4a–4b). For σ equal to 0 and ϵ equal to, say, $1/(8 \cdot 16\pi^2)$, the modulus exhibits relative minima near $\lambda_r = \pi^2, 4\pi^2, 9\pi^2, 16\pi^2 \ldots$, which are the eigenvalues of adiabatic theory (Figure 4a). None of these minima, however, is exactly nil, since the baroclinic modes of adiabatic theory are not exact solutions of the vertical structure equation if this is augmented with even the slightest amount of vertical density diffusion. The only vanishing eigenvalue is $\lambda_r = 0$, which corresponds to the undamped wave (21). For $\sigma = -1$ and $\epsilon = \epsilon_r(1 - i\sigma) = (1 - i\sigma)/(8 \cdot 16\pi^2)$, the modulus of $r_+ r_-(r_1 + r_2)$ is zero for $\lambda_r = 0$ and, for $K = 0$, $\lambda_r = 16\pi^2$ (Figure 4b). The first mode corresponds to the undamped wave (21) and the second mode corresponds to the critically damped wave in the long-wave limit (55a–55b).

4.2 Variable Stratification

4.2.1 General Solution

For $E_h = 0$ and $B_z \neq 0$, the vertical structure equation (13) is

$$iE_v \left(\frac{\hat{p}_{zzz}}{B} \right)_z - R\omega \left(\frac{\hat{p}_z}{B} \right)_z + \frac{R\omega K^2 + \beta k}{f^2} \hat{p} = 0, \tag{56}$$

where the dependence of B on depth is again made momentarily implicit. Inserting a WKB solution of the form (24) into this equation yields

$$\begin{aligned} iE_v \left[\frac{(S_0')^4}{\epsilon^4} + \ldots \right] - iE_v \frac{B_z}{B} \left[\frac{(S_0')^3}{\epsilon^3} + \ldots \right] \; - \\ R\omega \left[\frac{(S_0')^2}{\epsilon^2} + 2\frac{S_0' S_1'}{\epsilon} + \frac{S_0''}{\epsilon} + \ldots \right] + R\omega \frac{B_z}{B} \left[\frac{S_0'}{\epsilon} + \ldots \right] \; + \\ \left[R\omega K^2 + \beta k \right] \frac{B}{f^2} \;\; = \;\; 0. \end{aligned} \tag{57}$$

The eikonal equation is taken as

$$-R\omega \frac{(S_0')^2}{\epsilon^2} + \left[R\omega K^2 + \beta k \right] \frac{B}{f^2} = 0. \tag{58}$$

The solution of this equation is, with the choice $\epsilon = \sqrt{R}$ and apart from an additive constant,

$$S_0(z) = \pm i \sqrt{-\frac{R\omega K^2 + \beta k}{f^2 \omega}} \int_{-1}^{z} \sqrt{B(z')} dz'. \tag{59}$$

This first contribution to the pressure function $\hat{p}(z)$ assumes the parameter relationship $E_v < \mathrm{O}(R^2)$ and is purely adiabatic. In order for the second contribution to incorporate the effects of vertical mixing the transport equation must be taken as (since $B_z/B = \mathrm{O}(1)$)

$$iE_v \frac{(S_0')^4}{\epsilon^4} - R\omega \left(2\frac{S_0' S_1'}{\epsilon} + \frac{S_0''}{\epsilon} \right) + R\omega \frac{B_z}{B} \frac{S_0'}{\epsilon} = 0. \tag{60}$$

Its solution is

$$S_1(z) = \ln \left[\frac{R\omega K^2 + \beta k}{B(z) f^2 \omega} \right]^{-1/4} \pm \frac{E_v}{2R^{5/2}\omega} \left(-\frac{R\omega K^2 + \beta k}{f^2 \omega} \right)^{3/2} \int_{-1}^{z} B(z')^{3/2} dz', \tag{61}$$

to within another additive constant. This second contribution to $\hat{p}(z)$ corresponds to the parameter relationship $E_v = \mathrm{O}(R^{5/2})$. Its first part is formally identical to the solution of the transport equation for $E_h \neq 0$ and $E_v = 0$ (Eq. 30), whereas its second part represents the influence of vertical mixing. A general solution of the vertical structure equation (56) is constructed from the first two contributions to $\hat{p}(z)$,

$$\begin{aligned} \hat{p}_{\mathrm{WKB}}(z) \;=\; & B(z)^{1/4} \left[C_1 \cos \left(\int_{-1}^{z} \sqrt{\ell B(z')} dz' \right) + C_2 \sin \left(\int_{-1}^{z} \sqrt{\ell B(z')} dz' \right) \right] \\ & \left[C_3 \exp \left(\frac{\epsilon}{2} \int_{-1}^{z} \sqrt{[\ell B(z')]^3} dz' \right) + C_4 \exp \left(-\frac{\epsilon}{2} \int_{-1}^{z} \sqrt{[\ell B(z')]^3} dz' \right) \right], \end{aligned} \tag{62}$$

where ℓ is defined by (32) with ϖ replaced by ω and ϵ is defined by (40a). For future analysis it is convenient to express this solution as

$$\tilde{p}_{\mathrm{WKB}}(z) = C_1 e^{\phi_+(z)} + C_2 e^{-\phi_+(z)} + C_3 e^{\phi_-(z)} + C_4 e^{-\phi_-(z)}, \tag{63}$$

where $\tilde{p}_{\mathrm{WKB}}(z) = \hat{p}_{\mathrm{WKB}}(z)/B(z)^{1/4}$ and

$$\phi_\pm(z) = i \int_{-1}^{z} \sqrt{\ell B(z')} dz' \pm \frac{\epsilon}{2} \int_{-1}^{z} \sqrt{[\ell B(z')]^3} dz', \tag{64}$$

so that

$$\phi_\pm'(z) = i\sqrt{\ell B(z)} \pm \frac{\epsilon}{2} \sqrt{[\ell B(z)]^3}. \tag{65}$$

For simplicity the particular (but again oceanographically relevant) case where $B_z = 0$ at $z = -1, 0$ is only considered below.

4.2.2 Free Wave Solutions

When recast in terms of $\tilde{p}$ the boundary conditions (14a–14b) are

$$i\epsilon\tilde{p}_{zzz} - \tilde{p}_z = 0 \qquad \text{at } z = 1, 0, \tag{66a}$$

$$\tilde{p}_{zz} = 0 \qquad \text{at } z = 1, 0. \tag{66b}$$

Applying these conditions to the general solution (63) gives the eigenvalue relation

$$\left(i\epsilon\left[\phi'_+(-1)\right]^2 - 1\right)\phi'_+(-1)\phi'_-(0)r_1 + \left(i\epsilon\left[\phi'_-(-1)\right]^2 - 1\right)\phi'_-(-1)\phi'_+(0)r_2 = 0, \tag{67}$$

where

$$\begin{aligned} r_1 &= \left(i\epsilon\left[\phi'_-(0)\right]^2 - 1\right)\left(\left[\phi'_+(-1)\phi'_-(0)\right]^2 - \left[\phi'_-(-1)\phi'_+(0)\right]^2 \cosh\phi_-(0)\cosh\phi_+(0)\right) \\ &+ \left(i\epsilon\left[\phi'_+(0)\right]^2 - 1\right)\left[\phi'_-(-1)\right]^2 \phi'_-(0)\phi'_+(0)\sinh\phi_-(0)\sinh\phi_+(0), \end{aligned} \tag{68}$$

and

$$\begin{aligned} r_2 &= \left(i\epsilon\left[\phi'_+(0)\right]^2 - 1\right)\left(\left[\phi'_-(-1)\phi'_+(0)\right]^2 - \left[\phi'_+(-1)\phi'_-(0)\right]^2 \cosh\phi_-(0)\cosh\phi_+(0)\right) \\ &+ \left(i\epsilon\left[\phi'_-(0)\right]^2 - 1\right)\left[\phi'_+(-1)\right]^2 \phi'_-(0)\phi'_+(0)\sinh\phi_-(0)\sinh\phi_+(0). \end{aligned} \tag{69}$$

A first root of (67) is $\ell = 0$, which yields the dispersion relation $R(\omega_r + i\omega_i) = -\beta k/K^2$. This undamped wave was also found for uniform density stratification (section 4.2.1). A question of interest is whether the presence of variable stratification allows also for the damped waves found with uniform N (Eqs. 52a–52b). This indeed is the case at least in the situation where N near the bottom approaches N near the surface, i.e., if $B(-1) = B(0) = B$, with $B(z)$ still being completely arbitrary between these two levels. Albeit particular, this vertical density distribution mimics the oceanic situation where N near the bottom and the surface are both relatively small compared to values in the thermocline. In this situation, $\phi'_-(-1) = \phi'_-(0) = \phi'_-$ and $\phi'_+(-1) = \phi'_+(0) = \phi'_+$, so the eigenvalue relation (67) becomes

$$\left(\phi'_-\phi'_+\right)^2 (r_1 + r_2) = 0, \tag{70}$$

where

$$r_1 = 2\phi'_-\phi'_+\left(i\epsilon\phi'^2_+ - 1\right)\left(i\epsilon\phi'^2_- - 1\right)\left(1 - \cosh\phi_-(0)\cosh\phi_+(0)\right), \tag{71a}$$

$$r_2 = \left(\phi'^2_+\left[i\epsilon\phi'^2_- - 1\right]^2 + \phi'^2_-\left[i\epsilon\phi'^2_+ - 1\right]^2\right)\sinh\phi_-(0)\sinh\phi_+(0). \tag{71b}$$

Note the formal similarity with the exact relation obtained for constant N (Eq. 47). The modified relation (70) has the root $\phi'_- = \sqrt{\ell B}(i - \epsilon\ell B/2) = 0$ (the situation $\phi'_+ = \sqrt{\ell B}(i+\epsilon\ell B/2) = 0$ allows a negative ω_i for $\sigma^2 > 1$ and is not considered below). Omitting the possibility $\ell = 0$ already considered, this root implies

$$(\epsilon_r\ell_i + \epsilon_i\ell_r)\,B = 2, \tag{72a}$$

$$(\epsilon_r\ell_r - \epsilon_i\ell_i)\,B = 0, \tag{72b}$$

where ℓ_r and ℓ_i are the real and imaginary parts of ℓ. In contrast to the situation $\ell = 0$, this system requires the imaginary part of ω to be different from zero ($\ell_i \neq 0$ and $\epsilon_i \neq 0$). It leads to the dispersion relation

$$R\,(\omega_r + i\omega_i) = -\frac{\beta k}{K^2}\frac{1-\sigma^2}{1+\sigma^2} - iE_v\frac{BK^2}{2f^2}\frac{\sigma^2}{1-\sigma^2}, \tag{73}$$

which is identical to that found for uniform stratification (Eq. 52a) save for a constant value for the second term on the right-hand side. In the long-wave limit, the wave is critically damped at a rate

$$R\omega_i = -\frac{1}{2|f|}\sqrt{E_vB\beta|k|}. \tag{74}$$

The results above obtained with $B(-1) = B(0)$ obviously hold also for uniform stratification. Note that for uniform stratification the decay rate for $K \to 0$ in the WKB approximation (Eq. 74) is underestimated by a factor of $2\sqrt{2}$ compared to the exact result (55a). The underestimation arises from the omission in $\hat{p}_{\mathrm{WKB}}(z)$ of contributions from the most structured components which are the most prone to attenuation by vertical mixing (as indicated by the dispersion relation for an unbounded medium (11a)).

5 Discussion

Our major results are summarized. It is found that the linear hydrostatic system augmented with density diffusion in the horizontal or vertical direction supports the un-

damped Rossby wave $R(\omega_r + i\omega_i) = -\beta k/K^2$, which is the barotropic mode of adiabatic theory. This result is expected, since fluid parcels in a barotropic wave do not experience any buoyancy forces and are therefore insensitive to the vortex stretching induced by mixing (Eq. 8). On the other hand, density diffusion modifies the properties of baroclinic waves. Horizontal diffusion does not alter the frequency of the waves and only generates damping (Eqs. 22a–22b and Eqs. 37a–37b). The capability of horizontal diffusion to damp baroclinic waves stems from its tendency to counteract the deformation of isopycnal surfaces which is produced by the passage of these waves. The damping rate increases (i) linearly with the horizontal diffusivity and (ii) nonlinearly with the horizontal wave number and mode number (Eqs. 22a–22b). When the horizontal wavelength is considerably smaller than the internal radius of deformation, the damping rate becomes insensitive to the horizontal scale of the wave (Figure 1). This behavior is identical to the decay caused by bottom friction on a barotropic wave in a layer of uniform density (*Pedlosky*, 1987). Similar results are obtained for variable background stratification with uniform N only near the bottom and the surface (Eqs. 37a–37b).

Vertical diffusion, on the other hand, modifies both the real and imaginary parts of the complex frequency of Rossby waves. For increasing values of $\sigma = \omega_i/\omega_r$ the wave period must increase in order for the wave to maintain both its scale and orientation in the horizontal plane (Eq. 53; Figure 3). For a fixed σ the rate of damping increases linearly with vertical diffusivity and quadratically with the horizontal wave number (Eqs. 52a–52b; Figure 2b). A complete description of the wave is obtained for the long-wave limit. In this limit the wave is critically damped at a rate which increases linearly with the square roots of vertical diffusivity and zonal wave number (Eqs. 55a–55b). Similar results are found for variable background stratification with (i) uniform N near the bottom and the surface; and (ii) identical N at these two levels (Eq. 73).

The remainder of this section discusses the oceanographic implications of our results. The relevance of the discussion obviously depends on Fickian diffusion being an appropriate model for the effects of small-scale buoyancy transport on Rossby waves and on the accuracy of the available estimates of (κ_h, κ_v) in the ocean. Density diffusion can have

at least three effects on the waves: (i) an attenuation of amplitude; (ii) a modification of phase speed; and (iii) a change in vertical structure. The three effects are discussed in turn below.

5.1 Amplitude Attenuation

The importance of amplitude attenuation for the wave is measured by the ratio of wave decay rate to wave frequency (σ). Consider first σ in the presence of horizontal diffusion. For uniform background stratification the ratio σ is given by (from Eq. 22b)

$$\sigma = \kappa_h \frac{K_*^2}{\beta_* k_*} \left(\frac{n\pi}{L_D} \right)^2 . \tag{75}$$

This expression also holds for variable stratification provided that L_D is based on the depth-integral of N (Eq. 37b). Oceanic observations are used below in order to derive plausible estimates of σ. It is important to note that the horizontal diffusivity κ_h and the wave numbers (K_*, k_*) in Eq. (75) cannot be chosen independently, since κ_h depends on the scale of motion it is intended to parameterize. Estimates of horizontal dispersion on several scales are available from the North Atlantic Tracer Release Experiment (NATRE) which took place near 26°N, 28°W (*Ledwell et al.*, 1998). A passive tracer was released along an isopycnal surface near 300 m depth and surveyed over a period of 30 months as it dispersed in different directions. *Ledwell et al.* (1998) concluded that Fickian diffusion may perhaps be an appropriate model for tracer dispersion at lateral scales from 300 km to 1000 km, i.e., larger than the scales of the mesoscale eddy field. The eddy diffusivity estimated from the tracer dispersion at these scales was of the order of 1000 $\mathrm{m^2\ s^{-1}}$, which is consistent with estimates for this oceanic region based on other approaches such as deep float drifting (see references in *Ledwell et al.* (1998); *Ollitrault and Colin de Verdière* (2002)). For simplicity consider an estimate of σ at NATRE for the first baroclinic mode ($n = 1$) with zero meridional wave number ($K_* = k_*$). Expression (75) is used with a climatologic estimate of L_D near 26°N, 28°W (Fig. 6 of *Chelton et al.* (1998)). Thus, assuming $\kappa_h = 10^3\ \mathrm{m^2\ s^{-1}}$ (*Ledwell et al.*, 1998), $L_D = 4.5 \times 10^4$ m (*Chelton et al.*, 1998), and $\beta_* = 2.1 \times 10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$, it comes $\sigma = 1.2$ for $k_* = 5 \times 10^{-6}\ \mathrm{m^{-1}}$ (long wave) and

$\sigma = 2.3$ for $k_* = 10^{-5}$ m^{-1} (short wave). It seems therefore that horizontal mixing could cause significant attenuation over the wave period, even for the first mode.

Consider then σ in the presence of vertical diffusion. For uniform stratification σ should satisfy (from Eq. 52b),

$$\left(1-\sigma^2\right)^2 = \sigma\left(1+\sigma^2\right)\kappa_v \frac{K_*^4}{\beta_* k_*}\left(\frac{N}{f_*}\right)^2, \tag{76}$$

where it has been assumed that $\sigma \neq 0$. The factor $\psi = \kappa_v(K_*^4/(\beta_* k_*))(N/f_*)^2$ is very small for waves with $K_* \approx k_*$, as shown for example by observations at NATRE. The vertical (actually diapycnal) diffusivity estimated for the first 6 months of the tracer survey was $(0.12 \pm 0.02) \times 10^{-4}$ m^2 s^{-1}, whilst for the subsequent 24 months it was $(0.17 \pm 0.02) \times 10^{-4}$ m^2 s^{-1} (*Ledwell et al.*, 1998). The vertical tracer distribution remained very close to Gaussian for the full 30 months, as the root mean square dispersion grew from 5 to 50 m. Here a value $\kappa_v = 0.15 \times 10^{-4}$ m^2 s^{-1} is adopted. It is further assumed that (i) $N = 7 \times 10^{-4}$ s^{-1}, which the buoyancy frequency observed at NATRE near 300 m during the May 1993 survey (Fig. 3 of *Ledwell et al.* (1998)), and (ii) $K_* = k_* = 10^{-5}$ m^{-1}, i.e., a wave with a relatively small zonal scale, which should lead to a conservative (i.e., upper) estimate of ψ. With $f_* = 6.4 \times 10^{-5}$ s^{-1} and $\beta = 2.1 \times 10^{-11}$ m^{-1} s^{-1}, it comes $\psi = \mathrm{O}(10^{-7})$. A similar result is found if one considers instead the diapycnal diffusivities at mid-latitudes inferred from lowered ADCP/CTD profiles, which are also of O(10^{-5} m^2 s^{-1}) on average (*Kunze et al.*, 2006). Given the smallness of ψ, the quartic (76) can safely be approximated by $(1-\sigma^2)^2 = 0$, which has the obvious roots $\sigma = \pm 1$ (Figure 5). Thus, the damping caused by vertical diffusion is nearly critical also for waves with small horizontal scales. A similar result can be derived for variable stratification using Eq. (73).

Note that our estimates of σ for both horizontal and vertical mixing are sensitive to the assumed orientation of the wave $\mathbf{K}_*/K_*$. In the presence of horizontal mixing, the amplitude of σ is relatively large for a wave whose phase advances primarily in the meridional direction (Eq. 75). Damping by vertical mixing can also be supercritical for such a wave (Figure 5). This behavior arises from the anisotropy, not in the influence of mixing, but in the planetary β-effect: if phase propagation is mostly meridional, the horizontal motion

of fluid parcels in the quasigeostrophic wave is chiefly zonal, so the parcels experience only small changes in planetary vorticity, i.e., the wave frequency is low. Thus, for waves with comparable horizontal scales (K_*), the ratio of decay rate to frequency is larger for those waves whose phase advances primarily in the north-south direction.

5.2 Change in Phase Speed

Longitude-time sections (Hovmöller diagrams) are commonly used to characterize Rossby waves. The discussion is thus focused on the influence of mixing on the zonal phase speed C,

$$C = \left(\frac{\partial x}{\partial t}\right)_p = -\frac{\partial p/\partial t}{\partial p/\partial x}. \tag{77}$$

With the pressure field $p(x, y, z, t)$ given by Eq. (12), the phase speed has the general form

$$C = \frac{\omega_r}{k}\left[1 + \sigma \cot\left(kx + ly + S(z) - \omega_r t\right)\right], \tag{78}$$

where $S(z)$ is the phase of $\hat{p}(z)$. Thus, in the presence of damping ($\sigma \neq 0$), the isobars in longitude-time sections are deflected from the direction defined by ω_r/k, the amount of deflection being proportional to the ratio of wave decay rate to wave frequency. Consider first the particular form of C in the presence of horizontal mixing. In the long-wave limit the dispersion relation (37a) becomes (from Eq. 22a):

$$R\left(\omega_r + i\omega_i\right) = -\beta k \left(\frac{\overline{B^{1/2}}}{n\pi f}\right)^2 - iE_h K^2. \tag{79}$$

The phase speed C_n can thus be written as

$$C_n = c_n\left[1 + \sigma_n \cot\left(k\left[x - c_n t\right] + ly + S(z)\right)\right], \tag{80}$$

where $c_n = -(\beta/R)(\overline{B^{1/2}}/n\pi f)^2$ is the phase (or group) speed of the nth baroclinic mode and $\sigma_n = E_h(n\pi f/\overline{B^{1/2}})^2 K^2/(\beta k)$ is the value of σ for that mode. Note that c_n is always negative, i.e., long waves in adiabatic theory propagate their phase and energy to the west. The term proportional to σ_n in Eq. (80) is the change in phase speed brought about by horizontal mixing. Let us define $\theta = k\left[x - c_n t\right] + ly + S(z)$. It comes

$$\frac{\partial C_n}{\partial t} = \frac{c_n^2}{\sin^2\theta} E_h \left(\frac{n\pi f}{\overline{B^{1/2}}}\right)^2 \frac{K^2}{\beta}. \tag{81}$$

The rate of change $\partial C_n/\partial t$ is always positive, which shows that horizontal mixing tends to decelerate westward propagating waves.

A similar deceleration occurs in the presence of vertical mixing. It is illustrated for a wave with $k < 0$ and $\omega_r > 0$ (Figure 6). The meridional coordinate y is taken as zero without loss of generality. The pressure perturbation is assumed to be nil at $x = 1$ and $t = 0$, so $S = |k| + \pi/2$. The zonal phase speed is then

$$C = \frac{|\omega|}{\sqrt{2}k}\left[1 - \cot\left(|k|\,(1-x) - \frac{|\omega|t}{\sqrt{2}} + \frac{\pi}{2}\right)\right], \tag{82}$$

where it has been assumed that the wave is critically damped. The region below the line $|k|(1-x) - |\omega|t/\sqrt{2} = 0$ represents undisturbed fluid, whereas the region above that line represents fluid affected by the wave disturbance (Figure 6). The wave is decelerated since

$$\frac{\partial C}{\partial t} = \frac{\omega^2}{2|k|\sin^2\theta} > 0, \tag{83}$$

where, here, $\theta = |k|(1-x) - |\omega|t/\sqrt{2} + \pi/2$. The rate of change $(\partial x/\partial t)_p$ vanishes along the line $|k|(1-x) - |\omega|t/\sqrt{2} + \pi/4 = 0$ (Figure 6). In the region below this line the pressure increases as the wave moves to the west, whilst in the region above this line the pressure decreases as the wave becomes attenuated.

5.3 Change in Vertical Structure

Horizontal diffusion does not modify the vertical structure of (baroclinic) Rossby waves. Indeed, in the presence of horizontal diffusion, the vertical structure equation remains of second order, so the eigenvalues ($n^2\pi^2$, with $n = 1, 2, \ldots$) are those of adiabatic theory. The vertical distribution of the pressure in the wave is therefore unaltered (Eqs. 19 and 31).

Vertical diffusion, on the other hand, raises the order of the vertical structure equation and can therefore influence the vertical structure of baroclinic waves. Such influence can be examined from the third factor in the WKB solution (62),

$$C_3 \exp\left(\frac{\epsilon}{2}\int_{-1}^{z}\sqrt{[\ell B(z')]^3}dz'\right) + C_4 \exp\left(-\frac{\epsilon}{2}\int_{-1}^{z}\sqrt{[\ell B(z')]^3}dz'\right). \tag{84}$$

Two fundamentally different effects are present, which correspond to the real and imaginary parts of the argument

$$\frac{\epsilon}{2}\int_{-1}^{z}\sqrt{[\ell B(z')]^3}dz'. \tag{85}$$

The real part describes an exponentially growing and decaying pair, whereas the second part is an oscillation. The presence of an oscillation indicates that vertical mixing may influence the wave structure far from the horizontal boundaries. Such remote influence justifies a posteriori using the WKB approach as a general solution method, as it provides a global approximation of $\hat{p}(z)$ (a boundary-layer approach, for example, would be inappropriate). It is revealing to consider the long-wave limit with uniform N, in which case

$$\frac{\epsilon}{2}\int_{-1}^{z}\sqrt{[\ell B(z')]^3}dz' = \frac{E_v}{2\left(R|\omega|\right)^{5/2}}\left(\frac{B\beta|k|}{f^2}\right)^{3/2}\left(\cos\frac{5\phi}{2} - i\sin\frac{5\phi}{2}\right)(z+1)\,, \tag{86}$$

where $|\omega| = |\omega_r|\sqrt{1+\sigma^2}$ and $\tan\phi = \sigma$. The length scales $\delta_{\Re}$ and $\delta_{\Im}$, defined by

$$(\delta_{\Re}, \delta_{\Im}) = \delta\left(\sec\frac{5\phi}{2}, \csc\frac{5\phi}{2}\right), \tag{87}$$

where

$$\delta = \frac{2\left(R|\omega|\right)^{5/2}}{E_v}\left(\frac{f^2}{B\beta|k|}\right)^{3/2}, \tag{88}$$

determine, respectively, the exponential and oscillatory influences on the vertical structure of the wave. In dimensional form, the common factor δ is

$$\delta_* = \frac{2}{\kappa_v}\left(\frac{|f_*|}{N}\right)^3\sqrt{\frac{|\omega_*|^5}{\left(\beta_*|k_*|\right)^3}}. \tag{89}$$

This relation can be used to estimate δ_* for a range of values of $|\omega_*|$ and $|k_*|$. The δ_* values obtained in this way can then be usefully represented in the dispersion diagram for the long baroclinic waves of adiabatic theory (Figure 7). Assuming $\kappa_v = 10^{-5}$ m^2 s^{-1}, $|f_*|/N = 0.1$, and $\beta_* = 10^{-11}$ m^{-1} s^{-1}, it is found that the length scale δ_* generally exceeds the ocean depth by several orders of magnitude (Figure 7). Thus, vertical diffusion appears to be too small to significantly affect the vertical structure of baroclinic waves, except for the waves with very small frequencies, i.e., for the high modes. If κ_v is taken as 10^{-4} m^2 s^{-1}, as estimated for some regions at abyssal depths (e.g., *Kunze et al.* (2006)),

vertical diffusion still appears to be too weak to notably modify the vertical structure of the gravest mode ($n = 1$) (note also that use of the abyssal diffusivity of 10^{-4} m^2 s^{-1} may be questioned for studying the influence of diffusion on baroclinic modes given their near-surface intensification). This result is consistent with numerical solutions of the eigenvalue problem (*Farneti and Killworth*, 2005).

5.4 Comparison to Other Effects

The influences of buoyancy mixing on Rossby waves which are investigated in this paper are briefly compared with other 'secondary' effects on Rossby waves which have been examined in earlier work. Many of these effects were studied in the context of the 'too fast' propagation of long waves (first mode), which has been postulated by *Chelton and Schlax* (1996). *Killworth et al.* (1997), for example, showed that a background zonal flow with vertical shear can speed up long waves as the flow modifies the potential vorticity gradient which is perceived by these waves (see also *Dewar* (1998)). It is found here that density diffusion would rather slow down the westward propagation of long waves (Eqs. 80–82).

Of particular interest to the present study is the work of *Qiu et al.* (1997). These authors used linear shallow water theory to explore the role of horizontal friction (parameterized as eddy diffusion or Newtonian damping) in the propagation of long baroclinic Rossby waves. They found that the layer thickness anomalies generated by free waves emanating from the eastern boundary decay westward at a scale which decreases with latitude and wave frequency. Our study shows that buoyancy mixing could also contribute to the dissipation of boundary-generated waves. Indeed, it is found that the damping factor σ can be of order 1 for both horizontal and vertical mixing (section 5.1), implying that the wave amplitude would be substantially attenuated over a wave period (by $e^{-2\pi}$ for $|\sigma| = 1$). Provided that mixing can be described as Fickian diffusion and that existing estimates of (κ_h, κ_v) are sufficiently accurate, the adiabatic theory of Rossby waves would therefore need to be re-assessed.

6 Conclusions

Linear hydrostatic theory is used in order to study the influences of buoyancy mixing (in the form of density diffusion) on the radiation of Rossby waves in a mid-latitude ocean. The ocean is vertically bounded, but the wave disturbances are assumed to occur far from lateral boundaries. Free wave solutions of the vorticity equation are sought by imposing impermeability and insulation conditions at the bottom and at the surface. No background flow is considered and the bottom is flat.

It is found that the vorticity equation supports the undamped wave $R(\omega_r + i\omega_i) = -\beta k/K^2$, which is the barotropic mode of adiabatic theory. On the other hand, density diffusion modifies the properties of baroclinic modes. Horizontal diffusion damps these modes but does not alter their vertical structure. The damping rate increases (i) linearly with the horizontal diffusivity, and (ii) nonlinearly with the horizontal wave number and mode number. When the horizontal scale of the waves is much smaller than the internal radius of deformation, the damping rate becomes insensitive to K. Vertical diffusion produces both damping and a change in the vertical structure of baroclinic waves. A complete description of the wave is obtained for the long-wave limit. In this limit the wave is critically damped at a rate which increases linearly with the square roots of vertical diffusivity and zonal wave number. Observational estimates of diffusitivies suggest that horizontal and vertical mixing strongly attenuates baroclinic waves in the ocean. The vertical structure of the gravest modes, however, would not be notably affected.

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Figure 1. Damping rate of baroclinic waves as a function of horizontal wave number for the first three baroclinic modes. The damping rate is scaled by the ratio L_D^2/κ_h and the horizontal wave number is scaled by L_D, where L_D is the internal deformation radius and κ_h the horizontal diffusivity.

Figure 2. Dispersion relation in the presence of vertical mixing for uniform background stratification. Panels (a) and (b) show, respectively, the real and imaginary part of ω as a function of the zonal wave number. The meridional component of the wave vector is taken as 1.

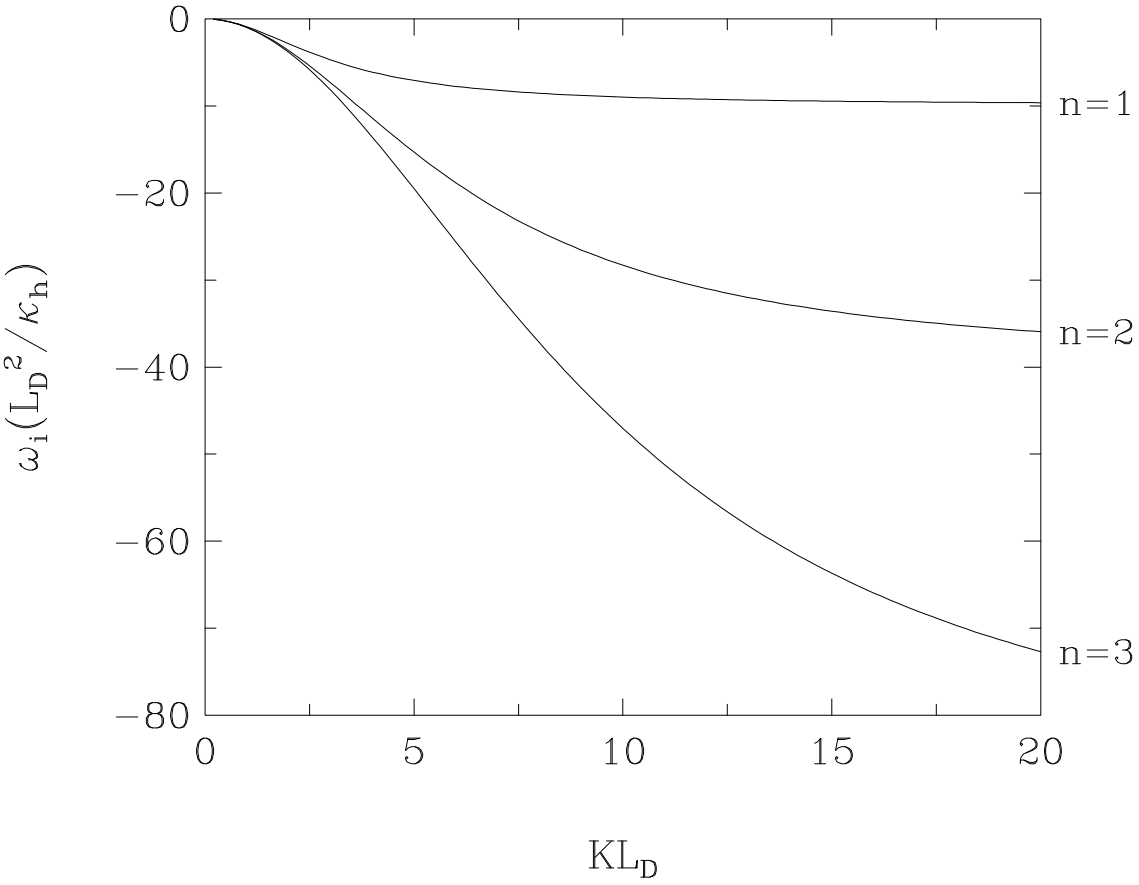
Figure 3. Propagation diagram for Rossby waves in the presence of vertical mixing. The diagram is shown for different values of the ratio of wave decay rate to wave frequency ($\sigma = 0$, 0.2, 0.4, and 0.6). The ratio $\beta/(-2R\omega_r)$ is arbitrarily taken as 1 (its amplitude is much larger in the oceanic situation).

Figure 4. Eigenfunction in the presence of vertical mixing for (a) $\sigma = 0$ and (b) $\sigma = -1$. For both panels $\epsilon_r = 1/(8 \cdot 16\pi^2)$ and $BK^2/f^2 = 0.001$ (the curves with $K = 0$ would be indistinguishable). The eigenfunction is normalized to its maximum value in the displayed λ_r interval (0–200). For reference the eigenvalues of adiabatic theory are shown (vertical dashed lines).

Figure 5. Representation of the functions $g_1(\sigma) = (1 - \sigma^2)^2$ and $g_2(\sigma, \psi) = \sigma(1 + \sigma^2)\psi$, where σ is the ratio of the wave decay rate to wave frequency and $\psi = \kappa_v(K_*^4/(\beta_* k_*))(N/f_*)^2$. The function g_2 is displayed for different arbitrary values of $\psi = 0$, 0.1, and 1. Wave solutions are defined by the points of intersection $g_1 = g_2$.

Figure 6. Field of pressure ($p/\hat{p}$) in a critically damped Rossby wave (for $k = -1$). The zero isoline, which is defined by $|k|(1-x) - |\omega|t/\sqrt{2} = 0$, separates the region undisturbed by wave motion below from the region affected for the passage of the wave above. The dotted line, which is defined by $|k|(1 - x) - |\omega|t/\sqrt{2} + \pi/4 = 0$, separates the region where the pressure increases as the wave phase moves westward from the region where the pressure decreases owing to damping by vertical mixing.

Figure 7. Length scale δ_* describing the influences of vertical diffusion on the vertical structure of baroclinic waves (solid lines; the line labels are the exponents q of $\delta_* = 10^q$ m). The length scale is depicted in the frequency–zonal wave number space and is computed assuming $\kappa_v = 10^{-5}$ m^2 s^{-1}, $|f_*|/N = 0.1$, and $\beta_* = 10^{-11}$ m^{-1} s^{-1}. The relation between frequency and zonal wave number of adiabatic theory is also shown for the first four modes (dashed lines).



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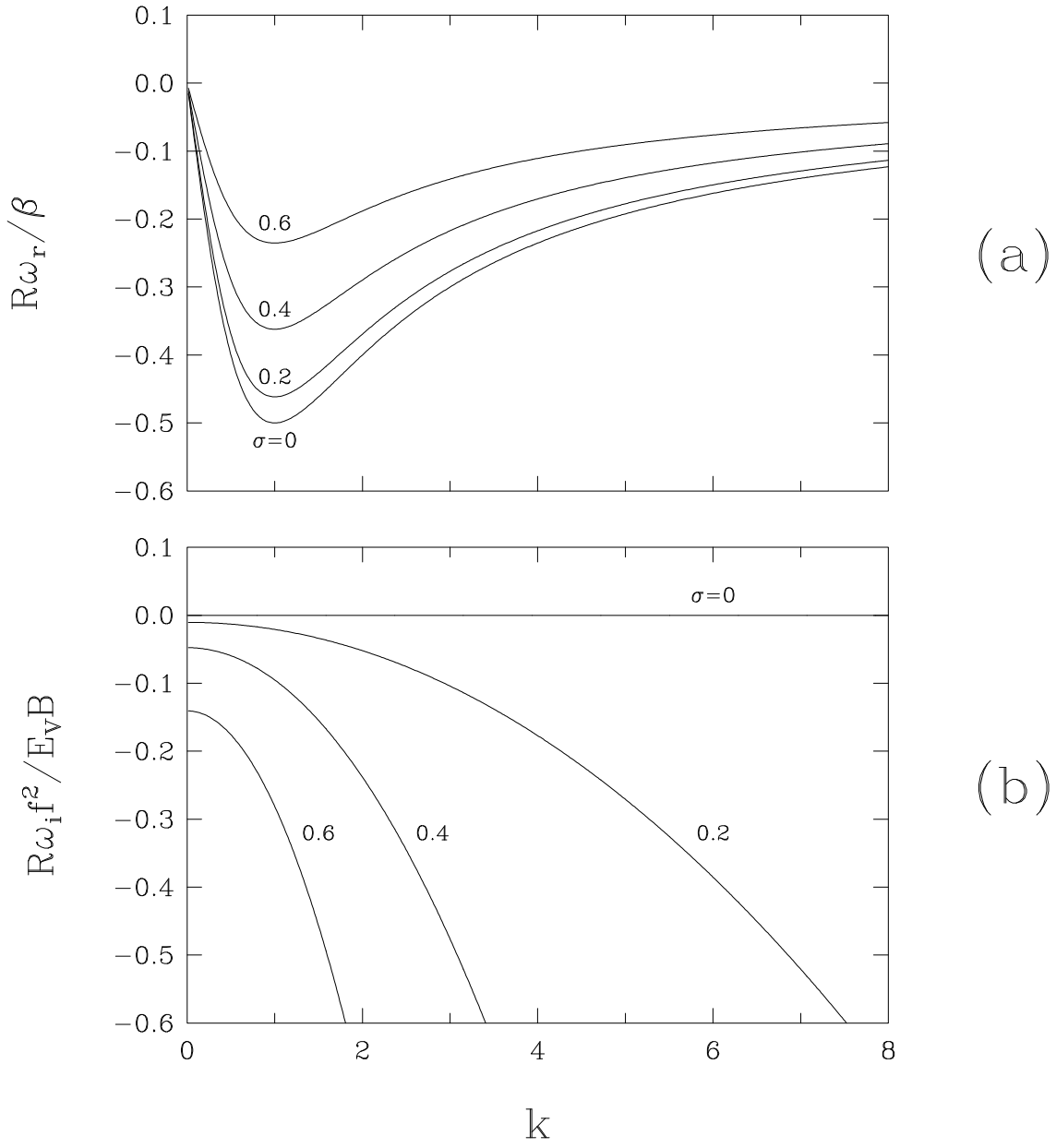

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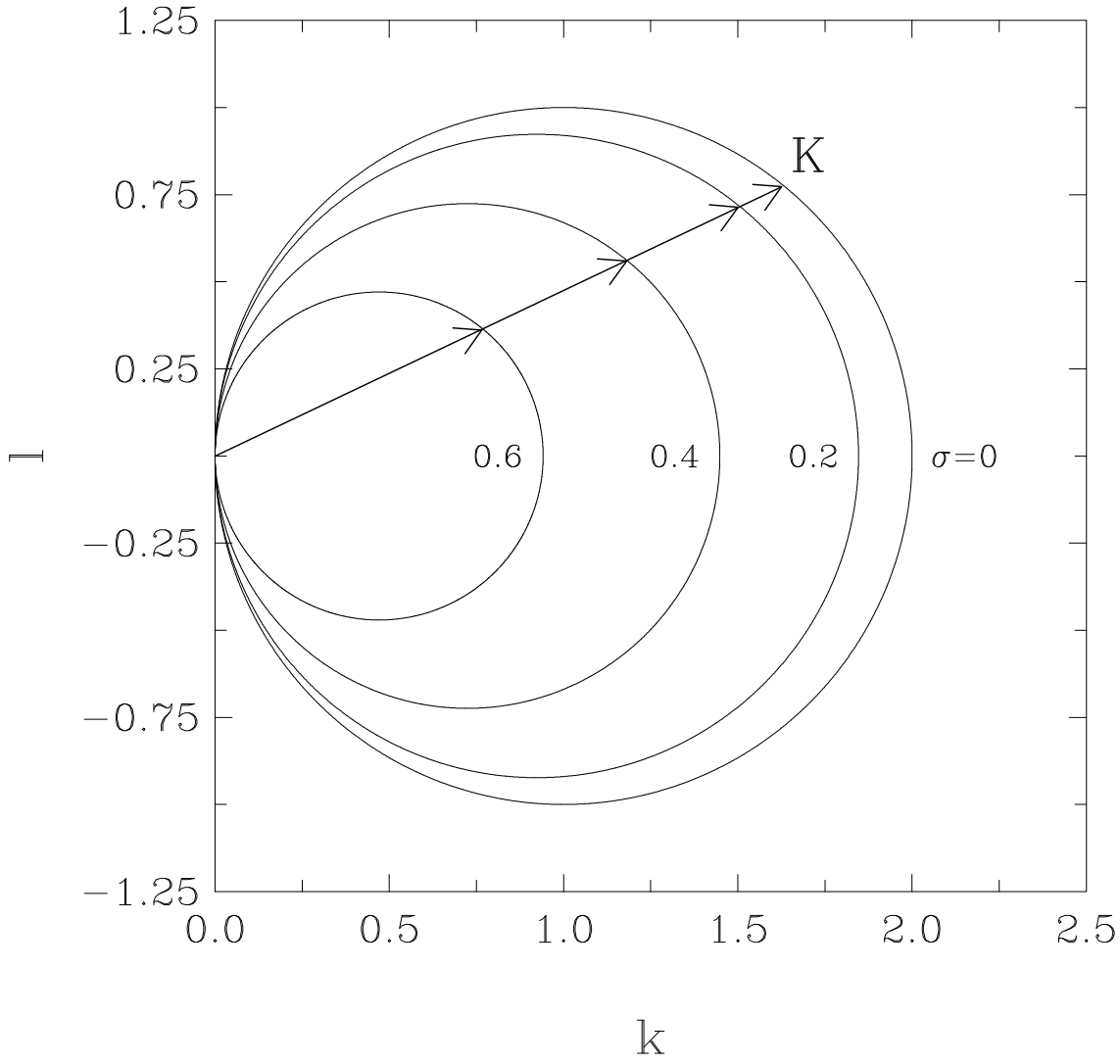

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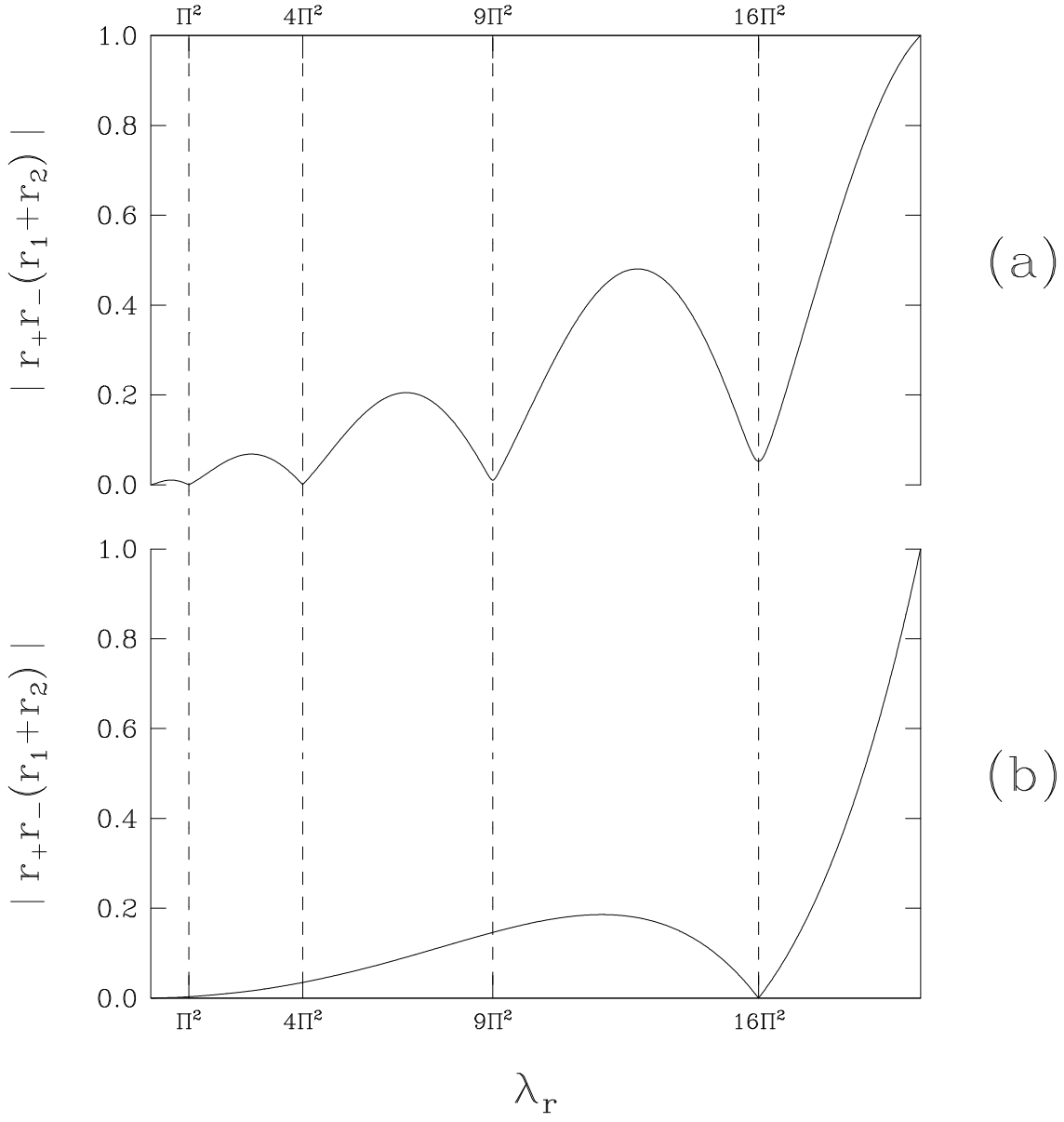

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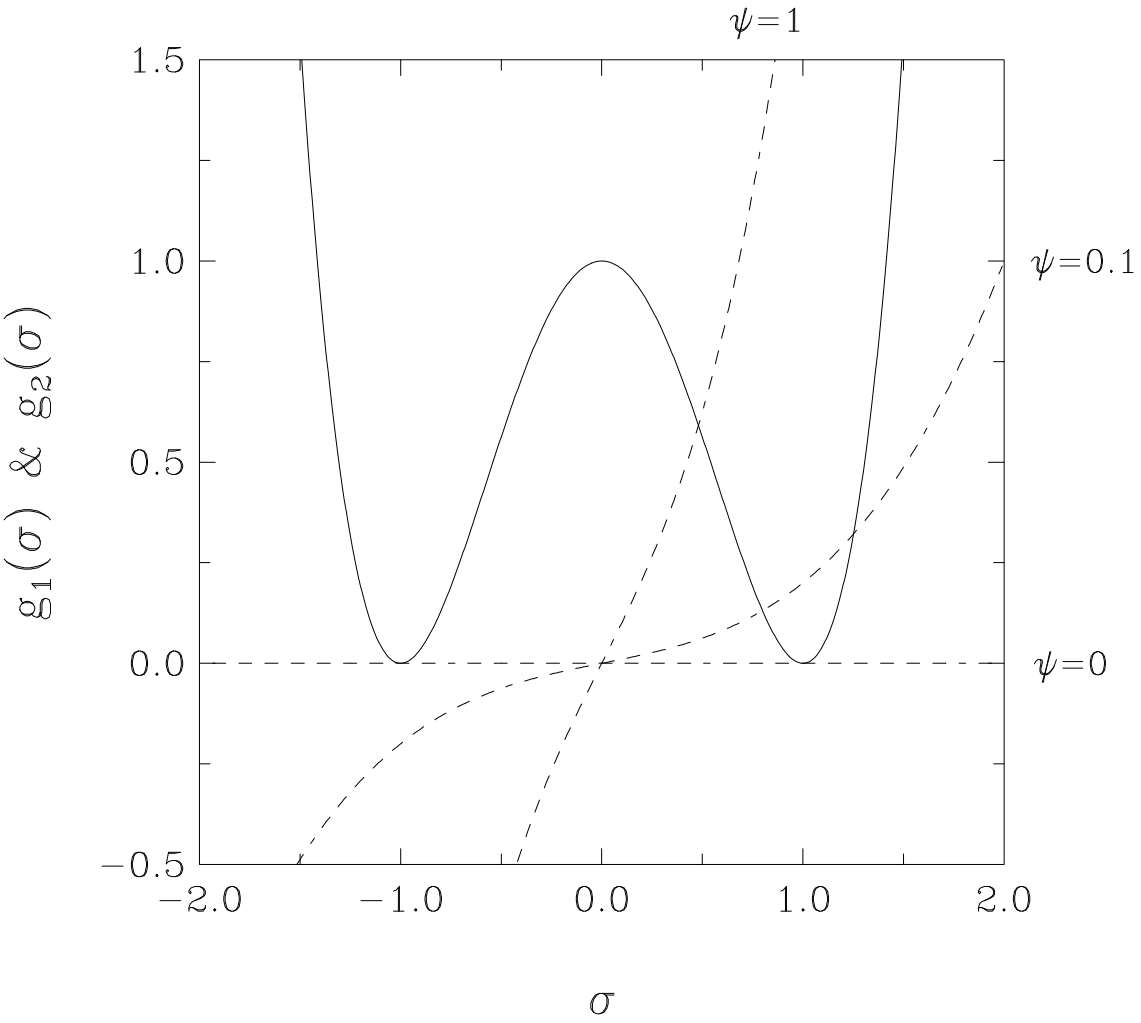

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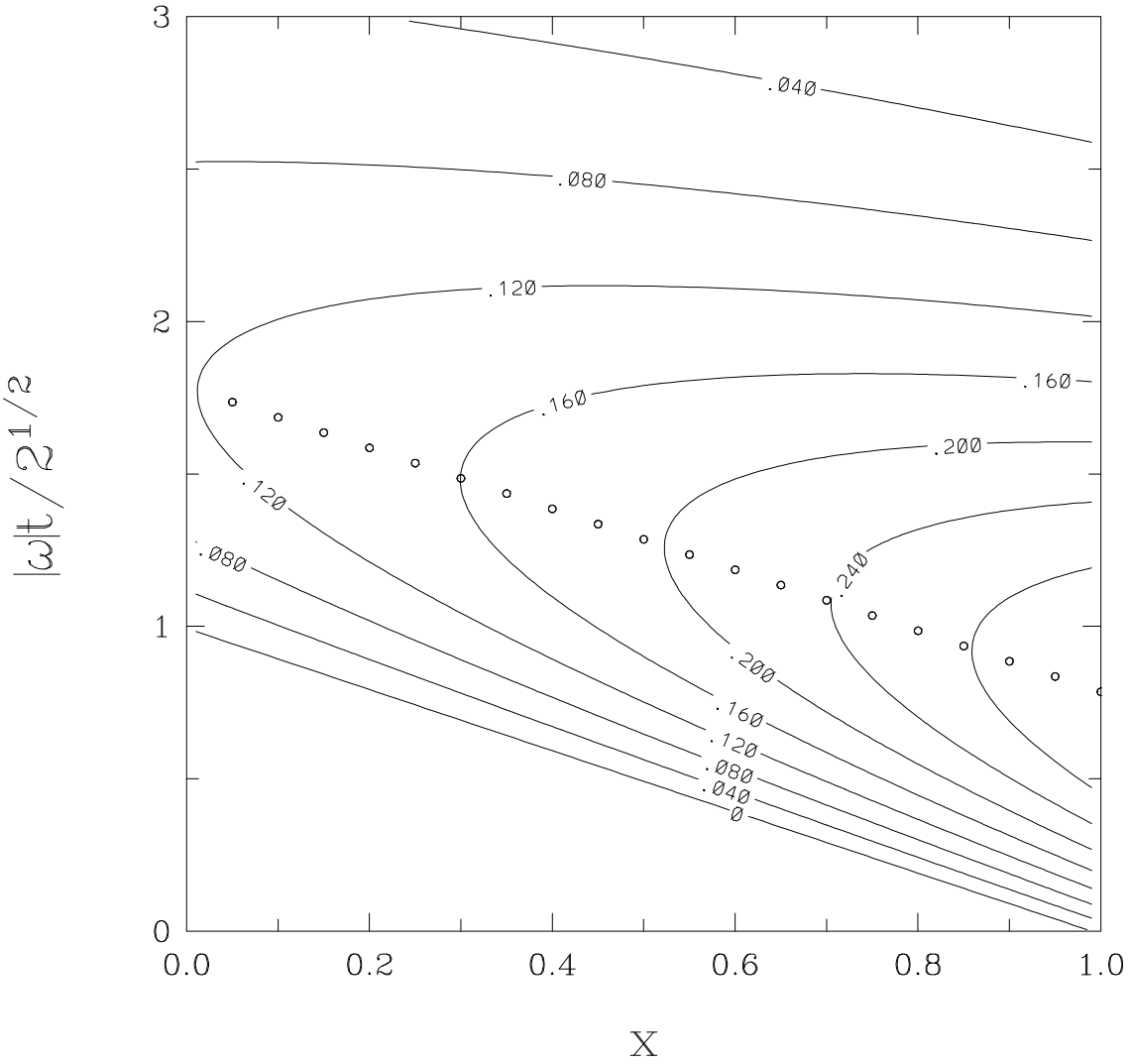

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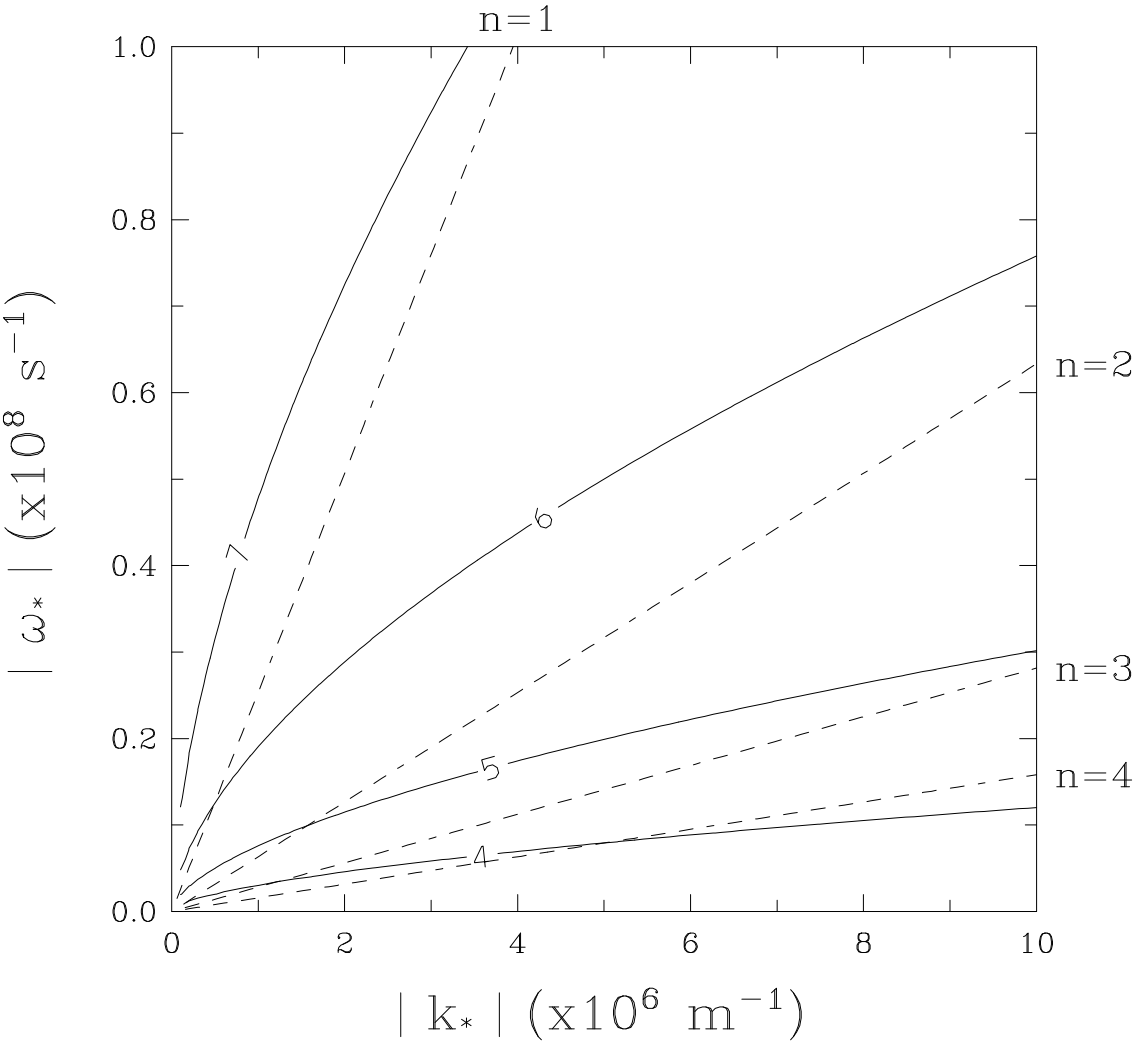

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